FOURIER COEFFICIENTS OF MODULAR FORMS ON G_2

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Abstract. We develop a theory of Fourier coefficients for modular forms on the split exceptional group G_2 over $\mathbb{Q}$.

Introduction

One of the most surprising aspects in the classical theory of modular forms f on the group $SL_2(\mathbb{Z})$ is the wealth of information carried by the Fourier coefficients $a_n(f)$, for $n \geq 0$. The Fourier coefficients of Eisenstein series were calculated by Hecke and Siegel, and are instrumental in the study of zeta functions at negative integers. The Fourier coefficients of theta series have been studied since Jacobi; they give many deep results on Euclidean lattices, such as the unicity of the Leech lattice. Finally, the action of Hecke operators on Fourier coefficients goes back to Mordell, and allows one to show that the Mellin transform of an eigenform is an L-function with Euler product. For an introduction to these basic results, the reader can consult [S] or [R].

Siegel developed a theory of Fourier coefficients $c_N(f)$ for holomorphic forms f on the symplectic group $Sp_{2g}(\mathbb{Z})$. Here the coefficients, for forms of even weight, are indexed by positive semi-definite, integral even quadratic spaces N of rank g. There is an analogous theory for holomorphic forms on tube domains, where the Fourier coefficients are indexed by orbits on integral elements in the corresponding homogeneous cone.

On the other hand, one has a less refined notion of Fourier coefficients for a general automorphic form f on a general reductive group G. Given any parabolic subgroup $P = M \cdot N$ of G and a unitary character χ of $N(\mathbb{A})$ trivial on $N(\mathbb{Q})$, the χ-th Fourier coefficient of f is the function on $G(\mathbb{A})$ given by

$$f_\chi(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng) \cdot \overline{\chi(n)} dn.$$

This notion of Fourier coefficients is useful for many purposes, such as the definition of cusp forms, but being functions rather than numbers, it is often difficult to extract arithmetic information from the coefficients f_χ. For arithmetic applications, it is thus desirable to have a refined theory of Fourier coefficients analogous to that for the holomorphic forms discussed above.

In this paper, we develop such a theory of Fourier coefficients for certain modular forms on the exceptional Chevalley group $G_2(\mathbb{Z})$. Here the symmetric space $X = G_2(\mathbb{R})/\mathrm{SO}_4$ does not have an invariant complex structure; there are thus no holomorphic modular forms. The real components of the automorphic representations we will consider are in the quaternionic discrete series [GW]. For forms of even weight, we will show that the Fourier coefficients

$c_A(f)$ are indexed by totally real cubic rings A: commutative rings with unit, which are free of rank 3 over $\mathbb{Z}$ and such that the $\mathbb{R}$-algebra $A \otimes \mathbb{R}$ is isomorphic to $\mathbb{R}^3$.

The definition of the Fourier coefficients requires some background on the Heisenberg parabolic subgroup $P \subset G_2$; this is provided in the first 3 sections. We then determine the orbits of its Levi factor, which is isomorphic to GL_2, on the space of binary cubic forms. These orbits correspond to cubic rings, and the orbits of primitive forms (namely those for which the gcd of the coefficients is equal to 1) correspond to the Gorenstein cubic rings over $\mathbb{Z}$. We then combine our results on orbits with a recent result of Wallach [W], on the uniqueness of certain linear forms on quaternionic discrete series representations of $G_2(\mathbb{R})$, to give a definition of $c_A(f)$ in Section 8. Once the coefficients have been defined, it is natural to ask if the Fourier coefficients $c_A(f)$ determine the form f. Unlike the case of modular forms on $SL_2(\mathbb{Z})$, this is not automatic, but we show in Section 8 that it is the case if f is a cusp form.

As an illustration of the theory, we calculate the Fourier coefficients for Eisenstein series (Section 9) and analogs of theta series (Section 10). There is a natural family of Eisenstein series E_{2k} (of even weight $2k$) which was first investigated by Jiang and Rallis [JR]. Assuming an extension of their local results, we show that for a maximal cubic ring A, the A-th Fourier coefficient of the Eisenstein series E_{2k} is the non-zero rational number $\zeta_A(1-2k)$. The analogs of theta series are construced via the dual pair correspondence arising from the restriction of the minimal representation of the quaternionic form of the exceptional group E_8 [Ga]. The A-th Fourier coefficients of the analogs of theta series count embeddings of the ring A into integral exceptional Jordan algebras, just as the coefficients of Siegel theta series count embeddings of quadratic spaces over $\mathbb{Z}$.

The rest of the paper studies the action of spherical Hecke operators on Fourier coefficients. We give some background on the general theory in Section 11 and then work out in Section 13 the relative Satake transform when $G = G_2$ and $L = GL_2$ is the Levi factor of the Heisenberg parabolic subgroup P. Using this transform, we determine the action of the two generators of the spherical Hecke algebra at p on the Fourier coefficients. This involves the determination of single coset representatives for the double cosets corresponding to the two generators and the computations are carried out in Section 14. The resulting formulas in Section 15 are analogs of the well-known formula

$$a_n(T_p|f) = a_{np}(f) + p^{2k-1} a_{n/p}(f)$$

for the action of the Hecke operator T_p on the Fourier coefficients of a holomorphic modular form f of weight $2k$ on $SL_2(\mathbb{Z})$. Finally, we show in the last section that if f is a Hecke eigenform, then the primitive coefficients (i.e. those at Gorenstein cubic rings) and the Hecke eigenvalues determine the rest of the coefficients and hence f (if f is a cusp form). This is the analog of the classical result that if f is a holomorphic cuspidal Hecke eigenform on $SL_2(\mathbb{Z})$, then f is determined by $a_1(f)$ and its Hecke eigenvalues.

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Contents

1. Maximal Parabolic Subgroups

We begin by reviewing some material on maximal parabolic subgroups in simple algebraic groups (c.f. [Bo], [BoT] and [Sp]). Let G be a simple algebraic group of adjoint type over an algebraically closed field k. Let $\mathfrak{g} = \mathrm{Lie}(G)$, and let $T \subset B \subset G$ be a maximal torus, contained in a Borel subgroup. Let Δ be the set of simple roots for T determined by B.

The maximal parabolic subgroups P of G which contain B are associated to simple roots α. If $\theta = \Delta - \{\alpha\}$, and W_θ is the subgroup of the Weyl group of G generated by the simple reflections in θ, then $P = \bigcup_{w \in W_\theta} BwB$. There is a unique Levi factor L of P which contains T; it has Lie algebra

$$\mathrm{Lie}(L) = \mathrm{Lie}(T) \oplus \left(\bigoplus_{m_\alpha(\beta)=0} \mathfrak{g}_\beta \right)$$

where $m_\alpha(\beta)$ is the multiplicity of α in the root β, and $\mathfrak{g}_\beta$ is the one-dimensional root space corresponding to β. The unipotent radical U of P has Lie algebra

$$V = \mathrm{Lie}(U) = \bigoplus_{m_\alpha(\beta)>0} \mathfrak{g}_\beta .$$

The center of L, which is isomorphic to $\mathbb{G}_m$, acts on V and gives a grading of k-vector spaces

$$V = \bigoplus_{n \geq 1} V_n, \tag{1.1}$$

$$V_n = \bigoplus_{m_\alpha(\beta)=n} \mathfrak{g}_\beta .$$

Each subspace V_n is a linear representation of L, and the following proposition describes its structure.

Proposition 1.2. *The representation V_n of L is non-zero when $1 \leq n \leq$ the multiplicity of α in the highest root β_0. In this case, the representation V_n is indecomposable.*

If the characteristic p of k is 2 or 3, and there are two roots β_1 and β_2 which satisfy

$$m_\alpha(\beta_1) = m_\alpha(\beta_2) = n$$

$$||\beta_1||^2 = p \cdot ||\beta_2||^2$$

then V_n has a unique irreducible L-submodule, generated by the short root spaces, and the quotient module (which is generated by the long root spaces) is irreducible. In all other cases, when $V_n \neq 0$, the representation V_n of L is irreducible.

Proof. This is a consequence of the results of [ABS], which show that the restriction of V_n to the normalizer of a maximal torus in L has at most 2 irreducible constituents, corresponding to te different lengths of roots β with $m_\alpha(\beta) = n$. The submodule is then determined using a Chevalley basis of $\mathfrak{g}$. □

Now let β_0 be the highest root. If G is not of type A, there is a unique simple root α with $\langle \beta_0^\vee, \alpha \rangle = 1$. Furthermore, α has multiplicity 2 in β_0, and is a long root provided that G is not of type C. The associated maximal parabolic subgroup $P \subset G$ is called the Heisenberg parabolic, and the Lie algebra V of U has a 2-step gradation:

$$\begin{cases} V = V_1 \oplus V_2, \\ V_2 = \mathfrak{g}_{\beta_0}. \end{cases} \tag{1.3}$$

The Lie bracket gives an alternating form on V_1, with values in V_2, and hence gives an L-linear map:

$$f : \wedge^2 V_1 \longrightarrow V_2. \tag{1.4}$$

Proposition 1.5. *If the characteristic p of k is 2 and G is of type C, then $f = 0$ and the Lie algebra V is abelian. In all other cases, $f \neq 0$.*

Assume that $f \neq 0$. If V_1 is an irreducible L-module, then the alternating form f is nondegenerate, and V_2 is the center of V. If V_1 contains a non-trivial L-submodule V_1^{short}, then this is the radical of the alternating form f, and $V_1^{short} \oplus V_2$ is the center of V.

Proof. When $\mathrm{char}(k) = 2$ and G is of type C, all roots in V_1 are short, and the Chevalley relations show that $f = 0$. In all other cases, α and $\alpha' = \beta_0 - \alpha$ are long roots in V_1 with $f(\alpha \wedge \alpha') = [\alpha, \alpha'] \neq 0$.

If $f \neq 0$, the radical is an L-submodule of V_1, and hence is zero when V_1 is irreducible. When V_1 is reducible, we can use the bracket laws on a Chevalley basis to determine the radical of f, and a direct computation proves the proposition. $\square$

2. The Unipotent Radical

We now use some results of Demazure [D, Pg. 438-440] and Serre [S, Pg. 530-531] to convert our knowledge of the Lie algebra $V = \mathrm{Lie}(U) = \oplus_{n\geq 1} V_n$ to information on the unipotent radical U of P.

The unipotent group U has a canonical filtration by L-stable, characteristic subgroups

$$U = U_1 \supset U_2 \supset .... \supset U_d \supset \{1\},$$

where U_i is the product of all root subgroups U_β with $m_\alpha(\beta) \geq i$. Note that in [D], our subgroup U_i is denoted U_{i-1}, so that $U = U_0$. We have:

$$\mathrm{Lie}(U_i) = \oplus_{n\geq i} V_n.$$

Demazure proved that the successive quotients U_i/U_{i+1} are vector groups over k, and that the L-action on then is k-linear. Finally he proved that these representations of L are isomorphic to the representations of L on the k-vector spaces

$$\mathrm{Lie}(U_i/U_{i+1}) = V_i.$$

Hence the results of the previous section show that they are indecomposable $k[L]$-modules.

The commutator gives a map

$$U_i \times U_j \to U_{i+j}.$$

Projecting to the quotient U_{i+j}/U_{i+j+1}, we get an L-bilinear form:

$$U_i/U_{i+1} \times U_j/U_{j+1} \to U_{i+j}/U_{i+j+1}.$$

Results of Serre on the canonical exponential show that this form can be identified with the Lie bracket $V_i \times V_j \to V_{i+j}$. Indeed, if k has characteristic zero (or chartacteristic $p > h$, the Coxeter number of G), there is an isomorphism of unipotent groups:

$$\exp : \mathrm{Lie}(U) \longrightarrow U,$$

where the group structure on $V = \mathrm{Lie}(U)$ is given by the Campbell-Baker-Hausdorff formula [B, Ch. II, §6]:

$$v +_H w = v + w + \frac{1}{2}[v, w] + \frac{1}{12}[v, [v, w]] + \frac{1}{12}[w, [w, v]] + ...$$

The identity of the group $\mathrm{Lie}(U)$ is $v = 0$, and the inverse of v is $-v$. The isomorphism exp is characterized by the fact that its derivative is the identity map on $\mathrm{Lie}(U)$. Moreover, exp is L-equivariant, and the above shows that it can be defined over $\mathbb{Q}$. The exponential induces an isomorphism of subgroups $\mathrm{Lie}(U_i) \to U_i$, and the isomorphism

$$\exp : V_i \longrightarrow U_i/U_{i+1}$$

over $\mathbb{Q}$ has no denominators. It thus gives an isomorphism of these vector groups over $\mathbb{Z}$, and hence a canonical L-isomorphism in all characteristics. Further, since

$$v +_H w +_H (-v) +_H (-w) = [v, w] + ...$$

we see that the commutator map on the graded quotients is indeed the Lie bracket map:

$$V_i \times V_j \longrightarrow V_{i+j}.$$

In the case of the Heisenberg parabolic P, we have filtration:

$$U = U_1 \supset U_2 = U_{\beta_0} \supset \{1\}.$$

Translating the results of Proposition 1.5, we have:

Corollary 2.1. *If char$(k) = 2$ and G is of type C_{n+1}, then $L = GSp_{2n}$ and U is abelian of dimension $2n+1$. In all other cases, the commutator subgroup of U is U_2, and $U^{ab} = U/(U,U)$ is isomorphic to V_1 as an L-module. When this L-module is irreducible (for example, when char$(k) \neq 2,3$), the center of U is isomorphic to U_2.*

3. The Heisenberg Parabolic in G_2

We specialize the results of the preceding two sections to the group $G = G_2$ and the Heisenberg parabolic P. If the simple roots of G are $\Delta = \{\alpha, \alpha'\}$ with α long, then P is associated to α and the Levi factor L of P is simorphic to GL_2, with simple root α'. We have 4 root spaces contributing to V_1:

$$\{\alpha, \alpha+\alpha', \alpha+2\alpha', \alpha+3\alpha'\},$$

and a single root space

$$\beta_0 = 2\alpha + 3\alpha'$$

contributing to V_2.

The pairing

$$f : \wedge^2 V_1 \to V_2$$

is nondegenerate and V_1 is irreducible, provided $\mathrm{char}(k) \neq 3$. If the characteristic of k is 3, then the L-submodule V_1^{short} is spanned by the 2 root spaces:

$$\{\alpha+\alpha', \alpha+2\alpha'\}$$

and this submodule is the radical of the non-zero pairing f. In all cases, $U^{ab} \cong V_1$ is a 4-dimensional representation of L.

Proposition 3.1. *The representation L on the space Hom$(U, \mathbb{G}_a)$ is isomorphic to the twisted representation of GL_2 on the space of binary cubic forms*

$$p(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

over k, where

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2$$

acts by the formula:

$$\gamma \cdot p(x,y) = \frac{1}{det(\gamma)} \cdot p(Ax+Cy, Bx+Dy).$$

Proof. Since $\mathrm{Hom}(U, \mathbb{G}_a) = \mathrm{Hom}(U^{ab}, \mathbb{G}_a)$, the representation of L on the character group is isomorphic to V_1^*. This is indecomposable, and irreducible if $\mathrm{char}(k) \neq 3$. Our identification of the unique submodule shows that $\mathrm{Hom}(U, \mathbb{G}_a) \cong S^3(k^2)$ as representations of $L^{der} \cong SL_2$. Since the center of L acts by a fundamental character on V_1, and hence on V_1^*, we may choose an isomorphism $L \cong GL_2$ so that the central element

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

acts by multiplication by A. This gives the result. □

Remark: (i) The twisted representation of GL_2 on the space of binary cubic forms is faithful, whereas the usual action has kernel μ_3.

(ii) Under the choice of the isomorphism $L \cong GL_2$ in the above proof, the modulus character δ_P of P is given by:

$$\delta_P(g) = \det(g)^{-3}, \quad \text{for } g \in L.$$

Here is another way to see that linear forms on V_1 correspond to binary cubic forms $p(x,y)$. Define the cubic mapping $\theta : k^2 \to V_1 \otimes \det$ by the formula

$$\theta(x,y) = x^3 e_\alpha + x^2 y e_{\alpha+\alpha'} + xy^2 e_{\alpha+2\alpha'} + y^3 e_{\alpha+3\alpha'}. \tag{3.2}$$

Here $\langle e_\alpha, e_{\alpha+\alpha'}, e_{\alpha+2\alpha'}, e_{\alpha+3\alpha'}\rangle$ is a Chevalley basis, normalized by:

$$\begin{aligned}
[e_{\alpha'}, e_\alpha] &= e_{\alpha+\alpha'},\\
[e_{\alpha'}, e_{\alpha+\alpha'}] &= 2e_{\alpha+2\alpha'},\\
[e_{\alpha'}, e_{\alpha+2\alpha'}] &= 3e_{\alpha+3\alpha'}.
\end{aligned}$$

A short computation shows that one can choose an isomorphism $L \cong GL_2$ so that θ is L-equivariant. Hence, linear forms p on V_1 give cubic forms $p \circ \theta(x,y)$ on k^2.

A Borel subgroup of L stabilizes a unique complete flag

$$0 \subset W_1 \subset W_2 \subset W_3 \subset W_4 = V_1$$

with $\dim W_i = i$. It also stabilizes a unique line l in the standard representation k^2. The line W_1 is equal to $\theta(l)$; more generally, we have the following:

Proposition 3.3. *A linear form p vanishes on the subspace W_i if and only if the corresponding cubic form $p\circ\theta$ on k^2 vanishes to order $\geq i$ on the line l.*

Proof. It suffices to check this for the Borel subgroup B of upper triangular matrices which stabilizes the line $l = \langle (x,0)\rangle$, since all the Borel subgroups are conjugate. The complete flag in V^* stabilized by B is given by:

$$0 \subset \langle x^3\rangle \subset \langle x^3, x^2y\rangle \subset \langle x^3, x^2y, xy^2\rangle \subset V_1^*.$$

Hence the linear forms p vanishing on W_1 are those of the form $ax^3 + bx^2y + cxy^2$, which vanishes to order ≥ 1 on l. The p vanishing on W_2 are those of the form $ax^3 + bx^2y$, and these cubic forms vanish to order ≥ 2 on l. Finally, the p vanishing on W_3 have the form ax^3, and vanish to order ≥ 3 on l. □

We now consider $G = G_2$ as a Chevalley group over $\mathbb{Z}$, with Heisenberg parabolic $P = L\cdot U$. The Levi factor is now isomorphic to the group scheme GL_2 over $\mathbb{Z}$, and V_1 and V_2 are free $\mathbb{Z}$-modules (of ranks 4 and 1) on which L acts. By our results over fields, the bracket $f : \wedge^2 V_1 \to V_2$ is surjective, $[U,U] \cong U_{\beta_0}$, and $\mathrm{Hom}(U, \mathbb{G}_a)$ is isomorphic to the free $\mathbb{Z}$-module $\mathrm{Hom}(V_1, \mathbb{G}_a) = \mathrm{Hom}(U(\mathbb{Z}), \mathbb{Z})$. We have shown the following (c.f. [Sp, Pg. 160-161]):

Proposition 3.4. *The representation of $L(\mathbb{Z})$ on the module Hom$(U(\mathbb{Z}), \mathbb{Z})$ is isomorphic to the twisted representation of $GL_2(\mathbb{Z})$ on the space of binary cubic forms over $\mathbb{Z}$.*

Thus, the set of $L(\mathbb{Z})$-orbits on $\mathrm{Hom}(U(\mathbb{Z}), \mathbb{Z})$ is in canonical bijection with the set of $GL_2(\mathbb{Z})$-orbits on the space of binary cubic forms. In the next section, we identify these orbits with the isomorphism classes of rings A of rank 3 over $\mathbb{Z}$.

4. **Binary Cubic Forms and Cubic Rings**

We recall that the twisted action of

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2(\mathbb{Z})$$

on the element

$$p(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

in the space of binary cubic forms with integer coefficients is given by:

$$\gamma \cdot p(x, y) = \frac{1}{\det(\gamma)} \cdot p(Ax + Cy, Bx + Dy). \tag{4.1}$$

As remarked earlier, this twisted action is faithful. In this section, we will parametrize the $GL_2(\mathbb{Z})$-orbits.

We say that a commutative, associative ring A (with unit 1) is a *cubic ring* if A is a free $\mathbb{Z}$-module of rank 3.

Proposition 4.2. *There is a bijection, to be given below, between the set of $GL_2(\mathbb{Z})$-orbits on the space of binary cubic forms with integer coefficients and the set of isomorphism classes of cubic rings A.*

Proof. If A is a cubic ring, choose a basis $A = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \alpha + \mathbb{Z} \cdot \beta$ over $\mathbb{Z}$. By adding integers to α and β, we may arrange that the product

$$\alpha\beta = n$$

lies in $\mathbb{Z}$. Call this a *good basis* for A (c.f. [DF, Pg. 103-105]). We will first establish a bijection between cubic rings with a good basis and binary cubic forms.

A cubic ring with a good basis is determined up to isomorphism by the products:

$$\begin{cases} \alpha\beta = n \\ \alpha^2 = m + b\alpha - a\beta \\ \beta^2 = l + d\alpha - c\beta \end{cases} \tag{4.3}$$

with a, b, c, d, l, m, n in $\mathbb{Z}$. Since A is associative,we have

$$\alpha^2 \cdot \beta = \alpha \cdot \alpha\beta,$$
$$\alpha \cdot \beta^2 = \alpha\beta \cdot \beta.$$

Writing these out, we find that:

$$\begin{cases} n = -ad \\ m = -ac \\ l = -bd, \end{cases} \tag{4.4}$$

but that the integers (a, b, c, d) are arbitrary. To A with the good basis $(1, \alpha, \beta)$, we associate the binary cubic form $p(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, and to the form p, we associate the cubic ring with multiplication given by (4.3) and (4.4). This is the first bijection.

Now consider the operation where a good basis $(1, \alpha, \beta)$ of A is replaced by another good basis $(1, \alpha', \beta')$. Write

$$\begin{pmatrix} 1 & 0 & 0 \\ u & A & B \\ v & C & D \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha' \\ \beta' \end{pmatrix}$$

with

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2(\mathbb{Z}),$$

and u and v integers determined by γ (and the fact that $\alpha'\beta' = n'$ is an integer). After some calculation, best suppressed here, we find that the form

$$p'(x,y) = a'x^3 + b'x^2y + c'xy^2 + d'y^3$$

associated to $(1, \alpha', \beta')$ is equal to $\gamma \cdot p$. This completes the proof of the Proposition. $\square$

As an example, the orbit of $p = (0,0,0,0)$ gives the cubic ring

$$A = \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2, \alpha\beta),$$

and the orbit of $p = (1,0,0,0)$ gives the cubic ring

$$A = \mathbb{Z}[\alpha]/(\alpha^3).$$

Remarks: Deligne has observed that the bijection of orbits and rings established in Proposition 4.2 holds over any base scheme S. There is an equivalence of categories between the following two kinds of objects, with morphisms being the isomorphisms:

a) A vector bundle V of rank 2, with p in $Sym^3(V) \otimes \wedge^2(V)^{-1}$;

b) A vector bundle A of rank 3, with a (commutative) algebra structure.

The key point is that we are not looking at the action of $GL(V)$ on binary cubic forms (ie. on elements of $Sym^3(V)$) where the subgroup μ_3 in the center acts trivially, but at the twisted action on $Sym^3(V) \otimes \wedge^2(V)^-1$, which is faithful.

The invariants and covariants of the form $p(x,y)$, studied in the XIX-th century by Eisenstein, Hermite and others [DF, Pg. 167], can all be given in terms of the cubic ring A (see the recent article [HM]). For example, the discriminant Δ of $p(x,y)$, defined by

$$\Delta = b^2c^2 + 18abcd - 4ac^3 - 4db^3 - 27a^2d^2, \tag{4.5}$$

is equal to the discriminant of A over $\mathbb{Z}$. Indeed, if $\mathrm{Tr} : A \to \mathbb{Z}$ is the trace form, we have:

$$\mathrm{disc}(A/\mathbb{Z}) = \det \begin{pmatrix} \mathrm{Tr}(1) & \mathrm{Tr}(\alpha) & \mathrm{Tr}(\beta) \\ \mathrm{Tr}(\alpha) & \mathrm{Tr}(\alpha^2) & \mathrm{Tr}(\alpha\beta) \\ \mathrm{Tr}(\beta) & \mathrm{Tr}(\alpha\beta) & \mathrm{Tr}(\beta^2) \end{pmatrix}$$

for any $\mathbb{Z}$-basis $(1, \alpha, \beta)$ of A. Using a good basis, we find

$$\begin{cases} \mathrm{Tr}(1) = 3 \\ \mathrm{Tr}(\alpha) = b \\ \mathrm{Tr}(\beta) = -c \\ \mathrm{Tr}(\alpha^2) = b^2 - 2ac \\ \mathrm{Tr}(\beta^2) = c^2 - 2bd \\ \mathrm{Tr}(\alpha\beta) = -3ad. \end{cases}$$

From this, we obtain the identity $\mathrm{disc}(A/\mathbb{Z}) = \Delta$.

Eisenstein defined a quadratic covariant of $p(x, y)$:

$$q(x, y) = (b^2 - 3ac)x^2 + (bc - 9ad)xy + (c^2 - 3bd)y^2. \tag{4.6}$$

We have

$$\mathrm{disc}(q) = -3\Delta,$$

and q is positive-definite when $\Delta > 0$. If $p(x, y)$ is associated to A with a good basis $(1, \alpha, \beta)$, and we write

$$\gamma = x\alpha + y\beta = \frac{\mathrm{Tr}(\gamma)}{3} + \gamma_0$$

with $\gamma_0 \in \frac{1}{3} \cdot A$ of trace zero, then we have:

$$q(x, y) = \frac{3}{2}\mathrm{Tr}(\gamma_0^2). \tag{4.7}$$

Similarly, Hermite defined a cubic covariant of $p(x, y)$:

$$\begin{aligned} n(x, y) &= (2b^3 - 9abc + 27a^2d)x^3 \\ &+ (3b^2c - 9ac^2 + 27abd)x^2y \\ &+ (-3bc^2 + 18b^2d - 27acd)xy^2 \\ &+ (-2c^3 + 9bcd - 27ad^2)y^3, \end{aligned} \tag{4.8}$$

with

$$\Delta(n) = 3^6 \cdot \Delta^3.$$

With the above notation, we have the formula

$$n(x, y) = 27 \cdot \mathbb{N}(\gamma_0). \tag{4.9}$$

The relation between the covariants (of degree 6):

$$n(x, y)^2 + 27 \cdot \Delta \cdot p(x, y)^2 = 4 \cdot q(x, y)^3,$$

follows from the formula for the discriminant of γ_0 in terms of the coefficients of its characteristic polynomial.

5. Primitivity and Gorenstein Cubic Rings

One invariant of the $GL_2(\mathbb{Z})$-orbit of $p(x,y)$ is the content $e \geq 0$ of the form p, defined as the non-negative generator of the ideal $\mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c + \mathbb{Z}d$ of $\mathbb{Z}$. We say that $p(x,y)$ is *primitive* if $e = 1$. Every non-zero form p may be written uniquely as

$$p = e \cdot p_0, \tag{5.1}$$

with $e \geq 1$ the content of p, and p_0 primitive. If the cubic ring A corresponds to the $GL_2(\mathbb{Z})$-orbit of $p(x,y)$, then we also say that A has content e.

We say the cubic ring is Gorenstein if the A-module $\mathrm{Hom}(A,\mathbb{Z})$ is projective. For example, if $A = \mathbb{Z}[\gamma]$ is generated by a single element, it is Gorenstein. Indeed, $\mathrm{Hom}(A,\mathbb{Z}) = A \cdot f$ is free, with basis given by the map:

$$\begin{cases} f(1) = 0 \\ f(\gamma) = 0 \\ f(\gamma^2) = 1. \end{cases}$$

Proposition 5.2. *The form $p(x,y)$ is primitive if and only if the associated cubic ring A is Gorenstein. If $p = e \cdot p_0$, with $e \geq 1$, then $A = \mathbb{Z} + eA_0$.*

Before proving this result, we give a useful description of the function $|p(x,y)|$, using the rank 2 $\mathbb{Z}$-module $A/\mathbb{Z}$.

Lemma 5.3. *The element $\gamma = m\alpha + n\beta (\mathrm{mod}\ \mathbb{Z})$ in $A/\mathbb{Z}$ generates a subring of finite index in A if and only if $p(m,n) \neq 0$. In this case, $|p(m,n)|$ is the index of $\mathbb{Z}[\gamma]$ in A.*

Proof. Since

$$\gamma^2 = m^2\alpha^2 + 2mn\alpha\beta + n^2\beta^2 + k(m\alpha + n\beta) + l,$$

where k and l are integers, we have

$$\gamma^2 \equiv (bm^2 + dn^2 + km)\alpha + (-am^2 - cn^2 + kn)\beta$$

in $A/\mathbb{Z}$.

The index of $\mathbb{Z}[\gamma]$ in A is finite if and only if the matrix

$$M = \begin{pmatrix} m & bm^2 + dn^2 + km \\ n & -am^2 - cn^2 + kn \end{pmatrix}$$

has non-zero determinant, in which case the index is $|\det(M)|$. Since $\det(M) = -p(m,n)$, the lemma is proved. □

We now give the proof of Proposition 5.2. Let l be a prime number, and put $A_l = A \otimes \mathbb{Z}_l$. We will show that $p(x,y) \neq 0 (\mathrm{mod}\ l)$ is equivalent to the fact that $\mathrm{Hom}(A_l, \mathbb{Z}_l)$ is a free A_l-module.

If $p(x,y) \equiv 0 (\mathrm{mod}\ l)$, then the ring

$$A/lA = \mathbb{Z}/l\mathbb{Z}[\alpha,\beta]/(\alpha^2, \beta^2, \alpha\beta)$$

is not Gorenstein over $\mathbb{Z}/l\mathbb{Z}$. Indeed, it is a local ring with maximal ideal $m = (\alpha, \beta)$. Since $m^2 = 0$, the kernel of m on A/lA has dimension 2, but A/m has dimension 1.

Now assume that $p(x,y) \neq 0(\text{mod } l)$. Since p has at most 3 distinct roots (mod l), we can find $(m,n) \in \mathbb{Z}_l^2$ such that $p(m,n)$ is a unit in $\mathbb{Z}_l$, *unless* $l = 2$ and $p(x,y)$ is equivalent to the form $x^2y - xy^2$ over $\mathbb{Z}_2$. In the latter case, $A_2 \cong \mathbb{Z}_2^3$, and $\text{Hom}(A_2, \mathbb{Z}_2) \cong A_2$ by the trace form. So we may assume that $p(m,n)$ is a unit in $\mathbb{Z}_l$. By Lemma 5.3, $A_l = \mathbb{Z}_l[\gamma]$ with $\gamma = m\alpha + n\beta$. Hence $\text{Hom}(A_l, \mathbb{Z}_l)$ is a free module, by the remarks preceding Proposition 5.2. This proves the first assertion in Proposition 5.2.

If $(1, \alpha_0, \beta_0)$ is a good basis for A_0, with form $p_0(x,y)$, then $(1, \alpha = e\alpha_0, \beta = e\beta_0)$ is a good basis for $A = \mathbb{Z} + eA_0$. The associated form is $p = e \cdot p_0$, by the formulas in (4.3). This proves the second asssertion of Proposition 5.2.

For our calculations with Hecke operators in §15, we will need a local variant of the content e. We say that the p-depth of A is n, if e is divisible by p^n and not by p^{n+1}. Assume, for the rest of the section, that the p-depth of A is zero. Let $A_n = \mathbb{Z} + p^n A$ which has p-depth n, and let $q(x,y)$ be a binary cubic form in the orbit corresponding to A. For all $n \geq 0$, the abelian group A_n/A_{n+1} is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$, so there are $(p+1)$ free abelian groups B with $A_{n+1} \subset B \subset A_n$. How many of these lattices are cubic rings?

Proposition 5.4. *There is one-to-one correspondence between the solutions of $q(x,y) \equiv 0$ (mod p) and the cubic rings B with $A_1 \subset B \subset A$.*
If $n \geq 1$, any lattice B with $A_{n+1} \subset B \subset A_n$ is a cubic ring.

Proof. Let $\langle 1, \alpha, \beta \rangle$ be the good basis for A corresponding to the cubic form $q(x,y)$. Any lattice B between A_1 and A is spanned by 1, $p\alpha$, $p\beta$, and an additional element $a\alpha + b\beta$ for some integers a and b (which are well-defined (mod p)). One checks that B is a ring if and only if $q(a,b) = 0$ (mod p). The statement for $n \geq 1$ is clear. □

Proposition 5.5. *Let B be a ring such that $A_{n+1} \subset B \subset A_n$. Then the p-depth of B lies between $n-1$ and $n+2$. If the p-depth of B is $n-1+i$, then $q(x,y)$ (mod p) has a zero of order i.*

Proof. We can find a good basis $\langle 1, \alpha, \beta \rangle$ of A such that $\langle 1, p^n\alpha, p^{n+1}\beta \rangle$ is a good basis of B. If $q(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ is the cubic form associated to the above good basis for A, the cubic form associated to the above good basis for B is
$$ap^{n-1}x^3 + bp^n x^2y + cp^{n+1}xy^2 + dp^{n+2}y^3.$$
The first assertion now follows from the fact that A has p-depth 0. Moreover, if the form attached to B is divisible by p^{n-1+i}, then $(1,0)$ is a zero of $q(x,y)$ (mod p) of order i. □

Corollary 5.6. *If A/pA is a cubic field and B is a cubic ring such that $A_{n+1} \subset B \subset A_n$, then the p-depth of B is $n-1$.*

Fix an arbitrary cubic ring A' (not necessarily of p-depth zero), and a binary cubic form $q'(x,y)$ in the $GL_2(\mathbb{Z})$-orbit corresponding to A'. We conclude this section by describing another way of parametrizing the cubic rings B which contain or are contained in A' with index p.

By base extension, the action of $GL_2(\mathbb{Z})$ on the binary cubic forms over $\mathbb{Z}$ gives rise to a rational representation of $GL_2(\mathbb{Q})$ on the $\mathbb{Q}$-vector space of binary cubic forms over $\mathbb{Q}$. Now we have:

Proposition 5.7.

(i) *Let S_1 be the set of left cosets $GL_2(\mathbb{Z})\gamma$ contained in $GL_2(\mathbb{Z})\begin{pmatrix} p & \\ & 1 \end{pmatrix}GL_2(\mathbb{Z})$, for which $\gamma \cdot q'(x,y)$ has integer coefficients. Then there is a bijection between S_1 and the set of cubic rings B such that $B \subset_p A'$.*

(ii) *Let S_2 be the set of left cosets $GL_2(\mathbb{Z})\gamma$ contained in $GL_2(\mathbb{Z})\begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}GL_2(\mathbb{Z})$, for which $\gamma \cdot q'(x,y)$ has integer coefficients. Then there is a bijection between S_2 and the set of cubic rings B such that $A' \subset_p B$.*

Proof. (i) We associate to a left coset $GL_2(\mathbb{Z})\gamma$ in S_1 the cubic ring B corresponding to the orbit of $\gamma \cdot q'(x,y)$. The resulting map of sets is injective. Indeed, if $GL_2(\mathbb{Z})\gamma'$ is another coset mapping to B, then $\gamma \cdot q'$ and $\gamma' \cdot q'$ are in the same $GL_2(\mathbb{Z})$-orbit, and thus $GL_2(\mathbb{Z})\gamma = GL_2(\mathbb{Z})\gamma'$. The proof of the surjectivity is similar to the proof of Proposition 5.5.

(ii) The proof is similar to that for (i); we omit the details. $\square$

6. **Quaternionic Discrete Series**

We now consider the restriction of certain discrete series representation of the real Lie group $G_2(\mathbb{R})$ to the Heisenberg parabolic subgroup. Wallach has recently studied this situation in a more general setting [W], and we simply state his results for G_2 here.

The discrete series π_k we will consider here were discussed in [GW]. They are parametrized by integers $k \geq 2$, and have infinitesimal character $\rho + (k-2)\beta_0$, with β_0 the highest root. In this paper, π_k will denote the Casselman-Wallach globalization, which is a smooth Fréchet representation of moderate growth (c.f. [C], [W2] and [W3]). The maximal compact subgroup K of $G_2(\mathbb{R})$ is $SU_4 = (SU_2 \times SU_2)/\langle \pm 1 \rangle$, with the long root SU_2 as the first factor. The representation π_k is admissible for the long root SU_2, and its underlying Harish-Chandra module decomposes as a K-module:

$$(\pi_k)_K \cong \bigoplus_{n \geq 0} S^{2k+n}(\mathbb{C}^2) \otimes S^n(S^3\mathbb{C}^2).$$

The minimal K-type is the representation

$$S^{2k} \otimes S^0, \qquad \text{of } (SU_2 \times SU_2)/\langle \pm 1 \rangle,$$

of dimension $2k+1$. Finally the subgroup $K_0 = U_2 = (U_1 \times SU_2)/\langle \pm 1 \rangle$ of K has highest weight $(\det)^k$ on the minimal K-type. Moreover, the representations π_k are non-generic: they have Gelfand-Kirillov dimension 5.

We also have the continuation of quaternionic discrete series π_0 and π_1 constructed in [GW]. They have the same properties as the representations π_k above, although the infinitesimal characters $\rho - \beta_0$ and $\rho - 2\beta_0$ are no longer regular. The representation π_1 is a limit discrete series , and π_0 has a trivial minimal K-type. Both are unipotent in the sense of Vogan [V].

Following the techniques in [GW, §6], one can show that π_k is a submodule of a degenerate C^∞-principal series representation $\mathrm{Ind}_{P(\mathbb{R})}^{G_2(\mathbb{R})} \lambda_k$, with λ_k a 1-dimensional representation of $P(\mathbb{R})/U(\mathbb{R}) \cong GL_2(\mathbb{R})$. Indeed, we have:

$$\lambda_k = (\mathrm{sign})^k \cdot |\det|^{-k-1},$$

where (sign) is the unique quadratic character of $GL_2(\mathbb{R})$. Here, we recall that we have chosen an isomorphism $L \cong GL_2$ so that the modulus character of P is $\delta_P = \det^{-3}$.

The real vector space $\mathrm{Hom}(U(\mathbb{R}), \mathbb{R})$ is isomorphic to the group of characters $\mathrm{Hom}(U(\mathbb{R}), S^1)$ under the map taking f to $\chi = e^{2\pi i f}$. This isomorphism takes the lattice $\mathrm{Hom}(U(\mathbb{Z}), \mathbb{Z})$ to the subgroup of characters χ which are trivial on $U(\mathbb{Z})$. This subgroup is a representation of $L(\mathbb{Z})$, isomorphic to the action of $GL_2(\mathbb{Z})$ on the space of binary cubic forms with integer coefficients.

The full character group is a representation of $L(\mathbb{R}) \cong GL_2(\mathbb{R})$, isomorphic to the representation on the space of binary cubic forms with real coefficients. We say a character χ of $U(\mathbb{R})$ is *generic* when the cubic form $p(x, y)$ associated to χ has discriminant $\Delta \neq 0$. The generic characters break up into two $L(\mathbb{R})$-orbits: those with $\Delta > 0$, corresponding to the real cubic algebra $\mathbb{R}^3$, and those with $\Delta < 0$, corresponding to the cubic algebra $\mathbb{R} \times \mathbb{C}$. A representative χ for the orbit with $\Delta > 0$ is given by $\chi = e^{2\pi i f}$, where $f : U(\mathbb{R}) \to \mathbb{R}$ is non-zero on the two short root spaces $\mathfrak{g}_\gamma$ with $m_\alpha(\gamma) = 1$, and zero on the two long root spaces with $m_\alpha(\gamma) = 1$.

Here is Wallach's result for G_2.

Proposition 6.1. *Let χ be a generic character of $U(\mathbb{R})$, and let $k \geq 0$.*

If $\Delta(\chi) < 0$, then the complex vector space of continuous linear maps $Hom_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi))$ is zero.

If $\Delta(\chi) > 0$, then the complex vector space of continuous linear maps $Hom_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi))$ is 1-dimensional, and it affords the representation $(sign)^k$ for $\Sigma_3 = Stab(\chi) \subset GL_2(\mathbb{R})$.

7. Modular Forms on G_2 of weight k

We fix a quaternionic discrete series representation π_k of $G_2(\mathbb{R})$, with $k \geq 2$, or a continuation with $k = 0$ or 1. Let $\mathcal{A}$ be the space of automorphic forms on G_2. More precisely, $\mathcal{A}$ is the space of smooth functions φ on the adelic group

$$G_2(\mathbb{A}) = G_2(\mathbb{R}) \times G_2(\hat{\mathbb{Q}})$$

which satisfy the following conditions:

(i) φ is left invariant under $G_2(\mathbb{Q})$;

(ii) φ is right invariant under some open compact subgroup K_f of $G_2(\hat{\mathbb{Q}})$;

(iii) φ is annihilated by an ideal J of finite codimension in $Z(\mathfrak{g})$, the center of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of $G_2(\mathbb{R})$;

(iv) φ is of moderate growth on $G_2(\mathbb{R})$ (c.f. [BoJ] and [W2, Pg. 252]).

Note that this definition of $\mathcal{A}$ differs from that in much of the literature, since we are not assuming that φ is K-finite; instead, we will let $\mathcal{A}_K$ be the subspace of $\mathcal{A}$ consisting of K-finite functions. As a result, $\mathcal{A}$ is a representation of the adelic group $G_2(\mathbb{A})$. For fixed K_f and J, let $\mathcal{A}(J, K_f)$ be the subspace of $\mathcal{A}$ consisting of those functions φ for which conditions (ii) and (iii) are satisfied with respect to the given J and K_f. By a fundamental theorem of Harish-Chandra (c.f. [BoJ]), its underlying $(\mathfrak{g}, K)$-module $\mathcal{A}(J, K_f)_K$ is admissible and finitely generated; further, $\mathcal{A}(J, K_f)$ is the Casselman-Wallach globalization of $\mathcal{A}(J, K_f)_K$. Let $\mathcal{A}_0 \subset \mathcal{A}$ be the subspace of cusp forms. We now make the following definition.

Definition: The space of modular forms of weight k and level 1 for G_2 is the complex vector space

$$M_k = \mathrm{Hom}_{G_2(\mathbb{R}) \times G_2(\hat{\mathbb{Z}})}(\pi_k \otimes \mathbb{C}, \mathcal{A}). \tag{7.1}$$

The subspace of cusp forms is:

$$M_k^0 = \mathrm{Hom}_{G_2(\mathbb{R}) \times G_2(\hat{\mathbb{Z}})}(\pi_k \otimes \mathbb{C}, \mathcal{A}_0). \tag{7.2}$$

By the fundamental theorem of Harish-Chandra alluded to above, M_k is finite-dimensional. Moreover, it affords a representation of the spherical Hecke algebra

$$\mathcal{H}(G_2(\hat{\mathbb{Q}})//G_2(\hat{\mathbb{Z}})) \cong \hat{\bigotimes}_l \mathcal{H}(G_2(\mathbb{Q}_l)//G_2(\mathbb{Z}_l)).$$

In Sections 9 and 10, we shall give some examples of elements of M_k, and in Section 15, we shall study the action of $\mathcal{H}(G_2(\hat{\mathbb{Q}})//G_2(\hat{\mathbb{Z}}))$ on M_k in greater detail.

8. Fourier Coefficients

In this section, we are going to define, for any $f \in M_k$, a collection of Fourier coefficients $c_A(f) \in \mathbb{C}$. The coefficients are indexed by those cubic rings A with $A \otimes \mathbb{R} = \mathbb{R}^3$, and depend linearly on f.

For vectors $v \in \pi_k$, we may view $f(v)$ as a function on the double coset space

$$G_2(\mathbb{Q})\backslash G_2(\mathbb{A})/G_2(\hat{\mathbb{Z}}),$$

which is identified with the single coset space

$$G_2(\mathbb{Z})\backslash G_2(\mathbb{R})$$

by the strong approximation theorem: $G_2(\hat{\mathbb{Q}}) = G_2(\mathbb{Q}) \cdot G_2(\hat{\mathbb{Z}})$. Let χ be a character of $U(\mathbb{R})$ which is trivial on $U(\mathbb{Z})$, and define a continuous linear form on π_k by the integral:

$$l_\chi(v) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v)(u)\overline{\chi(u)}du. \tag{8.1}$$

Here du is a Haar measure on the unipotent group $U(\mathbb{R})$, and the quotient $U(\mathbb{Z})\backslash U(\mathbb{R})$ is compact.

Proposition 8.2. *The linear form l_χ lies in the complex vector space $Hom_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi))$. If $\chi' = \gamma \cdot \chi$ with γ in $L(\mathbb{Z})$, then $l_{\chi'} = \gamma \cdot l_\chi$.*

Proof. For $g \in U(\mathbb{R})$, we must show that $l_\chi(gv) = \chi(g)l_\chi(v)$. But $f(gv)$ is the function on $G_2(\mathbb{Z})\backslash G_2(\mathbb{R})$ defined by:

$$f(gv)(h) = f(v)(hg).$$

Hence,

$$\begin{aligned} l_\chi(gv) &= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(gv)(u)\overline{\chi(u)}du \\ &= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v)(ug)\overline{\chi(u)}du \\ &= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v)(u')\overline{\chi(u'g^{-1})}du', \quad u' = ug,\, du' = du \\ &= \chi(g)l_\chi(v), \end{aligned}$$

as required. Now assume that $\chi' = \gamma \cdot \chi$, with γ in $L(\mathbb{Z})$, so that

$$\chi'(u) = \chi(\gamma^{-1}(u)) = \chi(\gamma^{-1}u\gamma).$$

Then

$$\begin{aligned}
l_{\chi'}(v) &= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v)(u)\overline{\chi'(u)}du \\
&= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v)(u)\overline{\chi(\gamma^{-1}u\gamma)}du \\
&= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v)(\gamma u'\gamma^{-1})\overline{\chi(u')}du', \qquad u' = \gamma^{-1}u\gamma,\ du' = du \\
&= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(\gamma^{-1}v)(u')\overline{\chi(u')}du \\
&= l_\chi(\gamma^{-1}v) = \gamma \cdot l_\chi(v)
\end{aligned}$$

□

If $\Delta(\chi) < 0$, $l_\chi = 0$ by Proposition 6.1. If $\Delta(\chi) > 0$, l_χ lies in the 1-dimensional spacecomplex vector space $\mathrm{Hom}_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi))$. Fix a character χ_0 with $\Delta(\chi_0) > 0$, and a basis vector l_0 of $\mathrm{Hom}_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi_0))$. Since χ is in the $L(\mathbb{R})$-orbit of χ_0, we may write $\chi = g \cdot \chi_0$, with $g \in L(\mathbb{R})$ well-defined up to right mulitplication by $\Sigma_3 = \mathrm{Stab}(\chi_0)$. If k is even, this finite group fixes l_0. If k is odd, Σ_3 acts on $\mathrm{Hom}_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi_0))$ by the sign character. In any case, the linear form $\lambda_k(g) \cdot (g \cdot l_0)$ gives a well-defined basis element of $\mathrm{Hom}_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi))$. Hence we may write

$$l_\chi = c_\chi(f) \cdot \lambda_k(g) \cdot (g \cdot l_0), \tag{8.3}$$

for some constant $c_\chi(f)$.

If the weight k is even, it follows by Proposition 8.2 that $c_\chi(f)$ depends only on the $L(\mathbb{Z})$-orbit of the character χ. We have seen in Propositions 3.4 and 4.2 that the $L(\mathbb{Z})$-orbits of such χ's are indexed canonically by the cubic rings A with $\mathrm{disc}(A) > 0$, so that $A \otimes \mathbb{R} = \mathbb{R}^3$. Hence, if the orbit of χ corresponds to A, we write $c_A(f)$ for the constants $c_\chi(f)$ in this orbit, and call $c_A(f)$ the A-th Fourier coefficient of f.

When k is odd, it is no longer the case that the constant $c_\chi(f)$ depends only on the $L(\mathbb{Z})$-orbit of χ. Indeed, if $\chi' = \gamma \cdot \chi$, where $\gamma \in L(\mathbb{Z})$, an easy calculation shows that $c_{\chi'}(f) = \det(\gamma) \cdot c_\chi(f)$. As a consequence, $c_\chi(f)$ depends not only on the cubic ring A (which index the orbit of χ), but also on an orientation of A, i.e. the choice of a basis element e of $\wedge^3 A$. Hence, when k is odd, we denote the Fourier coefficients of f by $c_{A,e}(f)$. Since $c_{A,-e}(f) = -c_{A,e}(f)$, we shall abuse notation and write $c_A(f)$ for the pair of numbers $\pm c_{A,e}(f)$. It is interesting to note that a similar complication arises for the Siegel modular forms of odd weight.

If we replace the basis l_0 by the basis $l_0' = \alpha l_0$, then

$$c_A(f) = \alpha c_A'(f)$$

for all A. Also, it follows from definition that

$$c_A(\alpha f + \beta g) = \alpha c_A(f) + \beta c_A(g).$$

When k is odd, we have $c_A(f) = 0$ whenever the stabilizer of χ in $GL_2(\mathbb{Z})$ contains an involution; for example, when $A = \mathbb{Z} + B$, with B an order in a quadratic field.

Having defined the Fourier coefficients $c_A(f)$ for modular forms $f \in M_k$, we can ask a number of natural questions. The first question which suggests itself is whether f is determined by its Fourier coefficients. We can show this is true for cusp froms.

Proposition 8.4. *If $f \in M_k^0$ is a cusp form and $c_A(f) = 0$ for all A, then $f = 0$.*

The proof of this proposition makes use of the other standard maximal parabolic subgroup $Q = M \cdot N$ of G over $\mathbb{Z}$. Hence we begin by describing the structure of Q briefly. Its Levi factor M is isomorphic to GL_2, and its unipotent radical N is a 3-step nilpotent group over $\mathbb{Z}$:

$$N = N_1 \supset N_2 \supset N_3 \supset \{1\}.$$

This filtration is the one introduced in §2 for a general maximal parabolic subgroup. The center N_3 of N is 2-dimensional, and one can choose an isomorphism $M \cong GL_2$ so that the action of M on $\mathrm{Hom}(N_3, \mathbb{G}_a)$ is the standard representation of GL_2 twisted by the determinant character. Similarly, $W_1 \cong N_1/N_2$ is also 2-dimensional, and the action of M on $\mathrm{Hom}(W_1, \mathbb{Z})$ is the standard representation of GL_2. Note that $U \cap N$ is the 4-dimensional commutator subgroup of the unipotent radical of the Borel subgroup $B = P \cap Q$. Moreover, we have the inclusions

$$\begin{cases} U_2 \subset N_3, \\ N_2 \subset U \cap N, \end{cases}$$

where we recall from §2 that $U_2 = [U, U]$ is the center of the unipotent radical U of P.

We now begin the proof of the proposition. Take any non-zero $v \in \pi_k$, and let $\varphi = f(v) \in \mathcal{A}_0$. Using strong approximation, we shall regard φ as a function on $G_2(\mathbb{Z})\backslash G_2(\mathbb{R})$. Note that φ is a non-generic cusp form, and we need to show that $\varphi = 0$. We first note the following lemma:

Lemma 8.5. *The automorphic form φ vanishes if and only if its constant term along U_2*

$$\varphi_{U_2}(g) = \int_{U_2(\mathbb{Z})\backslash U_2(\mathbb{R})} \varphi(ug)du, \qquad g \in G_2(\mathbb{R}),$$

vanishes as a function on $G(\mathbb{R})$.

Proof. Clearly, if φ vanishes, so does φ_{U_2}. To prove the converse, consider the Fourier expansion of φ along the compact abelian group $N_3(\mathbb{Z})\backslash N_3(\mathbb{R})$:

$$\varphi(g) = \sum_{\psi} \varphi_\psi(g)$$

where the sum extends over the characters $\psi \in \mathrm{Hom}(N_3(\mathbb{Z})\backslash N_3(\mathbb{R}), S^1)$, and

$$\varphi_\psi(g) = \int_{N_3(\mathbb{Z})\backslash N_3(\mathbb{R})} \varphi(ng) \cdot \overline{\psi(n)}dn.$$

If $\varphi_{U_2} = 0$, then $\varphi_\psi = 0$ for any ψ which restricts to the trivial character on the subgroup $U_2(\mathbb{R})$. But any other character of $N_3(\mathbb{Z})\backslash N_3(\mathbb{R})$ is conjugate under $M(\mathbb{Z})$ to a ψ of the above type. Hence $\varphi_\psi = 0$ for all ψ, and the lemma is proved. □

To prove Proposition 8.4, it remains to show that $\varphi_{U_2} = 0$. For any $g \in G(\mathbb{R})$, the function

$$u \mapsto \varphi_{U_2}(ug), \qquad \text{for } u \in U(\mathbb{R})$$

descends to a function on $V_1(\mathbb{R})$, where $V_1 \cong U/U_2$. Considering the Fourier expansion of φ_{U_2} along the compact abelian group $V_1(\mathbb{Z})\backslash V_1(\mathbb{R})$, we have

$$\varphi_{U_2}(g) = \sum_{\chi} \varphi_{\chi}(g)$$

where the sum is over the characters $\chi \in \mathrm{Hom}(V_1(\mathbb{Z})\backslash V_1(\mathbb{R}), S^1)$, and

$$\varphi_{\chi}(g) = \int_{V_1(\mathbb{Z})\backslash V_1(\mathbb{R})} \varphi_{U_2}(vg) \cdot \overline{\chi(v)} dv.$$

Proposition 6.1 implies that $\varphi_\chi = 0$ for all χ satisfying $\Delta(\chi) < 0$, and by assumption, $\varphi_\chi = 0$ for those χ such that $\Delta(\chi) > 0$. Hence, to show that φ_{U_2} vanishes, it remains to see that $\varphi_\chi = 0$ for degenerate χ, i.e. those for which $\Delta(\chi) = 0$.

We have yet to use the fact that φ is a non-generic cusp form. This is equivalent to the assertion that the constant term $\varphi_{U\cap N}$ of φ along $U \cap N$ vanishes identically. To see this, consider the Fourier expansion of $\varphi_{U\cap N}$ along $U_B^{ab} = (U \cap N)\backslash U_B$, where U_B is the unipotent radical of the Borel subgroup B. We deduce that φ is cuspidal if and only if $(\varphi_{U\cap N})_\psi = 0$ for any degenerate character ψ of $U_B^{ab}(\mathbb{Z})\backslash U_B^{ab}(\mathbb{R})$ (i.e. those ψ which restrict to the trivial character of the root subgroup corresponding to one of the two simple roots). On the other hand, φ is non-generic if and only if $(\varphi_{U\cap N})_\psi = 0$ for any nondegenerate character ψ. Hence ψ is a non-generic cusp form if and only if $(\varphi)_{U\cap N} = 0$.

We now claim that in fact φ_{N_2} is already identically zero. To see this, we consider its Fourier expansion along $W_1(\mathbb{Z})\backslash W_1(\mathbb{R})$, where $W_1 \cong N_1/N_2$:

$$\varphi_{N_2}(g) = \sum_{\phi} (\varphi_{N_2})_\phi(g).$$

The fact that $\varphi_{U\cap N} = 0$ implies that $(\varphi_{N_2})_\phi = 0$ for any ϕ which restricts to the trivial character on the subgroup $U(\mathbb{R}) \cap N(\mathbb{R})$. But any other character is conjugate under $M(\mathbb{Z})$ to a ϕ of the above type. Hence we conclude that $\varphi_{N_2} = 0$. In particular, this implies that $\varphi_\chi = 0$ for any character $\chi \in \mathrm{Hom}(V_1(\mathbb{Z})\backslash V_1(\mathbb{R}), S^1)$ which restricts to the trivial character of $N_2(\mathbb{R}) \subset U(\mathbb{R})$.

Finally, we observe that any degenerate character χ is conjugate under $L(\mathbb{Z})$ to a character which is trivial on $N_2(\mathbb{R})$, and hence $\varphi_\chi = 0$ for all χ. Proposition 8.4 is proved completely.

We conclude this section with another question: are there bounds for the Fourier coefficients of f in terms of the discriminants of the cubic rings? Recall that the discriminant $disc(A)$ of a cubic ring A was defined in Section 4. The following Proposition gives the analog of the Hecke bound for cusp forms:

Proposition 8.6. *Let $f \in M_k^0$ be a cusp form. Then for any totally real cubic ring A,*

$$|c_A(f)| \leq C_f \cdot |disc(A)|^{\frac{k+1}{2}}$$

for some constant C_f depending only on f.

Proof. Recall that the Fourier coefficient $c_A(f)$ is defined by the equation (8.3):

$$l_\chi = c_A(f) \cdot \lambda_k(g) \cdot (g \cdot l_0)$$

where $\chi = g \cdot \chi_0$ is a character in the $GL_2(\mathbb{Z})$-orbit corresponding to A. Assume without loss of generality that $\Delta(\chi_0) = 1$ (here, $\Delta(\chi_0)$ was defined in (4.5)). Then we see that

$$|disc(A)| = |det(g)|^2.$$

Pick any $v_0 \in \pi_k$ such that $l_0(v_0) = 1$. Evaluating the above equation at the vector $g \cdot v_0$, we obtain

$$|c_A(f)| = |l_\chi(g \cdot v_0)| \cdot |disc(A)|^{\frac{k+1}{2}}.$$

On the other hand, as f is cuspidal, $f(v_0)$ is bounded as a function on $G_2(\mathbb{Z})\backslash G_2(\mathbb{R})$. Since

$$l_\chi(g \cdot v_0) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v_0)(ug) \cdot \overline{\chi(u)} du,$$

we conclude that $|l_\chi(g \cdot v_0)|$ is bounded above by a constant independent of A. The proposition is proved. $\square$

9. Eisenstein Series of weight $2k \geq 4$

To show that the theory of Fourier coefficients developed above is non-empty, we give some examples of modular forms of weight k and study their Fourier coefficients. In this section, we consider a natural family of Eisenstein series E_{2k} of even weight $2k \geq 4$, and show under some hypothesis that their Fourier coefficients are given by:

$$c_A(E_{2k}) = \zeta_A(1-2k)$$

for maximal cubic rings A.

As we mentioned before, there is an embedding

$$i : \pi_{2k} \hookrightarrow \mathrm{Ind}_{P(\mathbb{R})}^{G_2(\mathbb{R})} \lambda_{2k},$$

which is well-defined up to scaling. The character λ_{2k} is the archimedean component of a global character

$$\chi_k : P(\mathbb{A}) \to \mathbb{C}^\times,$$

which is unramified at every finite place; indeed one has

$$\chi_k = |\det|^{-2k-1}.$$

We can thus consider the global induced representation:

$$I(k) = \mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})} \chi_k = \hat{\otimes}_v I_v(k).$$

For a finite prime l, let $\Gamma_{k,l}$ be the unique vector in $I_l(k)$ which is fixed by $G_2(\mathbb{Z}_l)$, and which satisfies $\Gamma_{k,l}(1) = 1$. For $\varphi \in \pi_k$, set

$$\hat{\varphi} = i(\varphi) \bigotimes (\otimes_l \Gamma_{k,l}) \in I(k),$$

and form the Eisenstein series

$$E(\hat{\varphi}, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G_2(\mathbb{Q})} \hat{\varphi}(\gamma g) \tag{9.1}$$

for $g \in G_2(\mathbb{R})$. This converges absolutely when $2k > 2$, and defines an element of $\mathcal{A}$. Thus the map

$$E_{2k} : \varphi \mapsto E(\hat{\varphi}, g)$$

defines a non-zero element of M_{2k}.

We now consider the Fourier coefficients of E_{2k}. Much computation has been done by Jiang and Rallis [JR] in the adelic setting, and we begin by recalling their results. Let χ be a character of $U(\mathbb{R})$ which is trivial on $U(\mathbb{Z})$. By strong approximation, we can regard χ as a character of $U(\mathbb{A})$ which is trivial on $U(\mathbb{Q})$ and $U(\hat{\mathbb{Z}})$. Consider the automorphic form $E(g) = E(\hat{\varphi}, g)$ defined by (9.1), for $\hat{\varphi} \in I(k)$. We then compute $l_\chi(\varphi)$ following the approach of [JR]:

$$
\begin{aligned}
l_\chi(\varphi) &= \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(u)\cdot\overline{\chi(u)}du \\
&= \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \left(\sum_{\gamma\in P(\mathbb{Q})\backslash G_2(\mathbb{Q})} \hat{\varphi}(\gamma u)\right)\cdot\overline{\chi(u)}du
\end{aligned}
$$

Now the double coset space $P(\mathbb{Q})\backslash G_2(\mathbb{Q})/P(\mathbb{Q})$ has 4 representatives, say w_0, w_1, w_2, w_3, with

$$P(\mathbb{Q})w_0P(\mathbb{Q}) = P(\mathbb{Q})w_0U(\mathbb{Q})$$

the open P-orbit. Hence,

$$l_\chi(\varphi) = \sum_{i=0}^{3}\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \left(\sum_{\gamma\in P(\mathbb{Q})\backslash P(\mathbb{Q})w_iP(\mathbb{Q})} \hat{\varphi}(\gamma u)\right)\cdot\overline{\chi(u)}du.$$

Jiang and Rallis showed that the only non-zero term in the sum over w_i is the term corresponding to w_0. Hence

$$
\begin{aligned}
l_\chi(\varphi) &= \int_{U(\mathbb{A})} \hat{\varphi}(w_0u)\cdot\overline{\chi(u)}du \\
&= \left(\int_{U(\mathbb{R})} \varphi(w_0u)\cdot\overline{\chi(u)}du\right)\cdot\left(\prod_p\int_{U(\mathbb{Q}_p)} \Gamma_p(w_0u_p)\cdot\overline{\chi(u_p)}du_p\right)
\end{aligned}
$$

an absolutely convergent Euler product. Now we have [JR, Thm. 2]:

Proposition 9.2. *Assume that χ corresponds to a maximal cubic ring A. If $A\otimes\mathbb{Q}_p$ is one of the following:*

$$
\begin{cases}
\mathbb{Q}_p\times\mathbb{Q}_p\times\mathbb{Q}_p; \\
\mathbb{Q}_p\times K,\ \textit{where } K \textit{ is the unramified quadratic extension of } \mathbb{Q}_p \textit{ and } p\neq 2; \\
\textit{the unramified cubic extension of } \mathbb{Q}_p,\ \textit{with } \mathbb{Q}_p \textit{ containing all cube roots of unity,}
\end{cases}
$$

then

$$\int_{U(\mathbb{Q}_p)} \Gamma_p(w_0u_p)\cdot\overline{\chi(u_p)}du_p = c_p\cdot\zeta_{A\otimes\mathbb{Z}_p}(2k),$$

where c_p is an explicit universal constant independent of A.

For the rest of the section, we assume that the formula in Proposition 9.2 holds for all finite primes and use this to compute the Fourier coefficients $c_A(E_{2k})$ of E_{2k} for maximal A. Hence, after a rescaling, we have:

$$l_\chi(\varphi) = \zeta_A(2k)\cdot\int_{U(\mathbb{R})}\varphi(w_0u)\cdot\overline{\chi(u)}du,$$

and it remains to examine the archimedean factor. The archimedean integral converges for $k\geq 1$ and the linear form

$$\varphi\mapsto\int_{U(\mathbb{R})}\varphi(w_0u)\cdot\overline{\chi(u)}du \tag{9.3}$$

defines a non-zero element of $\mathrm{Hom}_{U(\mathbb{R})}(I_\infty(k), \mathbb{C}(\chi))$. We expect but do not know that its restriction to the submodule π_{2k} is non-zero. However, it is not difficult to see that the vanishing of this restriction for one such χ with $\Delta(\chi) > 0$ implies the vanishing of the restriction for all χ's with $\Delta(\chi) > 0$, in which case $c_A(E_{2k})$ is zero for all A. Since we do not expect this to be the case, we make the further assumption that the archimedean integral is non-zero when restricted to π_{2k} for some (and hence all) χ with $\Delta(\chi) > 0$.

To compute Fourier coefficient, we now fix a χ_0 with $\Delta(\chi_0) > 0$, and a non-zero $l_0 \in Hom_{U(\mathbb{R})}(\pi_{2k}, \mathbb{C}(\chi_0))$. We pick χ_0 to be the character corresponding to $p_0 = (0,1,1,0)$, and let l_0 be the linear form

$$l_0(\varphi) = \int_{U(\mathbb{R})} \varphi(w_0 u) \cdot \overline{\chi_0(u)} du. \tag{9.4}$$

By our assumption, this defines a non-zero element of $\mathrm{Hom}_{U(\mathbb{R})}(\pi_{2k}, \mathbb{C}(\chi_0))$. Now take any $g \in GL_2(\mathbb{R})$ such that $g \cdot \chi_0 = \chi$. We have:

$$\begin{aligned}(g \cdot l_0)(\varphi) &= \int_{U(\mathbb{R})} \varphi(w_0 u g^{-1}) \cdot \overline{\chi_0(u)} du \\ &= \int_{U(\mathbb{R})} \varphi(w_0 g^{-1} u') \cdot \overline{\chi_0(g^{-1} u' g)} \cdot \delta_P(g)^{-1} du' \quad \text{with } u' = gug^{-1} \\ &= \delta_P(g)^{-1} \cdot \int_{U(\mathbb{R})} \varphi((w_0 g^{-1} w_0) w_0 u') \cdot \overline{\chi(u')} du' \\ &= \delta_P(g)^{-1} \cdot \delta_P(g)^{(2k+1)/3} \cdot \zeta_A(2k)^{-1} \cdot l_\chi(\varphi)\end{aligned}$$

Here the last equality follows because $w_0 g^{-1} w_0 \in GL_2(\mathbb{R})$, and $\delta_P(w_0 g^{-1} w_0) = \delta_P(g)$.

Now the Fourier coefficient c_χ is defined by the equality:

$$l_\chi = c_\chi \cdot \delta_P^{(2k+1)/3}(g) \cdot (g \cdot l_0).$$

By the above computation, we see that:

$$c_\chi = \zeta_A(2k) \cdot \delta_P(g) \cdot \delta_P(g)^{-2(2k+1)/3} = \zeta_A(2k) \cdot |\det(g)|^{4k-1}$$

since $\delta_P = |\det|^{-3}$. On the other hand,

$$|\det(g)|^2 = \Delta(\chi) = \mathrm{disc}(A),$$

Hence,

$$c_\chi = \zeta_A(2k) \cdot \mathrm{disc}(A)^{(4k-1)/2}$$

which by the functional equation gives $\zeta_A(1-2k)$ up to a universal scalar. Concluding, under various local assumptions, we have seen that up to a universal scalar

$$c_A(E_{2k}) = \zeta_A(1-2k). \tag{9.5}$$

Hence, the Eisenstein series E_{2k} are analogs of Cohen's Eisenstein series of half-integral weight [Co]. Observe that since $\zeta_A(2k)$ is about the size of 1, $c_A(E_{2k})$ grows like $|\mathrm{disc}(A)|^{2k-\frac{1}{2}}$, which violates the bound in Proposition 8.6 satisfied by cusp forms.

Remarks: When $2k = 2$, the series (9.1) may not converge. However, by the theory of Langlands, the Eisenstein series can be meromorphically continued in the parameter k to the whole complex plane, provided φ is K-finite. If φ is spherical, the Eisenstein series does have a pole, and the residue at $k = 1$ is a constant function. However, for $\varphi \in \pi_2$, the corresponding Eisenstein series is holomorphic at $k = 1$. This gives an embedding of vector spaces $E_2 : \pi_2 \to \mathcal{A}(G)$. However, the map E_2 is not $G_2(\mathbb{A})$-equivariant. This is the analog of the fact that the classical Eisenstein series E_2 for $SL_2(\mathbb{Z})$ is non-holomorphic.

10. **Exceptional Theta Series of weight** 4

In this section, we give examples of theta series of weight 4 on G_2. Recall that if (Λ, q) is an even unimodular lattice of rank $8k$ over $\mathbb{Z}$, and if we set

$$a_n(q) = \#\{x \in \Lambda : q(x) = 2n\},$$

then the function

$$f(z) = \sum_{n \geq 0} a_n(q) e^{2\pi i n z}$$

is a modular form on $SL_2(\mathbb{Z})$ of weight $4k$. In the same token, by a theta series on G_2, we shall mean a modular form f whose Fourier coefficients $c_A(f)$ count the number of embeddings of A into certain cubic structures over $\mathbb{Z}$. This is an embedding problem studied in [GG], and we begin by describing it in greater detail.

Let R be Coxeter's order in the $\mathbb{Q}$-algebra of Cayley's octonions, and let J be the set of 3×3 Hermitian matrices with coefficients in R. Then J is a free $\mathbb{Z}$-module of rank 27, and the determinant map provides a natural cubic form $N_J : J \to \mathbb{Z}$. Let $X \in J$ be an element in the cone of positive definite matrices, which satisfies $N_J(X) = 1$. Then the triple (J, N_J, X) is a pointed cubic space over $\mathbb{Z}$. An example of such an X is the identity matric I. It was shown in [Gr] that on varying $X \in J$, one gets precisely two isomorphism classes of pointed cubic spaces. Suppose that the two classes are represented by $J_I = (J, N_J, I)$ and $J_E = (J, N_J, E)$; we refer the reader to [Gr] for the definition of the element E. These two spaces are isomorphic over $\mathbb{Q}$ and $\mathbb{Z}_p$ for all p, but are globally inequivalent. The automorphism groups G_I and G_E of J_I and J_E are groups over $\mathbb{Z}$ in the sense of [Gr]. They have isomorphic generic fibers G, which are split of type F_4 over $\mathbb{Q}_p$ and are anisotropic over $\mathbb{R}$.

Similarly, a cubic ring A gives rise to a pointed cubic space $(A, N_A, 1)$ where N_A is the norm map of A. The counting problem studied in [GG] is that of computing the number

$$N(A) = 91 \cdot N(A, I) + 600 \cdot N(A, E),$$

where

$$\begin{cases} N(A, I) = \#\{A \to J_I\} \\ N(A, E) = \#\{A \to J_E\}. \end{cases}$$

We shall see that these numbers occur as the Fourier coefficients of modular forms of weight 4 on G_2.

The construction of these exceptional theta series exploits the fact that $G_2 \times G$ is a dual reductive pair in the quaternionic form H of E_8 over $\mathbb{Q}$. The group H has $\mathbb{Q}$-rank 4 and is split over $\mathbb{Q}_p$ for all p. Indeed, there exists integral models H_I and H_E of H such that the embedding $G_2 \times G \hookrightarrow H$ extends to embeddings

$$\begin{cases} G_2 \times G_I \hookrightarrow H_I, \\ G_2 \times G_E \hookrightarrow H_E. \end{cases}$$

of group schemes over $\mathbb{Z}$. The models H_I and H_E are also groups over $\mathbb{Z}$, and in particular, $K_I = H_I(\hat{\mathbb{Z}})$ and $K_E = H_E(\hat{\mathbb{Z}})$ are hyperspecial maximal compact subgroups of $H(\hat{\mathbb{Q}})$.

Let $\Pi = \hat{\otimes}_v \Pi_v$ be the global minimal representation of $H(\mathbb{A})$. The local minimal representation Π_p of $H(\mathbb{Q}_p)$ is unramified. Let Γ_I (respectively Γ_E) be a non-zero vector of $\hat{\otimes}_p \Pi_p$

fixed by the maximal compact subgroup K_I (respectively K_E); these are unique up to scaling. On the other hand, the representation Π_∞ of the real Lie group $H(\mathbb{R})$ has minimal K-type $\mathrm{Sym}^8(\mathbb{C}^2) \otimes \mathbb{C}$, where the maximal compact subgroup of $H(\mathbb{R})$ is $K = (SU_2 \times E_7)/\langle \pm 1 \rangle$. In [HPS], the restriction of Π_∞ to the dual pair $G_2(\mathbb{R}) \times G(\mathbb{R})$ was competely determined. In particular, it was shown that

$$\Pi_\infty^{G(\mathbb{R})} \cong \pi_4$$

as a representation of $G_2(\mathbb{R})$. Hence we have a $G_2(\mathbb{R})$-equivariant embedding

$$\iota : \pi_4 \longrightarrow \Pi_\infty,$$

well-defined up to scaling.

In [Ga], an embedding

$$\Theta : \Pi \longrightarrow \mathcal{A}(H)$$

of Π into the space of automorphic forms on H was constructed. Now we can construct two modular forms θ_I and θ_E of weight 4 as follows. For $v \in \pi_4$, we set

$$\begin{cases} \theta_I(v) = \text{ the restriction of } \Theta(\iota(v) \otimes \Gamma_I) \text{ to } G_2, \\ \theta_E(v) = \text{ the restriction of } \Theta(\iota(v) \otimes \Gamma_E) \text{ to } G_2. \end{cases}$$

The following result shows that θ_I and θ_E are exceptional theta series:

Proposition 10.1. *For any Gorenstein A,*

$$\begin{cases} c_A(\theta_I) = N(A, J_I), \\ c_A(\theta_E) = N(A, J_E). \end{cases}$$

Proof. In [Ga2, Thm. 11.3], it was shown that there exists an integer $e \geq 1$ such that the Fourier coefficients $c_A(\theta_I)$ and $c_A(\theta_E)$ are zero unless A has content divisible by e; further, if $A = \mathbb{Z} + eA_0$ has content precisely equal to e (so that A_0 is Gorenstein), then

$$\begin{cases} c_A(\theta_I) = N(A_0, J_I), \\ c_A(\theta_E) = N(A_0, J_E). \end{cases}$$

Hence, it remains to show that e is equal to 1.

Consider the modular form

$$\theta = 91 \cdot \theta_I + 600 \cdot \theta_E.$$

If $A = \mathbb{Z} + eA_0$, then $c_A(\theta) = N(A_0)$ and it was shown in [GG, Thm. 3] that $N(A_0) = c \cdot \zeta_{A_0}(-3)$ for a non-zero constant c independent of A_0. In particular, $c_A(\theta) \neq 0$. To show that $e = 1$, it suffices to show that for some Gorenstein cubic ring A, the Fourier coefficient of θ at A is non-zero. It was shown in [Ga2, Thm. 15.5] that (after a suitable scaling)

$$\theta = E_4$$

which is the analog of the classical Siegel-Weil formula. This implies that θ is a Hecke eigenform with some non-zero Fourier coefficients. On the other hand, Theorem 16.12 (which is proved at the end of the present paper) shows that a Hecke eigenform with a non-zero Fourier coefficient must have a non-zero Gorenstein coefficient. The proposition is proved. □

Remark: The proof of the proposition, together with [GG, Thm. 3], implies that the Fourier coefficient $c_A(E_4)$ is equal to $\zeta_A(-3)$ for maximal A, i.e. that (9.5) holds unconditionally when $k = 2$.

The examples of modular forms we've given in this and the previous section are non-cuspidal. The cuspidal support of E_{2k} and θ is the Borel subgroup, whereas that of $\theta' = \theta_I - \theta_E$ should be the Heisenberg parabolic P. It will be nice to construct some cusp forms and compute their Fourier coefficients. In particular, we conclude this section with the following question.

Question: What is the smallest k for which M_k^0 is non-zero?

11. Unramified Hecke Algebra and Satake Isomorphism

The remainder of the paper is devoted to the study of the action of Hecke operators on M_k. We begin with some background on the spherical Hecke algebra, and for the next few sections, the setting will be entirely local. Let G be a simple split algebraic group of adjoint type over the ring of integers $\mathcal{O}$ of a local field F. We fix the uniformizing element ϖ. Let $T \subset B \subset G$ be a maximal torus, contained in a Borel subgroup, defined over $\mathcal{O}$. Define the characters and co-characters of T by

$$\begin{cases} X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) \\ X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) \end{cases}$$

These are free abelian of rank $l = \dim(T)$, and have pairing into $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$. The choice of B determines a set of positive roots $\Phi^+ \subset X^*(T)$, and this determines a positive Weyl chamber P^+ in $X_*(T)$ by

$$P^+ = \{\lambda \in X_*(T) | \langle \lambda, \alpha \rangle \geq 0, \text{ for all } \alpha \in \Phi^+\}.$$

Let $\hat{G}$ be the complex dual group of G. This is a simply connected reductive group over $\mathbb{C}$ whose root system is dual to G. If we fix a maximal torus $\hat{T} \subset \hat{B} \subset \hat{G}$ then we have isomorphisms

$$\begin{cases} X^*(\hat{T}) = X_*(T) \\ X_*(\hat{T}) = X^*(T) \end{cases}$$

which take the positive roots (coroots) corresponding to $\hat{B}$ to the positive coroots (roots) corresponding to B. Under these identifications the elements of P^+ index the irreducible representations of $\hat{G}$: $\lambda \in P^+$ is the highest weight for $\hat{B}$.

Let $K = G(\mathcal{O})$, which is a hyperspecial maximal compact subgroup of $G = G(F)$ containing $T(\mathcal{O})$. By definition, the Hecke algebra $\mathcal{H} = \mathcal{H}(G, K)$ is the set of all compactly supported, K-bi-invariant functions $f : G \to \mathbb{C}$, with multiplication defined by convolution (using Haar measure on G giving K volume 1). For λ in $X_*(T)$, the double coset $K\lambda(\varpi)K$ does not depend on the choice of uniformizing element ϖ of F. The Cartan decomposition implies that the characteristic functions

$$c_\lambda = \mathrm{char}(K\lambda(\varpi)K) \qquad \lambda \in P^+$$

give a basis of $\mathcal{H}$ over $\mathbb{C}$.

Let $R(\hat{G})$ be the Grothendieck ring (over $\mathbb{C}$) of finite dimensional representations of $\hat{G}$. The Satake transform gives an isomorphism of $\mathcal{H}$ and $R(\hat{G})$. To describe it, we start with the simplest case when $G = T$. Then $X_*(T) = T/T(\mathcal{O})$, and the algebra $\mathcal{H}_T$ is simply the group algebra of $X_*(T)$. Since $X_*(T)$ is isomorphic to $X^*(\hat{T})$, we have

$$\mathcal{H}_T \cong R(\hat{T}).$$

In general, let N be the unipotent radical of B, defined over $\mathcal{O}$. Let dn be the unique Haar measure on N such that the volume of $N(\mathcal{O})$ is one. Let δ be the modular function on T, so that

$$d(tnt^{-1}) = |\delta(t)| \cdot dn.$$

For f in $\mathcal{H} = \mathcal{H}(G, K)$, define the Satake transform

$$S = S_{G/T} : \mathcal{H} \to \mathcal{H}_T$$

by the integral

$$S(f)(t) = |\delta(t)|^{1/2} \int_N f(tn)dn.$$

The image is precisely the ring of Weyl group invariants in the group algebra of $X_*(T)$ [Gr2, Prop. 3.6]. Since the ring $R(\hat{G})$ is isomorphic to $\mathbb{C}[X_*(T)]^W$, the Satake transform gives an isomorphism of $\mathbb{C}$-algebras

$$S : \mathcal{H} \cong R(\hat{G}).$$

Matrices for this isomorphism, using the standard bases c_λ of $\mathcal{H}$ and χ_λ of $R(\hat{G})$ (the character of the irreducible representation with highest weight λ) are described in [Gr2]; their entries involve Kazhdan-Lusztig polynomials.

More generally, if $P = LU$ is a parabolic subgroup of G with $T \subset L$ and $B \subset P$, we can define a relative Satake transform

$$S_{G/L} : \mathcal{H} \to \mathcal{H}_L$$

by the integral

$$S_L(f)(l) = |\delta_P(l)|^{1/2} \int_U f(lu)du$$

where $|\delta_P| : L \to \mathbb{R}_+^\times$ is the modular function for L. Via the Satake isomorphisms

$$\begin{cases} \mathcal{H} \cong R(\hat{G}) \\ \mathcal{H}_L \cong R(\hat{L}) \end{cases}$$

the relative Satake transform $S_{G/L}$ corresponds to the restriction of representations from $\hat{G}$ to $\hat{L}$. In particular, it is injective and its image lies in the subalgebra of $R(\hat{L})$ consisting of elements invariant under $W_{\hat{G}/\hat{L}} = W_{G/L} = N_G(L)/L$. After reviewing the classical example of $G = GL_2$ in the next section, we will work out a less familiar example in Section 13, where $G = G_2$, P is the Heisenberg parabolic, and $L = GL_2$.

12. The Hecke Algebra of GL_2

In this section, let $G = GL_2$. The Hecke algebra $\mathcal{H}$ of G over $\mathbb{Q}_p$ is well-known. We review the basic facts, in a manner suitable for later use in this paper. A good reference is [Se, Chap. VII].

The dual group $\hat{G}$ is isomorphic to $GL_2(\mathbb{C})$. Let χ be the character of the standard representation, and let χ^* be the character of the dual representation. Let $\pi = \wedge^2\chi$ be the determinant character of χ, and let π^* be the inverse of the determinant. Then we have an isomorphism of $\mathbb{C}$-algebras

$$R(\hat{G}) \cong \mathbb{C}[\chi, \pi, \pi^*]/(\pi \cdot \pi^* = 1).$$

Some further relations are:

$$\begin{cases} \chi^* = \chi \cdot \pi^* \\ \chi = \chi^* \cdot \pi. \end{cases}$$

The outer involution $A \mapsto {}^tA^{-1}$ of $GL_2(\mathbb{C})$ induces an involution τ of $R(\hat{G})$, with $\tau(\chi) = \chi^*$ and $\tau(\pi) = \pi^*$.

In the Satake isomorphism $S : \mathcal{H} \cong R(\hat{G})$, we find the following formulas for the generators [Gr2, §5]:

$$\begin{cases} S\left(\text{char } K \begin{pmatrix} p & \\ & 1 \end{pmatrix} K\right) = p^{1/2} \cdot \chi \\ S\left(\text{char } K \begin{pmatrix} p & \\ & p \end{pmatrix} K\right) = \pi \\ S\left(\text{char } K \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix} K\right) = \pi^*. \end{cases}$$

Put $\varphi = p^{1/2} \cdot \chi$ so that

$$S\left(\text{char } K \begin{pmatrix} p & \\ & 1 \end{pmatrix} K\right) = \varphi.$$

Let $\varphi^* = \tau(\varphi) = p^{1/2} \cdot \chi^*$. Since $\varphi^* = \varphi \cdot \pi^*$, we have

$$S\left(\text{char } K \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix} K\right) = \varphi^*.$$

Let $\mathcal{L}$ be the set of all lattices in the standard representation $\mathbb{Q}_p^2$ of $G = GL_2(\mathbb{Q}_p)$. Since G acts transitively on $\mathcal{L}$, and $K = GL_2(\mathbb{Z}_p)$ is the stabilizer of the lattice $\mathbb{Z}_p^2$, we have $\mathcal{L} \cong G/K$. The Hecke algebra acts on functions $f : \mathcal{L} \to \mathbb{C}$ as follows [Se, Pg. 98]

$$\begin{cases} \varphi \circ f(\Lambda) = \sum_{p\Lambda \subset \Lambda' \subset \Lambda} f(\Lambda') \\ \varphi^* \circ f(\Lambda) = \sum_{\Lambda \subset \Lambda' \subset \frac{1}{p}\Lambda} f(\Lambda') \\ \pi \circ f(\Lambda) = f(p\Lambda) \\ \pi^* \circ f(\Lambda) = f(\frac{1}{p}\Lambda). \end{cases} \tag{12.1}$$

The first sum above is taken over the $p+1$ lattices properly included in Λ and properly containing $p\Lambda$, and the second sum is similarly defined. Finally, a short calculation gives the formula

$$(\varphi \cdot \varphi^*) \circ f(\Lambda) = \sum_{p\Lambda \subset \Lambda' \subset \frac{1}{p}\Lambda} f(\Lambda') + p + 1.$$

Here, the sum is taken over the p^2+p lattices Λ' between $p\Lambda$ and $\frac{1}{p}\Lambda$, with $\Lambda'/p\Lambda \cong \mathbb{Z}/p^2$. Indeed, we have:

$$S\left(\text{char } K \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix} K\right) = \varphi \cdot \varphi^* - (p+1) \in R(\hat{G}).$$

To compute the action of Hecke operators on the Fourier coefficients $a_n(f)$ of a holomorphic form f of weight $2k$ for $SL_2(\mathbb{Z})$, we need the decomposition of the double coset $GL_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & 1 \end{pmatrix} GL_2(\mathbb{Z}_p)$ into single cosets. An argument with elementary divisors [Se, Pg. 99-100] gives

$$GL_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} GL_2(\mathbb{Z}_p) \cup \bigcup_{j=0}^{p-1} \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \tag{12.2}$$

as a union of $p+1$ right $GL_2(\mathbb{Z}_p)$-cosets. From this, it follows that [Se, Pg. 100] that the Fourier coefficients of $T_p|f$, with $T_p = p^{-1}\varphi$ are given by:

$$a_n(T_p|f) = a_{np}(f) + p^{2k-1} a_{n/p}(f) \tag{12.3}$$

with $a_{n/p} = 0$ unless $n \equiv 0 (\mathrm{mod}\ p)$.

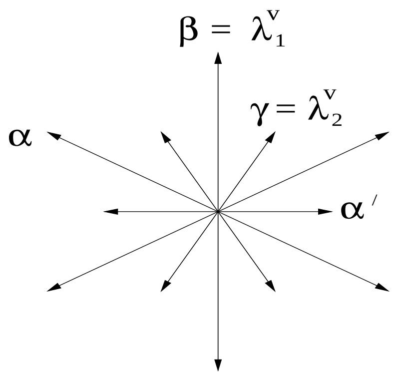
13. The Hecke Algebra of G_2

We now let $G = G_2$. Let P be the Heisenberg parabolic subgroup, with Levi factor $L \cong GL_2$. The two root spaces in the Lie algebra of L correspond to the short roots $\{\alpha', -\alpha'\}$ for G.

The dual group $\hat{G}$ is isomorphic to $G_2(\mathbb{C})$, and its representation ring $R(\hat{G})$ is isomorphic to the polynomial ring $\mathbb{C}[\chi_1, \chi_2]$, where χ_1 is the character of the 7-dimensional representation and χ_2 is the character of the 14-dimensional adjoint representation. Some useful identities in $R(\hat{G})$ are [Gr2, Pg. 234]:

$$\begin{cases} \wedge^2\chi_1 = \chi_1 + \chi_2 \\ \wedge^3\chi_1 = \chi_1^2 - \chi_2 \\ \wedge^{7-n}\chi_1 = \wedge^n\chi_1. \end{cases}$$

The highest weight λ_1 of χ_1 is identified with a dominant coroot for G. Since λ_1 is a short coroot, $\lambda_1 = \beta^\vee$, with $\beta = 2\alpha + 3\alpha'$ the (long) highest root. Similarly, the highest weight λ_2 for χ_2 is a dominant long coroot, so $\lambda_2 = \gamma^\vee$ with $\gamma = \alpha + 2\alpha'$. Here is a root diagram:



The Levi factor $\hat{L}$ in the dual parabolic $\hat{P}$ is isomorphic to $GL_2(\mathbb{C})$. Its Lie algebra has root spaces corresponding to the long coroots $\{\alpha'^\vee, -\alpha'^\vee\}$. Consulting the root diagram, and using the notation of the previous section, we find the following restriction formulas from $\hat{G}$ to $\hat{L}$:

$$\begin{cases} \mathrm{Res}(\chi_1) = \pi + \chi + 1 + \chi^* + \pi^* \\ \mathrm{Res}(\chi_2) = \mathrm{Res}(\chi_1) + \pi \cdot \chi + \chi \cdot \chi^* + \pi^* \cdot \chi^* - 1. \end{cases}$$

The image of restriction lies in the subring of τ-invariants of $R(\hat{L})$.

To study the Satake transform for G over $\mathbb{Q}_p$, put

$$\begin{cases} \varphi_1 = p^3\chi_1 \\ \varphi_2 = p^5\chi_2 \end{cases}$$

in $R(\hat{G})$. This is analogous to our normalization $\varphi = p^{1/2}\chi$ in the previous section. Then the calculations in [Gr2, §5] give the formulas:

$$\begin{cases} \varphi_1 = S(K\lambda_1(p)K) + 1 \\ \varphi_2 - \varphi_1 = S(K\lambda_2(p)K) + p^4 \end{cases}$$

in $R(\hat{G})$, where $K = G(\mathbb{Z}_p) \subset G = G_2(\mathbb{Q}_p)$.

We are now interested in obtaining the decomposition of $K\lambda_i(p)K$ into single K-cosets of the form ulK, with u in $U = U(\mathbb{Q}_p)$, the unipotent radical of P, and l in $L = L(\mathbb{Q}_p) \cong GL_2(\mathbb{Q}_p)$ its Levi factor. This is analogous to the decomposition of $GL_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & 1 \end{pmatrix}GL_2(\mathbb{Z}_p)$ obtained in (12.2), and is needed for the computation of the action of Hecke operators on modular forms. To accomplish this, we will use the relative Satake transform, and the following observation.

Proposition 13.1. *Fix t in G and l in L. Let $c[t]$ in $\mathcal{H}(G,K)$ be the characteristic function of the double coset KtK. Then*

$$S_{G/L}(c[t])(l) = |\delta_P(l)|^{1/2} \cdot \#\{\textit{distinct cosets of the form } ulK \textit{ in } KtK \textit{ with } u \in U\}.$$

Proof. Since, by definition,

$$S_{G/L}(c[t])(l) = |\delta_P(l)|^{1/2} \cdot \int_U c[t](lu)du,$$

where $\int_{U(\mathbb{Z}_p)} du = 1$, we have

$$S_{G/L}(c[t])(l) \cdot |\delta_P(l)|^{-1/2} = \#\{\text{distinct cosets of the form } lvU(\mathbb{Z}_p) \text{ in } lU \cap KtK, \text{ with } v \in U\}.$$

Since $U(\mathbb{Z}_p) = U \cap K$, the right hand side is equal to $\#\{$cosets of the form lvK in $KtK\}$. Since L normalizes U, putting $u = lvl^{-1}$, this is equal to $\#\{$cosets of the form ulK in $KtK\}$, as required. □

Since we have a commutative diagram of $\mathbb{C}$-algebra homomorphisms

$$\begin{array}{ccc} \mathcal{H}(G) & \xrightarrow{S_G} & R(\hat{G}) \\ {\scriptstyle S_{G/L}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ \mathcal{H}(L) & \xrightarrow[S_L]{} & R(\hat{L}) \end{array}$$

we can compute $S_{G/L}$ from our results on S_G and S_L, and the restriction of representations from $\hat{G}$ to $\hat{L}$.

Consider first the double coset $K\lambda_1(p)K$. We have seen that $S_G(c[\lambda_1(p)]) = p^3\chi_1 - 1$. Hence,

$$\mathrm{Res}(S_G(c[\lambda_1(p)])) = p^3(\pi + \chi + \chi^* + \pi^*) + (p^3 - 1)$$

in $R(\hat{L})$. But, by results in §12,

$$(13.2)\qquad \begin{cases} p^{1/2}\chi = S_L\left(c\begin{bmatrix} p & \\ & 1 \end{bmatrix}\right) \\ p^{1/2}\chi^* = S_L\left(c\begin{bmatrix} 1 & \\ & p^{-1} \end{bmatrix}\right) \\ \pi = S_L\left(c\begin{bmatrix} p & \\ & p \end{bmatrix}\right) \\ \pi^* = S_L\left(c\begin{bmatrix} p^{-1} & \\ & p^{-1} \end{bmatrix}\right). \end{cases}$$

We conclude that

$$S_{G/L}(c[\lambda_1(p)])(l) = \begin{cases} p^3 \quad \text{if } l \equiv \begin{pmatrix} p & \\ & p \end{pmatrix} \text{ or } \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}; \\ p^{5/2} \quad \text{if } l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}; \\ p^3 - 1 \quad \text{if } l \equiv 1; \\ 0, \quad \text{otherwise.} \end{cases}$$

Here, we have written $l \equiv t \in GL_2(\mathbb{Q}_p)$ to mean $l \in GL_2(\mathbb{Z}_p) \cdot t \cdot GL_2(\mathbb{Z}_p)$.

On the other hand, we have chosen the isomorphism $L \cong GL_2$ so that $\delta_P(l) = \det(l)^{-3}$. Hence,

$$|\delta_P(l)|^{1/2} = p^{\frac{3}{2}\cdot \mathrm{ord}_p(\det(l))}.$$

From this, and our determination of $S_{G/L}(c[\lambda_1(p)])$ above, we obtain the following

Corollary 13.3. *In the decomposition of $K\lambda_1(p)K$, the number of distinct cosets of the form ulK is given by*

$$\begin{cases} 1 \quad \textit{with } l \equiv \begin{pmatrix} p & \\ & p \end{pmatrix}; \\ p(p+1) \quad \textit{with } l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix}; \\ p^4(p+1) \quad \textit{with } l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}; \\ p^6 \quad \textit{with } l \equiv \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}; \\ p^3 - 1 \quad \textit{with } l \equiv 1; \\ 0 \quad \textit{otherwise.} \end{cases}$$

In particular, we see that the total number of distinct single K-cosets in $K\lambda_1(p)K$ is $p^6+p^5+p^4+p^3+p^2+p$, in agreement with [Gr2, Pg. 235]. To obtain an explicit decomposition, it remains to determine the elements u in ulK. This will be carried out in the next section.

Remarks: For a fixed

$$l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix}, \quad \text{respectively } l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix},$$

Proposition 13.1 and the ensueing computations actually show that there are precisely p (respectively p^4) single cosets of the form ulK in $K\lambda_1(p)K$. Since each of the double cosets

$$GL_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & 1 \end{pmatrix}GL_2(\mathbb{Z}_p) \quad \text{and } GL_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & 1 \end{pmatrix}GL_2(\mathbb{Z}_p)$$

contains $p+1$ single cosets, and $ulK = u'l'K$ implies that $lGL_2(\mathbb{Z}_p) = l'GL_2(\mathbb{Z}_p)$, this explains the numbers obtained in the second and third cases of Corollary 13.3.

In the remainder of this section, we will determine the l's which enter in the decomposition of $K\lambda_2(p)K$ into single K-cosets. Here we have [Gr2, Pg. 231]

$$S_G(c[\lambda_2(p)]) = p^5\chi_2 - p^3\chi_1 - p^4$$

so that

$$\mathrm{Res}(S_G(c[\lambda_2(p)])) = p^5(\pi\cdot\chi + (\chi\cdot\chi^*)_0 + \pi^*\cdot\chi^*) + (p^5-p^3)(\pi+\chi+\chi^*+\pi^*) + (p^5-p^3)$$

with $(\chi\cdot\chi^*)_0 = \chi\cdot\chi^* - \frac{p+1}{p}$ in $R(\hat{L})$. But by (13.2), as well as the formulas

$$\begin{cases} p^{1/2}\pi\cdot\chi = S_L\left(c\begin{bmatrix} p^2 & \\ & p \end{bmatrix}\right) \\ p^{1/2}\pi^*\cdot\chi^* = S_L\left(c\begin{bmatrix} p^{-1} & \\ & p^{-2} \end{bmatrix}\right) \\ p(\chi\cdot\chi^*)_0 = S_L\left(c\begin{bmatrix} p & \\ & p^{-1} \end{bmatrix}\right) \end{cases}$$

we obtain the following

Corollary 13.4. *In the decomposition of $K\lambda_2(p)K$, the number of distinct cosets of the form ulK is given by*

$$\begin{cases} 1\cdot(p+1) \quad \textit{with } l \equiv \begin{pmatrix} p^2 & \\ & p \end{pmatrix}; \\ p^4(p^2+p) \quad \textit{with } l \equiv \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix}; \\ p^9(p+1) \quad \textit{with } l \equiv \begin{pmatrix} p^{-1} & \\ & p^{-2} \end{pmatrix}; \\ (p^3-p)(p+1) \quad \textit{with } l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix}; \\ (p^6-p^4)(p+1) \quad \textit{with } l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}; \\ p^2-1 \quad \textit{with } l \equiv \begin{pmatrix} p & \\ & p \end{pmatrix}; \\ p^8-p^6 \quad \textit{with } l \equiv \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}; \\ p^5-p^3 \quad \textit{with } l \equiv 1; \\ 0 \quad \textit{otherwise.} \end{cases}$$

In particular, we obtain $p^{10}+p^9+p^8+p^7+p^6+p^5$ single cosets in all.

14. **Single Cosets Decompositions**

We now study the unipotent elements u which occur in the single cosets $ulK \subset K\lambda_i(p)K$. Since $ulK = lu'K$ with $u = lu'l^{-1}$, and u' is well-defined up to right multiplication by $K \cap U = U(\mathbb{Z}_p)$, we see that u is well-defined up to right multiplication by $lU(\mathbb{Z}_p)l^{-1}$.

Recall that for each root γ of T, we have the root group isomorphism $x_\gamma : \mathbb{G}_a \to U_\gamma$ over $\mathbb{Z}_p$, as well as the coroot $\gamma^\vee = h_\gamma : \mathbb{G}_m \to T$. Indeed, γ determines an embedding $SL_2 \hookrightarrow G$ over $\mathbb{Z}_p$ such that:

$$\begin{cases} x_\gamma(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ x_{-\gamma}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \\ h_\gamma(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \end{cases}$$

We will need to use the identity

$$h_\gamma(p)x_{-\gamma}(vp)x_\gamma(-t/p) \in K \tag{14.1}$$

for any root γ, where v and t are p-adic integers with $vt \equiv 1 (\mathrm{mod}\ p)$. Indeed, in the associated SL_2, this is the matrix product

$$\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ vp & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{t}{p} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & -t \\ v & \frac{1-vt}{p} \end{pmatrix}$$

which lies in $SL_2(\mathbb{Z}_p) \subset K$.

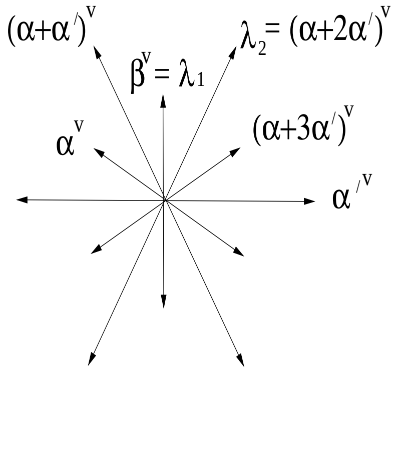
The coroots h_γ give us elements

$$h_\gamma(p) = \begin{pmatrix} p^a & \\ & p^b \end{pmatrix}$$

in $L \cong GL_2$, which we can identify by

$$\begin{cases} \langle \gamma^\vee, \beta \rangle = -a - b \\ \langle \gamma^\vee, \alpha' \rangle = a - b, \end{cases}$$

where α' is the basic short root and β is the highest root. Here is a coroot diagram:



We now begin by constructing the single cosets ulK in the double coset $K\lambda_1(p)K$.

Proposition 14.2. *If l lies in the double coset of either*

$$\begin{pmatrix} p & \\ & p \end{pmatrix}, \begin{pmatrix} p & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}, \text{ or } \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}$$

in $L \cong GL_2$, and u lies in $U(\mathbb{Z}_p)$, then ulK is contained in the K-double coset of $\lambda_1(p)$ in G. For each such l, the representatives u of the distinct right cosets of $U(\mathbb{Z}_p) \cap lU(\mathbb{Z}_p)l^{-1}$ in $U(\mathbb{Z}_p)$ give the distinct right cosets of the form ulK in $K\lambda_1(p)K$.

Proof. We have

$$\begin{cases} \begin{pmatrix} p & \\ & p \end{pmatrix} = -\beta^\vee(p) \\ \begin{pmatrix} p & \\ & 1 \end{pmatrix} = -\alpha^\vee(p) \\ \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix} = (\alpha + 3\alpha')^\vee(p) \\ \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix} = \beta^\vee(p) = \lambda_1(p) \end{cases}$$

Since these co-characters are in the Weyl group orbit of λ_1, and since representatives for the Weyl group elements can be taken in $N_G(T)(\mathbb{Z}_p) \subset K$, we see that $KlK = K\lambda_1(p)K$ for l in the 4 double cosets of GL_2 listed in the proposition. Hence ulK is contained in $K\lambda_1(p)K$ for any $u \in U(\mathbb{Z}_p)$. Moreover, for u and u' in $U(\mathbb{Z}_p)$, $ulK = u'lK$ if and only if $u^{-1}u'$ lies in $U(\mathbb{Z}_p) \cap lU(\mathbb{Z}_p)l^{-1}$. The index of the latter subgroup in $U(\mathbb{Z}_p)$ is

$$\begin{cases} 1 \text{ if } l \equiv \begin{pmatrix} p & \\ & p \end{pmatrix} \\ p \text{ if } l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix} \\ p^4 \text{ if } l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix} \\ p^6 \text{ if } l \equiv \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}. \end{cases}$$

By Corollary 13.3 and the remark following it, this is equal to the number of single cosets of the form ulK contained in $K\lambda_1(p)K$. Hence, by taking u to be distinct right coset representatives of $U(\mathbb{Z}_p) \cap lU(\mathbb{Z}_p)l^{-1}$ in $U(\mathbb{Z}_p)$, we obtain all such cosets, and the proposition is proved. □

Let $U^* \supset_p U(\mathbb{Z}_p)$ be the group obtained from $U(\mathbb{Z}_p)$ by adjoining the central elements $x_\beta(t/p)$ in U with $t \in \mathbb{Z}_p$.

Lemma 14.3. *If u lies in $U^* \setminus U(\mathbb{Z}_p)$, then uK is contained in $K\lambda_1(p)K$.*

Proof. We may assume that $u = x_\beta(t/p)$ with t a unit in $\mathbb{Z}_p$. Find v in $\mathbb{Z}_p$ such that $vt \equiv 1 (\text{mod } p)$. Then by (14.1), we see that u lies in the K-double coset of $h_\beta(p) = \lambda_1(p)$, as claimed. □

In fact, we can improve the above result slightly and obtain all single cosets uK (i.e. with $l \equiv 1$) contained in $K\lambda_1(p)K$. Recall the filtration

$$U(\mathbb{Z}_p) \supset U_2(\mathbb{Z}_p) \supset \{1\}$$

of $U(\mathbb{Z}_p)$ discussed in §1-2, with

$$U_2(\mathbb{Z}_p) \cong U_\beta(\mathbb{Z}_p) \cong \mathbb{Z}_p$$

and

$$U(\mathbb{Z}_p)/U_2(\mathbb{Z}_p) = V_1(\mathbb{Z}_p)$$

free of rank 4 over $\mathbb{Z}_p$. Let

$$m \subset \frac{1}{p} V_1(\mathbb{Z}_p)/V_1(\mathbb{Z}_p)$$

be a line stable under a Borel subgroup of $L(\mathbb{Z}/p\mathbb{Z}) = GL_2(\mathbb{Z}/p\mathbb{Z})$; we call such a line m a **singular line**. In the notations of §3, we have $m = W_1 = \theta(l)$. Let $V_1(m)$ be the corresponding $\mathbb{Z}_p$-module between $\frac{1}{p}V_1(\mathbb{Z}_p)$ and $V_1(\mathbb{Z}_p)$, and let $U(m)$ be the subgroup of U with

$$\begin{cases} U(m) \cap U_2 = \frac{1}{p} U_2(\mathbb{Z}_p) \\ U(m)/U(m) \cap U_2 = V_1(m). \end{cases} \tag{14.4}$$

Then $U(m)$ contains $U(\mathbb{Z}_p)$ with index p^2, and the $p+1$ subgroups $U(m) \subset U$ intersect in the group U^*.

Proposition 14.5. *If u lies in $U(m) \smallsetminus U(\mathbb{Z}_p)$, then uK is contained in $K\lambda_1(p)K$. The representatives u of the $p^2 - 1$ non-trivial cosets of $U(\mathbb{Z}_p)$ in $U(m)$ give distinct single cosets uK. As we vary the line m in $\frac{1}{p}V_1(\mathbb{Z}_p)/V_1(\mathbb{Z}_p)$, we obtain the $p^3 - 1$ distinct single cosets with $l \equiv 1$.*

Proof. If $u \in U^* \smallsetminus U(\mathbb{Z}_p)$, we have seen in the previous lemma that $uK \subset K\lambda_1(p)K$. Hence it remains to consider those $u \in U(m) \smallsetminus U^*$. We may assume, without loss of generality, that m is given by $\frac{1}{p}e_\alpha$, where α is the long basic root. Indeed, this is the highest weight vector for the Borel subgroup of L with root space $-\alpha'$. Then $U(m)$ is generated by U^* and $x_\alpha(t/p)$, with t in $\mathbb{Z}_p$. Since $\alpha^\vee$ is Weyl group conjugate to $\beta^\vee = \lambda_1$, the previous lemma shows that $x_\alpha(t/p)$ lies in $K\lambda_1(p)K$, whenever t is a unit in $\mathbb{Z}_p$. A commutation calculation in U then yields

$$U(\mathbb{Z}_p)x_\alpha(t/p)K = \bigcup_{a (\text{mod } p)} x_\beta(a/p)x_\alpha(t/p)K$$

which completes the proof that uK lies in the double coset of $\lambda_1(p)$ for all u in $U(m) \smallsetminus U(\mathbb{Z}_p)$.

Each of the $p+1$ lines m gives $p^2 - 1$ distinct single cosets, but the $p-1$ single cosets with $u \in U^* \smallsetminus U(\mathbb{Z}_p)$ are obtained with multiplicity $p+1$. Hence there are $(p+1)(p^2-1) - p(p-1) = p^3 - 1$ distinct single cosets in all. Comparing with Corollary 13.3, we see that we have obtained all the single cosets of the form uK in $K\lambda_1(p)K$. □

We now construct the single cosets ulK contained in $K\lambda_2(p)K$.

Proposition 14.6. *If l lies in the double coset of either*

$$\begin{pmatrix} p^2 & \\ & p \end{pmatrix}, \quad \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix}, \quad or \quad \begin{pmatrix} p^{-1} & \\ & p^{-2} \end{pmatrix}$$

in $L \cong GL_2$, and u lies in $U(\mathbb{Z}_p)$, then ulK is contained in $K\lambda_2(p)K$. For each such l, the representatives u of the distinct right cosets of $U(\mathbb{Z}_p) \cap lU(\mathbb{Z}_p)l^{-1}$ in $U(\mathbb{Z}_p)$ give the distinct right cosets of the form ulK contained in $K\lambda_2(p)K$.

Proof. This is similar to Proposition 14.2. We have

$$\begin{cases} \begin{pmatrix} p^2 & \\ & p \end{pmatrix} = -(\alpha+\alpha')^\vee(p) \\ \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix} = \alpha'^\vee(p) \\ \begin{pmatrix} p^{-1} & \\ & p^{-2} \end{pmatrix} = -(\alpha+2\alpha')^\vee(p) = -\lambda_2(p) \end{cases}$$

Since these co-characters are in the same Weyl group orbit as λ_2, we have ulK contained in $K\lambda_2(p)K$ for all $u \in U(\mathbb{Z}_p)$.

Again, we compute the index of $U(\mathbb{Z}_p) \cap lU(\mathbb{Z}_p)l^{-1}$ in $U(\mathbb{Z}_p)$ to be

$$\begin{cases} 1, \text{ if } l \equiv \begin{pmatrix} p^2 & \\ & p \end{pmatrix} \\ p^4, \text{ if } l \equiv \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix} \\ p^9, \text{ if } l \equiv \begin{pmatrix} p^{-1} & \\ & p^{-2} \end{pmatrix}. \end{cases}$$

By Corollary 13.4, this is the total number of single cosets of the form ulK in $K\lambda_2(p)K$, and thus the proposition is proved. $\square$

Next, we shall determine the single cosets in $K\lambda_2(p)K$, with

$$l \equiv \begin{pmatrix} p & \\ & p \end{pmatrix}, \begin{pmatrix} p & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}, \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}$$

in $L \cong GL_2$. It is more involved and requires greater care. The following lemma will be the main technical tool.

Lemma 14.7. *Let γ and γ' be a pair of long roots forming a 60° angle. Let t be a unit in $\mathbb{Z}_p$. Then*

$$Kx_\gamma(t/p)h_{\gamma'}(p)K = K\lambda_2(p)K = Kx_\gamma(-t/p^2)h_{-\gamma'}(p)K.$$

Proof. Let v be a unit in $\mathbb{Z}_p$ such that $vt \equiv 1 (\mathrm{mod}\ p)$. Then

$$Kx_\gamma(t/p)h_{\gamma'}(p)K = Kx_\gamma(t/p)h_{\gamma'}(p)x_{-\gamma}(-p^2v)K =$$

$$= Kx_\gamma(t/p)x_{-\gamma}(-pv)h_{\gamma'}(p)K = Kh_\gamma(p)h_{\gamma'}(p)K$$

where the last equality follows by the formula (14.1). Since γ and γ' form a $60°$ angle, $\delta = \frac{1}{3}(\gamma + \gamma')$ is a short root. Since $h_\gamma(p)h_{\gamma'}(p) = h_\delta(p)$, and $\delta^\vee$ is conjugated to λ_2, the first identity holds. To prove the second identity, note that inverse mapping preserves K-double cosets. Thus,

$$(x_\gamma(t/p)h_{\gamma'}(p))^{-1} = h_{-\gamma'}(p)x_\gamma(-t/p) = x_\gamma(-t/p^2)h_{-\gamma'}(p)$$

is in $K\lambda_2(p)K$. The lemma is proved. $\square$

The following lemma can easily be checked; we omit the proof.

Lemma 14.8. *Let $l = h_\gamma(p)$ where γ is a long root different from α and $-\alpha$. Let t be a unit in $\mathbb{Z}_p$, and set*

$$\begin{cases} u = x_\alpha(t/p) & \text{if } \langle\alpha, \gamma^\vee\rangle = 1 \\ u = x_\alpha(t/p^2) & \text{if } \langle\alpha, \gamma^\vee\rangle = -1. \end{cases}$$

Then the index of $U(\mathbb{Z}_p) \cap ulU(\mathbb{Z}_p)l^{-1}u^{-1}$ in $U(\mathbb{Z}_p)$ is

$$\begin{cases} 1 \text{ if } l = \begin{pmatrix} p & \\ & p \end{pmatrix} = -\beta^\vee(p) \\ p \text{ if } l = \begin{pmatrix} 1 & \\ & p \end{pmatrix} = -(\alpha + 3\alpha')^\vee(p) \\ p^4 \text{ if } l = \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix} = (\alpha + 3\alpha')(p) \\ p^6 \text{ if } l = \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix} = \beta^\vee(p). \end{cases}$$

To describe the single cosets we shall use the exponential map in a special case. A non-zero element w in V_1 is called singular if the line through w is singular. Assume now that w is singular, contained in $V_1(\mathbb{Z}_p)$, but not contained in $pV_1(\mathbb{Z}_p)$. Using the cubic map θ one can check that each such w is conjugated under $L(\mathbb{Z}_p) \cong GL_2(\mathbb{Z}_p)$ to e_α. In particular, since it is true for e_α,

$$\begin{cases} ad_w^2(\mathfrak{g}(\mathbb{Z}_p)) \subset 2 \cdot \mathfrak{g}(\mathbb{Z}_p) \\ ad_w^3 = 0 \end{cases}$$

where ad_w denotes the adjoint action on the Lie algebra $\mathfrak{g}$. Thus, the exponential map

$$exp(tw) = 1 + tad_w + t^2\frac{ad_w^2}{2}$$

is defined over $\mathbb{Z}_p$, and $exp(tw)$ is in K if and only if t is in $\mathbb{Z}_p$.

Proposition 14.9. *Let w be a singular element in $V_1(\mathbb{Z}_p)$, not contained in $pV_1(\mathbb{Z}_p)$.*
If

$$l \equiv \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}$$

put $u = \exp(w/p)$. Then ulK is contained in $K\lambda_2(p)K$. The family $U(\mathbb{Z}_p)ulK$ consists of p^6 disjoint single cosets.
If

$$l \equiv \begin{pmatrix} p & \\ & p \end{pmatrix}$$

put $u = \exp(w/p^2)$. Then ulK is contained in $K\lambda_2(p)K$. The family $U(\mathbb{Z}_p)ulK = ulK$ consists of one single coset.

Proof. Since w is $L(\mathbb{Z}_p) \cong GL_2(\mathbb{Z}_p)$ conjugated to e_α, we may assume that $w = e_\alpha$ where α is the long basic root. Then ulK is in $K\lambda_2(p)K$ by Lemma 14.7, where we take

$$\begin{cases} \gamma = \alpha \\ \gamma' = \beta. \end{cases}$$

Finally, the statements concerning the number of single cosets in $U(\mathbb{Z}_p)ulK$ follow from Lemma 14.8. □

In both cases the family $U(\mathbb{Z}_p)ulK$ depends on the choice of w modulo $pV_1(\mathbb{Z}_p)$. Since there are $p+1$ singular lines in $V_1(\mathbb{F}_p)$, each containing $p-1$ non-trivial elements, we see that the total number of single cosets of the form ulK given by Proposition 14.9 is

$$\begin{cases} p^2 - 1 \text{ if } l \equiv \begin{pmatrix} p & \\ & p \end{pmatrix} \\ p^8 - p^6 \text{ if } l \equiv \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}. \end{cases}$$

By Corollary 13.4 this is equal to the number of single cosets of the form ulK contained in $K\lambda_2(p)K$. In particular we have obtained all such cosets.

Proposition 14.10. *Let t be a unit in $\mathbb{Z}_p$.*
If

$$l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix}$$

put $u = x_\beta(t/p)$. Then ulK is contained in $K\lambda_2(p)K$. The family $U(\mathbb{Z}_p)ulK$ consists of p single cosets.
If

$$l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}$$

put $u = x_\beta(t/p^2)$. Then ulK is contained in $K\lambda_2(p)K$. The family $U(\mathbb{Z}_p)ulK$ consists of p^4 single cosets.

Proof. If we set

$$\begin{cases} \gamma = \beta \\ \gamma' = \alpha + 3\alpha' \end{cases}$$

in Lemma 14.7, then we see immediately that ulK is in $K\lambda_2(p)K$. Furthermore, since u is in the center of U, the number of single K-cosets in $U(\mathbb{Z}_p)ulK$ is equal to the index of $U(\mathbb{Z}_p) \cap lU(\mathbb{Z}_p)l^{-1}$ in $U(\mathbb{Z}_p)$ which is given in Proposition 14.2. □

In both cases the family $U(\mathbb{Z}_p)ulK$ depends on the choice of t modulo $p\mathbb{Z}_p$. As t runs through the $p-1$ non-trivial classes modulo $p\mathbb{Z}_p$, we see that the total number of single cosets of the form ulK constructed in Proposition 14.10 is

$$\begin{cases} p^2 - p \text{ if } l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix} \\ p^5 - p^4 \text{ if } l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix} \end{cases}.$$

This is not yet equal to the number of single cosets of the form ulK contained in $K\lambda_2(p)K$, which is $p^3 - p$ and $p^6 - p^4$ respectively. The remaining single cosets will be constructed in Proposition 14.12. To do so, we need the following lemma characterizing single $L(\mathbb{Z}_p) \cong GL_2(\mathbb{Z}_p)$ cosets in terms of the action on V_1.

Lemma 14.11. *(i) If*

$$l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix}$$

then there is unique singular line m_l in $V_1(\mathbb{F}_p)$ such that

$$l \cdot pV_1(\mathbb{Z}_p) + V_1(\mathbb{Z}_p) = V_1(m_l).$$

Furthermore, the set of all $t \equiv l$ such that $t \cdot pV_1(\mathbb{Z}_p) + V_1(\mathbb{Z}_p) = V_1(m_l)$ is a single right $GL_2(\mathbb{Z}_p)$-coset.

(ii) If

$$l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}$$

then there is unique singular line m_l in $V_1(\mathbb{F}_p)$ such that

$$l \cdot V_1(\mathbb{Z}_p) + V_1(\mathbb{Z}_p) = V(m_l).$$

Furthermore, the set of all $t \equiv l$ such that $t \cdot V_1(\mathbb{Z}_p) + V_1(\mathbb{Z}_p) = V_1(m_l)$ is a single right $GL_2(\mathbb{Z}_p)$-coset.

Proposition 14.12. *Let w be a singular element in $V_1(\mathbb{Z}_p)$. Assume that the reduction of w modulo p is not contained in m_l.*

If

$$l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix}$$

put $u = \exp(w/p)$. Then ulK is in $K\lambda_2(p)K$. The family $U(\mathbb{Z}_p)ulK$ consists of p single cosets.

If

$$l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}$$

put $u = \exp(w/p^2)$. *Then* ulK *is in* $K\lambda_2(p)K$. *The family* $U(\mathbb{Z}_p)ulK$ *consists of* p^4 *single cosets.*

Proof. Recall that $GL_2(\mathbb{Z}_p)$ acts transitively on the set of singular elements in $V_1(\mathbb{Z}_p)$, with non-trivial reduction modulo p. Thus, we may assume that $w = e_\alpha$. The stabilizer of w acts transitively on singular lines in $V_1(\mathbb{Z}_p)$ not containing e_α. In particular, we can assume that m_l is given by $e_{\alpha+3\alpha'}$. Putting

$$\begin{cases} \gamma = \alpha \\ \gamma' = -\alpha - 3\alpha' \end{cases}$$

in Lemma 14.7, we see immediately that ulK is contained in $K\lambda_2(p)K$. The number of single cosets in $U(\mathbb{Z}_p)ulK$ is p and p^4 respectively, by Lemma 14.8. □

In both cases the family $U(\mathbb{Z}_p)ulK$ depends on the choice of w modulo $pV_1(\mathbb{Z}_p)$. Since there are p singular lines in $V_1(\mathbb{F}_p)$ different from m_l, and each contains $p-1$ non-trivial elements, we see that the total number of single cosets of the form ulK given by Proposition 14.12 is

$$\begin{cases} p^3 - p^2 \text{ if } l \equiv \begin{pmatrix} p & \\ & 1 \end{pmatrix} \\ p^6 - p^5 \text{ if } l \equiv \begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix} \end{cases}.$$

These cosets, together with the cosets given in Proposition 14.10, give all single cosets of the form ulK contained in $K\lambda_2(p)K$, by Corollary 13.4.

It remains to deal with $l \equiv 1$. We have two families of single cosets. The first one is analogous to the family of single cosets constructed in Proposition 14.10 for the double coset $K\lambda_1(p)K$. Let

$$n \subset \frac{1}{p}V_1(\mathbb{Z}_p)/V_1(\mathbb{Z}_p)$$

be a plane stable under a Borel subgroup of $L(\mathbb{Z}/p\mathbb{Z}) = GL_2(\mathbb{Z}/p\mathbb{Z})$; we call such a plane a singular plane. Let $V_1(n)$ be the corresponding $\mathbb{Z}_p$-module between $\frac{1}{p}V_1(\mathbb{Z}_p)$ and $V_1(\mathbb{Z}_p)$, and let $U(n)$ be the subgroup of U with

$$\begin{cases} U(n) \cap U_2 = \frac{1}{p}U_2(\mathbb{Z}_p) \\ U(n)/U(n) \cap U_2 = V_1(n). \end{cases} \tag{14.13}$$

Let m be the unique singular line contained in n. Then $U(n)$ contains $U(m)$ with index p.

Proposition 14.14. *If* u *lies in* $U(n) \smallsetminus U(m)$, *then* uK *is contained in* $K\lambda_1(p)K$. *The representatives* u *of the* $p^3 - p^2$ *non-trivial cosets of* $U(\mathbb{Z}_p)$ *in* $U(n) \smallsetminus U(m)$ *give distinct single cosets* uK. *As we vary the plane* n *in* $\frac{1}{p}V_1(\mathbb{Z}_p)/V_1(\mathbb{Z}_p)$, *we obtain* $p^4 - p^2$ *distinct single cosets with* $l \equiv 1$.

Proof. We may assume, without loss of generality, that n is given by $\frac{1}{p}e_{\alpha+2\alpha'}$ and $\frac{1}{p}e_{\alpha+3\alpha'}$. Note that the unique singular line m in n is given by $\frac{1}{p}e_{\alpha+3\alpha'}$. Thus, we can assume that

$$u = x_{\alpha+2\alpha'}(t/p)x_{\alpha+3\alpha'}(a/p)x_{2\alpha+3\alpha'}(b/p)$$

where t is some unit in $\mathbb{Z}_p$ and a and b are in $\mathbb{Z}_p$. Assume first that $b = 0$. Then, the formula (14.1) implies that

$$KuK = Kh_{\alpha+2\alpha'}(p)x_{-\alpha-2\alpha'}(vp)x_{\alpha+3\alpha'}(a/p)K$$

for some unit v in $\mathbb{Z}_p$. The Chevalley commutation relations show that the commutator of $x_{-\alpha-2\alpha'}(vp)$ and $x_{\alpha+3\alpha'}(a/p)$ is in K. Since $x_{-\alpha-2\alpha'}(vp)$ is also in K, we see that the above double coset is in fact equal to

$$Kh_{\alpha+2\alpha'}(p)x_{\alpha+3\alpha'}(a/p)K = Kx_{\alpha+3\alpha'}(p^3a)h_{\alpha+2\alpha'}(p)K = Kh_{\alpha+2\alpha'}(p)K.$$

Since $(\alpha+2\alpha')^\vee$ is Weyl group conjugate to λ_2, we have shown that u lies in $K\lambda_2(p)K$, if $b = 0$. Next, assume that a is a unit and b any element in $\mathbb{Z}_p$. Then $x_\alpha(b/a)$ is in K, and the commutator with $x_{\alpha+3\alpha'}(a/p)$ gives $x_{2\alpha+3\alpha'}(b/p)$ (or $x_{2\alpha+3\alpha'}(-b/p)$). Note that the long basic root α is perpendicular to the short root $\alpha+2\alpha'$. In particular, $x_\alpha(b/a)$ commutes with $x_{\alpha+2\alpha'}(t/p)$. Thus, (ignoring the sign)

$$Kx_\alpha(b/a)x_{\alpha+2\alpha'}(t/p)x_{\alpha+3\alpha'}(a/p)x_\alpha(-b/a)K = Kx_{\alpha+2\alpha'}(t/p)x_{\alpha+3\alpha'}(a/p)x_{2\alpha+3\alpha'}(b/p)K.$$

and we have shown that u lies in $K\lambda_2(p)K$ if $b = 0$ or if a is a unit. Since the reflection about α fixes $\alpha+2\alpha'$, and switches $\alpha+3\alpha'$ and $2\alpha+3\alpha'$, it follows that u is in $K\lambda_2(p)K$ if $a = 0$ or if b is a unit, as well. Combining the two, we have completed the proof that uK lies in the double coset of $\lambda_2(p)$ for all u in $U(n) \smallsetminus U(m)$.

Each of the $p+1$ singular planes n gives $p^3 - p^2$ distinct single cosets. Hence there are $(p+1)(p^3-p^2) = p^4 - p^2$ distinct single cosets in all. $\square$

Proposition 14.15. *Let (m, m') be an ordered pair of distinct singular lines in $V_1(\mathbb{Q}_p)$, such that $m \cap V_1(\mathbb{Z}_p)$ and $m' \cap V_1(\mathbb{Z}_p)$ give distinct singular lines on reudction modulo p. Let w and w' be in $m \cap V_1(\mathbb{Z}_p)$ and $m' \cap V_1(\mathbb{Z}_p)$ respectively, but not in $pV_1(\mathbb{Z}_p)$. Let $u = \exp(w/p)\exp(w'/p)$. Then the family $U(\mathbb{Z}_p)uK$ is in $K\lambda_2(p)K$, and consists of p single cosets. The family $U(\mathbb{Z}_p)uK$ depends on the choice of w and w' modulo $pV_1(\mathbb{Z}_p)$. As we vary the $p(p+1)$ ordered pairs of distinct singular lines in $V_1(\mathbb{F}_p)$, we have $p^5 - p^4 - p^3 + p^2$ distinct single cosets in all.*

Proof. Since $L(\mathbb{Z}_p) \cong GL_2(\mathbb{Z}_p)$ acts transitively on the set of ordered pairs of distinct singular lines in $V_1(\mathbb{F}_p)$, we may assume that m is given by e_α and m' by $e_{\alpha+3\alpha'}$. Thus we may take $u = x_\alpha(t/p)x_{\alpha+3\alpha'}(t'/p)$ where t and t' are units in $\mathbb{Z}_p$. Note that α and $\alpha+3\alpha'$ are long roots. Let $SL_3 \subset G_2$, be the Chevalley subgroup generated by the long root subgroups. Then it is not difficult to check that

$$SL_3(\mathbb{Z}_p)uSL_3(\mathbb{Z}_p) = SL_3(\mathbb{Z}_p)\lambda_2(p)SL_3(\mathbb{Z}_p),$$

so that u lies in $K\lambda_2(p)K$. Finally, a commutation calculation in U gives

$$U(\mathbb{Z}_p)x_\alpha(t/p)x_{\alpha+3\alpha'}(t'/p)K = \bigcup_{a(\text{mod } p)} x_\beta(a/p)x_\alpha(t/p)x_{\alpha+3\alpha'}(t'/p)K,$$

a disjoint union of p single cosets.

Next, using the commutation relations in U, it is easy to see that these are disjoint families, as we run through ordered pairs of distinct singular lines in $V_1(\mathbb{F}_p)$. As each line contains $p-1$ non-trivial elements, we see that we have constructed $p(p+1)(p-1)^2p = p^5-p^4-p^3+p^2$ single cosets of the form uK in all.

$\square$

Combining the last two propositions, we see that we have constructed p^5-p^3 single cosets. By Corollary 13.4, these are all the single cosets of the form uK in $K\lambda_2(p)K$.

15. Action of Hecke Operators on Fourier Coefficients

Now we return to the global situation, and let G be the Chevalley group of type G_2 over $\mathbb{Z}$. Let t be an element of $G(\mathbb{Q}_p)$ and let $K = G(\mathbb{Z}_p)$. Once we have determined the decomposition $KtK = \bigcup t_i K$ into right cosets, we can define a Hecke operator t on the space $\mathcal{A}(G)^K$ of automorphic forms on $G(\mathbb{A})$ fixed by K by the formula

$$t|F(g) = \sum_i F(gt_i).$$

Here the t_i are viewed as elements of $G(\mathbb{A})$, which are equal to 1 at all places $v \neq p$. This action satisfies

$$(t_1 \cdot t_2)|F = t_1|(t_2|F).$$

Using strong approximation, we may view an automorphic form F on $G(\mathbb{A})$ fixed by $G(\hat{\mathbb{Z}})$ as a function F_∞ on $G(\mathbb{Z})\backslash G(\mathbb{R})$, given by

$$F_\infty(g_\infty) = F(g_\infty, 1, 1, ...).$$

In terms of the function F_∞, the action of the Hecke operator t looks a bit different. By strong approximation again, we may find an element s in $G(\mathbb{Q})$ which satisfy

$$\begin{cases} s_l \text{ is in } G(\mathbb{Z}_l) \text{ if } l \neq p; \\ s_p K = tK \text{ in } G(\mathbb{Q}_p). \end{cases} \tag{15.1}$$

Similarly, we can find elements s_i which approximate t_i in the above sense. The elements s and s_i are unique up to right multiplication by $G(\mathbb{Z})$, and viewing them in $G(\mathbb{R})$, we obtain a decomposition

$$G(\mathbb{Z})sG(\mathbb{Z}) = \bigcup s_i G(\mathbb{Z}). \tag{15.2}$$

We then have the formula

$$t|F_\infty(g_\infty) = \sum_i F_\infty(s_i^{-1} g_\infty)$$

for the action of the Hecke operator t on $F_\infty : G(\mathbb{Z})\backslash G(\mathbb{R}) \to \mathbb{C}$. Indeed, there is no loss of generality in taking t and t_i to be the p-adic components of s and s_i. Then

$$\begin{aligned} t|F_\infty(g_\infty) &= t|F(g_\infty, 1, 1, ...) \\ &= \sum_i F(g_\infty, 1, 1, ..., (s_i)_p, ...) \\ &= \sum_i F(s_i^{-1} \cdot (g_\infty, 1, 1, ..., (s_i)_p, ...) \quad \text{(since } F \text{ is left } G(\mathbb{Q})\text{-invariant)} \\ &= \sum_i F(s_i^{-1} \cdot g_\infty, 1, 1, ...) \quad \text{(since } s_i \in G(\mathbb{Z}_l) \text{ for all } l \neq p) \\ &= \sum_i F_\infty(s_i^{-1} g_\infty). \end{aligned}$$

Now let $f : \pi_k \otimes \mathbb{C} \to \mathcal{A}(G)$ be a modular form of weight k, so that the image of f is contained in the subspace of $\mathcal{A}(G)$ fixed by $G(\hat{\mathbb{Z}})$. For $v \in \pi_k$, let $F = f(v) \in \mathcal{A}(G)$, which we shall henceforth regard as a function on $G(\mathbb{Z})\backslash G(\mathbb{R})$ by restriction. We can now define an action of the Hecke operator $t \in G(\mathbb{Q}_p)$ on f by:

$$(t|f)(v) = t|F : G(\mathbb{Z})\backslash G(\mathbb{R}) \longrightarrow \mathbb{C}.$$

If χ is a character of $U(\mathbb{R})$ trivial on $U(\mathbb{Z})$, then we have defined the Fourier coefficient $c_\chi(t|f)$. Our goal is to express $c_\chi(t|f)$ in terms of the Fourier coefficients of f.

Let s and $s_i \in G(\mathbb{Q})$ be related to t and $t_i \in G(\mathbb{Q}_p)$ as in (15.1), so that we have a decomposition

$$G(\mathbb{Z})sG(\mathbb{Z}) = \bigcup_i s_i G(\mathbb{Z}).$$

It is now convenient to group the single $G(\mathbb{Z})$-cosets according to the $U(\mathbb{Z})$-orbits in which they lie, where $U(\mathbb{Z})$ acts by left multiplication on $G(\mathbb{Z})sG(\mathbb{Z})/G(\mathbb{Z})$. Hence we first write

$$G(\mathbb{Z})sG(\mathbb{Z}) = \bigcup_i U(\mathbb{Z})p_i G(\mathbb{Z}) = \bigcup_i \mathcal{O}_i,$$

where $p_i = u_i l_i$, with $u_i \in U(\mathbb{Q})$ and $l_i \in L(\mathbb{Q})$. Each $\mathcal{O}_i = U(\mathbb{Z})p_i G(\mathbb{Z})$ can further be decomposed into the union of $U_2(\mathbb{Z})$-orbits:

$$\mathcal{O}_i = \bigcup_j U_2(\mathbb{Z})p_{ij} G(\mathbb{Z}) = \bigcup_j \mathcal{O}_{ij}$$

where $p_{ij} = v_j p_i$ with $v_j \in U(\mathbb{Z})$. Finally, we write each $\mathcal{O}_{ij}$ as single $G(\mathbb{Z})$-cosets:

$$\mathcal{O}_{ij} = \bigcup_k z_k p_{ij} G(\mathbb{Z}),$$

with $z_k \in U_2(\mathbb{Z})$. Each $\mathcal{O}_{ij}/G(\mathbb{Z})$ is thus a homogeneous space for $U_2(\mathbb{Z})$, and the stabilizer of $p_{ij}G(\mathbb{Z})$ is $U_2(\mathbb{Z}) \cap l_i U_2(\mathbb{Z}) l_i^{-1}$. In particular, the number

$$m_i = \#\{\text{single } G(\mathbb{Z})\text{-cosets in } \mathcal{O}_{ij}\} = \# U_2(\mathbb{Z})/U_2(\mathbb{Z}) \cap l_i U_2(\mathbb{Z}) l_i^{-1}$$

depends only on i and not on j.

We are now ready to compute the Fourier coefficient $c_\chi(t|f)$. Consider first the constant term of $t|F$ along U_2. For $u \in U(\mathbb{R})$, we have:

$$\begin{aligned}
(t|F)_{U_2}(u) &= \int_{U_2(\mathbb{Z})\backslash U_2(\mathbb{R})} (t|F)(zu)dz \\
&= \sum_{i,j} \int_{U_2(\mathbb{Z})\backslash U_2(\mathbb{R})} \sum_k F(p_{ij}^{-1} z_k^{-1} zu)dz \\
&= \sum_{i,j} I_{ij}.
\end{aligned}$$

Consider the Fourier expansion of F along U_2:

$$F(g) = \sum_\psi F_\psi(g), \qquad g \in G(\mathbb{R}),$$

where ψ runs through the characters of $U_2(\mathbb{R})$ trivial on $U_2(\mathbb{Z})$ and

$$F_\psi(g) = \int_{U_2(\mathbb{Z})\backslash U_2(\mathbb{R})} F(zg) \cdot \overline{\psi(z)} dz.$$

Then for fixed i, j,

$$\begin{aligned} I_{ij} &= \int_{U_2(\mathbb{Z})\backslash U_2(\mathbb{R})} \sum_k \sum_\psi \psi(p_{ij}^{-1} z p_{ij}) \psi(p_{ij}^{-1} z_k^{-1} p_{ij}) F_\psi(p_{ij}^{-1} u) dz \\ &= \int_{U_2(\mathbb{Z})\backslash U_2(\mathbb{R})} \sum_\psi (p_{ij} \cdot \psi)(z) F_\psi(p_{ij}^{-1} u) \cdot \left(\sum_k (p_{ij} \cdot \psi)(z_k) \right) dz. \end{aligned}$$

Since

$$\sum_k (p_{ij} \cdot \psi)(z_k) = \begin{cases} m_i, \text{ if } p_{ij} \cdot \psi \text{ is trivial on } U_2(\mathbb{Z}); \\ 0, \text{ otherwise,} \end{cases}$$

we deduce that

$$I_{ij} = m_i \cdot F_{U_2}(p_{ij}^{-1} u),$$

and hence

$$(t|F)_{U_2}(u) = \sum_{i,j} m_i \cdot F_{U_2}(p_{ij}^{-1} u).$$

We can now regard F_{U_2} and $(t|F)_{U_2}$ as functions on $L(\mathbb{R}) \cdot V_1(\mathbb{R}) \cong P(\mathbb{R})/U_2(\mathbb{R})$. Then we have the Fourier expansion

$$F_{U_2}(g) = \sum_\psi F_{U_2,\psi}(g), \qquad g \in L(\mathbb{R}) \cdot V_1(\mathbb{R}),$$

where ψ runs through the characters of $V_1(\mathbb{R})$ trivial on $V_1(\mathbb{Z})$. Hence

$$\begin{aligned} (t|F)_\chi(1) &= \int_{V_1(\mathbb{Z})\backslash V_1(\mathbb{R})} (t|F)_{U_2}(v) \cdot \overline{\chi(v)} dv \\ &= \sum_i m_i \cdot \int_{V_1(\mathbb{Z})\backslash V_1(\mathbb{R})} \sum_j F_{U_2}(p_i^{-1} v_j^{-1} v) \cdot \overline{\chi(v)} dv \\ &= \sum_i m_i \cdot \int_{V_1(\mathbb{Z})\backslash V_1(\mathbb{R})} \sum_j \sum_\psi (l_i \cdot \psi)(v_j^{-1}) \cdot (l_i \cdot \psi)(v) \cdot F_{U_2,\psi}(p_i^{-1}) \cdot \overline{\chi(v)} dv \\ &= \sum_i m_i \cdot \int_{V_1(\mathbb{Z})\backslash V_1(\mathbb{R})} \sum_\psi F_{U_2,\psi}(p_i^{-1}) \cdot (l_i \cdot \psi)(v) \cdot \overline{\chi(v)} \cdot \left(\sum_j (l_i \cdot \psi)(v_j^{-1}) \right) dv. \end{aligned}$$

Since

$$\sum_j (l_i \cdot \psi)(v_j^{-1}) = \begin{cases} \#V_1(\mathbb{Z})/(V_1(\mathbb{Z}) \cap l_i V_1(\mathbb{Z}) l_i^{-1}), \text{ if } l_i \cdot \psi \text{ is trivial on } V_1(\mathbb{Z}); \\ 0, \text{ otherwise,} \end{cases}$$

and

$$n_i = m_i \cdot \#V_1(\mathbb{Z})/(V_1(\mathbb{Z}) \cap l_i V_1(\mathbb{Z}) l_i^{-1})$$

is the number of single $G(\mathbb{Z})$-cosets in $U(\mathbb{Z})p_iG(\mathbb{Z})$, we deduce that

$$(t|F)_\chi(1) = \sum_i n_i \cdot F_{l_i^{-1}\chi}(l_i^{-1}) \cdot \chi(u_i^{-1}), \tag{15.3}$$

where $F_{l_i^{-1}\chi}$ is equal to zero unless $l_i^{-1} \cdot \chi$ is trivial on $U(\mathbb{Z})$.

To relate these computations to the notion of Fourier coefficients defined in Section 8, let us fix a character χ_0 of $U(\mathbb{R})$ with $\Delta(\chi_0) > 0$, and a basis element l_0 in $\mathrm{Hom}_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi_0))$. Choose any $g \in L(\mathbb{R})$ such that $\chi = g \cdot \chi_0$. If $l_{f,\chi}$ denotes the linear functional on π_k defined by:

$$l_{f,\chi} : v \mapsto f(v)_\chi(1) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(v)(u) \cdot \overline{\chi(u)} du$$

then recall from (8.3) that $c_\chi(f)$ is defined by:

$$l_{f,\chi} = c_\chi(f) \cdot (\lambda_k(g) \cdot gl_0).$$

Now consider the linear functional

$$L_i : v \mapsto f(v)_{l_i^{-1}\chi}(l_i^{-1}),$$

whch is an element of $\mathrm{Hom}_{U(\mathbb{R})}(\pi_k, \mathbb{C}(\chi))$. A simple calculation shows that

$$L_i = c_{l_i^{-1}\chi}(f) \cdot \lambda_k(l_i)^{-1} \cdot (\lambda_k(g) \cdot gl_0). \tag{15.4}$$

From (15.3) and (15.4), we deduce that

$$l_{t|f,\chi} = \left(\sum_i n_i \cdot \lambda_k(l_i)^{-1} \cdot \chi(u_i)^{-1} \cdot c_{l_i^{-1}\chi}(f) \right) \cdot \lambda_k(g) \cdot gl_0,$$

and thus:

Proposition 15.5. *Suppose that*

$$G(\mathbb{Z})sG(\mathbb{Z}) = \bigcup_i U(\mathbb{Z})u_il_iG(\mathbb{Z}),$$

with $u_i \in U(\mathbb{Q})$ and $l_i \in L(\mathbb{Q})$. Then

$$c_\chi(t|f) = \sum_i n_i \cdot \lambda_k(l_i)^{-1} \cdot \chi(u_i)^{-1} \cdot c_{l_i^{-1}\chi}(f),$$

where

$$n_i = \#\{\textit{single } G(\mathbb{Z})\textit{-cosets in } U(\mathbb{Z})u_il_iG(\mathbb{Z})\},$$

and $c_{l_i^{-1}\chi}(f)$ is equal to zero unless $l_i^{-1}\chi$ is trivial on $U(\mathbb{Z})$.

Now fix a prime p, and consider the local Hecke algebra $\mathcal{H}_p$ at p. As we mentioned before,

$$\mathcal{H}_p \cong R(\hat{G}) \cong \mathbb{C}[\chi_1, \chi_2]$$

as $\mathbb{C}$-algebras, with χ_1 and χ_2 the two fundamental representations of the complex Lie group $\hat{G} = G_2(\mathbb{C})$. In the remainder of this section, we shall use Proposition 15.5 and the results of the previous section to obtain explicit formulas for the action of the Hecke operators χ_1 and χ_2 on the Fourier coefficients of modular forms in M_k.

Let A be a cubic ring with p-depth 0, corresponding to a character χ. Set $A_i = \mathbb{Z} + p^i A$, which has p-depth i, and corresponds to the character $\chi_i = \begin{pmatrix} p^i & \\ & p^i \end{pmatrix} \cdot \chi$. Let f be a modular form of weight k, and $c_i(f)$ the Fourier coefficient of f corresponding to A_i. To avoid issues about orientation, we assume for simplicity that the weight k is even. Then we have:

Proposition 15.6. *Let f be a modular form of even weight k and A_i the chain of cubic rings as above. If $i \geq 1$ then*

$$c_i(\chi_1|f) = p^{2k-1}c_{i-1}(f) + p^{k-1}\sum_{A_i \subset B \subset A_{i-1}} c_B(f) + c_i(f) + p^{-k}\sum_{A_{i+1}\subset B \subset A_i} c_B(f) + p^{1-2k}c_{i+1}(f).$$

where the inclusions of rings are proper. If $i = 0$, then

$$c_0(\chi_1|f) = p^{k-1}\sum_{A \subset_p B} c_B(f) + p^{-1}(n_A - 1)c_0(f) + p^{-k}\sum_{A_1 \subset B \subset A} c_B(f) + p^{1-2k}c_1(f),$$

where n_A is the number of rings B such that $A_1 \subset B \subset A$.

Proof. As we have noted before,

$$\chi_1|f = \frac{1}{p^3}(c[\lambda_1(p)]|f + f).$$

In Propositions 14.2 and 14.5, we have determined the decomposition of $G(\mathbb{Z}_p)\lambda_1(p)G(\mathbb{Z}_p)$ into single $G(\mathbb{Z}_p)$-cosets. Since the single coset representatives obtained there lie in $G(\mathbb{Z}[\frac{1}{p}])$, they serve as representatives for the single $G(\mathbb{Z})$-cosets in $G(\mathbb{Z})\lambda_1(p)G(\mathbb{Z})$ in view of (15.1) and (15.2). Moreover, the proofs of the propositions furnish us with a decomposition

$$G(\mathbb{Z})\lambda_1(p)G(\mathbb{Z}) = \bigcup_i U(\mathbb{Z})u_i l_i G(\mathbb{Z}),$$

and provide us with the number n_i of single $G(\mathbb{Z})$-cosets in each $U(\mathbb{Z})u_i l_i G(\mathbb{Z})$. Indeed, the representatives for the $(U(\mathbb{Z}), G(\mathbb{Z}))$-cosets in $G(\mathbb{Z})\lambda_1(p)G(\mathbb{Z})$ can be taken to be:

- $\begin{pmatrix} p & \\ & p \end{pmatrix}$;
- a set S_2 of representatives for the $p+1$ single cosets in $GL_2(\mathbb{Z})\begin{pmatrix} p & \\ & 1 \end{pmatrix}GL_2(\mathbb{Z})$;
- a set S_1 of representatives for the $p+1$ single cosets in $GL_2(\mathbb{Z})\begin{pmatrix} 1 & \\ & p^{-1} \end{pmatrix}GL_2(\mathbb{Z})$;
- $\begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}$;
- a set S^* of representatives for the $p-1$ non-trivial cosets of $U(\mathbb{Z})$ in U^*;
- for each of the $p+1$ singular lines $m \subset \frac{1}{p}V_1(\mathbb{Z})/V_1(\mathbb{Z})$, a set S_m of representatives for the $p-1$ non-trivial cosets of U^* in $U(m)$.

Here, the groups U^* and $U(m)$ over $\mathbb{Z}$ are defined analogously as the corresponding groups over $\mathbb{Z}_p$ introduced before Lemma 14.3 and in (14.4).

From this, and Proposition 15.5, we deduce that for $i \geq 1$,

$$c_i(t|f) = p^{2k-1}c_{i-1}(f) + p^{k-1}\sum_{l\in S_2} c_{l^{-1}\chi_i}(f) + c_i(f) + p^{-k}\sum_{l\in S_1} c_{l^{-1}\chi_i}(f) + p^{1-2k}c_{i+1}(f).$$

It remains to see that there are bijections

$$\{\text{cubic rings } A_i \subset B \subset A_{i-1}\} \longleftrightarrow \{l \in S_2 : l^{-1}\chi_i \text{ is trivial on } U(\mathbb{Z})\},$$

and

$$\{\text{cubic rings } A_{i+1} \subset B \subset A_i\} \longleftrightarrow \{l \in S_1 : l^{-1}\chi_i \text{ is trivial on } U(\mathbb{Z})\}.$$

In view of the correspondence between binary cubic forms over $\mathbb{Z}$ and the characters of $U(\mathbb{R})$ which are trivial on $U(\mathbb{Z})$, the required bijections follow from Proposition 5.7.

When $i = 0$, the only added subtlety is in the determination of the coefficient

$$\frac{1}{p^3}(1 + \#S^* + p\sum_m \sum_{u\in S_m} \chi(u))$$

of $c_A(f)$. Regarding χ as a character on $U(m)/U^*$, we see that

$$\sum_{u\in S_m} \chi(u) = \begin{cases} -1, & \text{if the restriction of } \chi \text{ to } U(m) \text{ is non-trivial;} \\ p-1, & \text{otherwise.} \end{cases}$$

On the other hand, if m corresponds to the line $l = (x_0 : y_0)$ in $\mathbb{P}^1(\mathbb{F}_p)$, and χ to the binary cubic form $q(x,y)$, then the triviality of the restriction of χ on $U(m)$ is equivalent to $q(x_0, y_0) = 0 (\mathrm{mod}\ p)$. Hence by Proposition 5.4, the number of m for which χ is trivial on $U(m)$ is equal to the number of rings B such that $A_1 \subset B \subset A$, and the proposition is proved. □

Corollary 15.7. *Assume that A/pA is a field. Then*

$$c_A(\chi_1|f) = -\frac{1}{p}c_A(f) + p^{1-2k}c_{A_1}(f).$$

Furthermore, for every cubic ring B, such that $A_2 \subset B \subset A_1$,

$$c_B(\chi_1|f) = p^{k-1}c_{A_1}(f) + p^{-k}c_{A_2}(f) + p^{1-2k}c_{B_1}(f).$$

Proof. This follows from the previous proposition and Corollary 5.6. □

Similarly, using Propositions 14.6, 14.9, 14.10, 14.12, 14.14 and 14.15, we can compute $c_i(\chi_2|f)$ in terms of the Fourier coefficients of f. We omit the proof and simply state the result:

Proposition 15.8. *If $i \geq 2$, then*

$$\begin{aligned} c_i(\chi_2|f) =& c_i(\chi_1|f) + p^{3k-2} \sum_{A_{i-1}\subset B\subset A_{i-2}} c_B(f) + p^{-1} \sum_{A_{i+1}\subset C\subset A_{i-1}} c_C(f) + \\ & p^{-1}c_i(f) + p^{1-3k} \sum_{A_{i+2}\subset B\subset A_{i+1}} c_B(f). \end{aligned}$$

Here each C is a ring such that $C/A_{i+1} \cong \mathbb{Z}/p^2\mathbb{Z}$.

When $i = 0$ or 1, the formulas expressing $c_i(\chi_2|f)$ in terms of the Fourier coefficients of f are more complicated. We highlight only the case when A/pA is a field:

Corollary 15.9. *Assume that A/pA a field. Then*

$$\begin{cases} c_0(\chi_2|f) = (\frac{1}{p}+\frac{1}{p^2})c_0(f) - p^{-2k}c_1(f) + p^{1-3k}\sum_{A_2\subset B\subset A_1} c_B(f) \\ c_1(\chi_2|f) = -p^{2k-2}c_0(f) + (1+\frac{1}{p})c_1(f) + p^{1-2k}c_2(f) + \sum_{A_3\subset B\subset A_2} p^{1-3k}c_B(f). \end{cases}$$

Proof. The formula for $c_1(\chi_2|f)$ follows from Proposition 15.5 and the results of the previous section, as soon as we show that there are no rings $A_2 \subset C \subset A$ such that $C/A_2 \cong \mathbb{Z}/p^2\mathbb{Z}$. Let $B = C \cap A_1$. Then $A_2 \subset B \subset A_1$ and B is contained in C with index p. By Corollary 5.6, C must be A_1. This is a contradition. Similar, even easier, considerations can be used to deal with $c_A(\chi_2|f)$. The corollary is proved. $\square$

16. Gorenstein Coefficients

Let $f \in M_k$ be a non-zero eigenform for the spherical Hecke algebra. In this section, we show that the Fourier coefficient $c_A(f)$ is non-zero for some Gorenstein ring A. Together with Proposition 8.4, this implies that if f is a non-zero cuspidal Hecke eigenform, then f is completely determined by its Hecke eigenvalues and its Gorenstein coefficients. This is the analog of the classical result that if $f(z) = \sum_{n\geq 1} a_n(f)e^{2\pi inz}$ is a non-zero cuspidal Hecke eigenform, then $a_1(f) \neq 0$. The proof of this statement is based on the formulas for the action of Hecke operators on $a_n(f)$. Indeed, the formula for each local operator T_p is

$$a_n(T_p|f) = a_{n/p}(f) + p^{-k}a_{pn}(f)$$

which shows that the coefficents $a_n(f)$ can be recovered from $a_1(f)$ and the Hecke eigenvalues. Our proof is based the same idea, exploiting the formulas for the Hecke operators χ_1 and χ_2 obtained in the previous Section.

Let A be a totally real cubic ring so that the cubic algebra $E = A \otimes \mathbb{Q}$ is étale. For every prime integer p, we define an equivalence relation between cubic rings in E as follows: $A \sim_p B$ if the intersection of A and B is contained in both A and B with index a power of p. This is equivalent to saying that $A \otimes \mathbb{Z}_l = B \otimes \mathbb{Z}_l$ as subrings of $E \otimes \mathbb{Q}_l$ for all $l \neq p$.

Proposition 16.1. *If the Fourier coefficients of f vanish for all rings of p-depth ≤ 1 in a $\sim_p$ equivalence class, then they vanish for all rings in the $\sim_p$ equivalence class.*

Proof. The proof is by induction on the depth. Assume that we have proved the vanishing for all rings of p-depth $\leq i$. Let $A_{i+1} = \mathbb{Z} + p^{i+1}A$ be a ring of p-depth $i+1$. Acting by the Hecke operator χ_1 on $c_{A_i}(f)$, and using the induction assumption, we obtain

$$0 = p^{k-1} \sum_{A_i \subset B \subset A_{i-1}} c_B(f) + p^{-k} \sum_{A_{i+1} \subset B \subset A_i} c_B(f) + p^{1-2k} c_{A_{i+1}}(f).$$

Thus, to show that $c_{A_{i+1}}(f) = 0$ we need to show that the first two summands on the right hand side are 0. Vanishing of the first sum is proved in the following lemma.

Lemma 16.2. *Assume that Fourier coefficents of f vanish for all rings of p-depth $\leq i$ in the $\sim_p$ class of A. Then*

$$\sum_{A_i \subset B \subset A_{i-1}} c_B(f) = 0.$$

Proof. Since $i \geq 1$, the ring A_{i-1} exists. Acting by the Hecke operator χ_1 on $c_{A_{i-1}}$, and using the assumption of the lemma, it follows that

$$0 = p^{k-1} \sum_{A_{i-1} \subset B \subset A_{i-2}} c_B(f) + p^{-k} \sum_{A_i \subset B \subset A_{i-1}} c_B(f).$$

By Proposition 5.5 the p-depth of each B in the first sum lies between $i-3$ and i. In particular, $c_B(f) = 0$ by the assumption of the lemma. The lemma follows. □

It remains to show vanishing of the second sum. This is much harder, however, and is accomplished in the following lemma.

Lemma 16.3. *Assume that Fourier coefficents of f vanish for all rings of p-depth $\leq i$ in the $\sim_p$ class of A. Then*

$$\sum_{A_{i+1}\subset B\subset A_i} c_B(f) = 0.$$

Proof. The idea this time is to use the operator χ_2 on $c_{A_{i-1}}$. After taking into account the induction assumption, we obtain

$$0 = p^{-1} \sum_{A_i\subset C\subset A_{i-2}} c_C(f) + p^{1-3k} \sum_{A_{i+1}\subset B\subset A_i} c_B(f),$$

where, as usual, the first sum is taken over all C such that C/A_i is cyclic of order p^2. Obviously, to prove the Lemma, we need to show that the first summand is 0. By intersecting each C with A_{i-1}, the first summand can be rewritten as

$$\sum_{A_i\subset C\subset A_{i-2}} c_B(f) = \sum_{A_i\subset B\subset A_{i-1}} \sum_{B\subset_p C} c_C(f) - (p+1)c_{A_{i-1}}(f).$$

Moreover, $c_{A_{i-1}}(f) = 0$, by the induction assumption, so it remains to show that the double sum on the right hand side is zero. By Proposition 5.5 the p-depth of each B between A_i and A_{i-1} lies between $i-2$ and $i+1$. Next assume that B has positive p-depth. Then, the ring B_{-1} such that $B = \mathbb{Z} + pB_{-1}$ exists, and has p-depth at most i. Applying the Hecke operator χ_1 to $c_{B_{-1}}(f)$, and using the induction assumption, it follows that

$$0 = p^{k-1} \sum_{B_{-1}\subset_p C} c_C(f) + p^{-k} \sum_{C\subset_p B_{-1}} c_C(f) + p^{1-2k} c_B(f).$$

We claim that the first sum on the right hand side is zero. By Proposition 5.5, any ring C containing B_{-1} with index p has p depth less or equal to $i+1$. It can be $i+1$ only when B_{-1} has p-depth i. In that case we can apply Lemma 16.2 to B_{-1} to show that the sum is zero. Otherwise, each individual term $c_C(f)$ is zero. In any case, the claim follows.

In the second sum we can replace $C \subset_p B_{-1}$ by $B \subset_p C$ since the rings C contained in B_{-1} with index p are precisely the rings containing B with index p. Summing the above equation over all B between A_i and A_{i-1} such that p-depth of B is positive, we obtain

$$0 = p^{-k} \sum_{A_i\subset B_+\subset A_{i-1}} \sum_{B\subset_p C} c_C(f) + p^{1-2k} \sum_{A_i\subset B_+\subset A_{i-1}} c_B(f),$$

where the subscript $+$ denotes that the sum is take over rings B with positive p-depth. However if B has p-depth 0, and $B \subset_p C$, then C has p-depth less then or equal to one. Thus $c_B(f) = c_C(f) = 0$ by the assumption of the proposition, and this implies that we can remove the subscript $+$ in the previous equation. Now, since the second sum is zero by Lemma 16.2, the first sum has to be zero as well. The lemma, and Proposition 16.1 follow. □

□

We shall now show that the vanishing of the Fourier coefficents of f for all rings of p-depth zero in a given $\sim_p$ class implies vanishing of all Fourier coefficents in the $\sim_p$ class. In view of 16.1 it suffices to show that $c_{A_1}(f) =$. The idea is similar to the one used in the proof of 16.1. The nessecary modifications of the proof will be based on the following proposition.

Proposition 16.4. *Each $\sim_p$ equivalence class of cubic rings contains a unique maximal element $A_{\max}$.*

Proof. We will define $A_{\max}$ in $A \otimes \mathbb{Q}$ by specifying its localizations in $A \otimes \mathbb{Q}_l$ for all primes l. For l not equal to p, we insist that the localization is equal to $A \otimes \mathbb{Z}_l$, for any ring A in the equivalence class. For $l = p$, we insist that the localization is equal to the integral closure of $\mathbb{Z}_p$ in $A \otimes \mathbb{Q}_p$. Since $A \otimes \mathbb{Q}_p$ is étale over $\mathbb{Q}_p$, the integral closure is free of rank 3 over $\mathbb{Z}_p$ and is maximal for this property. The above construction shows that $A_{\max}$ is the unique ring in the $\sim_p$ equivalence class which minimizes the power of p in the discriminant and thus it is the unique maximal element in he $\sim_p$ equivalence class of A. □

Let $f \in M_k$ be a Hecke eigenform. We shall now show that the vanishing of the Fourier coefficients of f for all rings of p-depth 0 in a given equivalence class implies the vanishing of all coefficients in the class. We first show the following:

Proposition 16.5. *Let A be a p-depth 0 ring in the $\sim_p$ equivalence class of $A_{\max}$. If the Fourier coefficients of f for all rings of p-depth 0 in the equivalence class of $A_{\max}$ vanish, then $c_{A_i}(f) = 0$ for $i = 1, 2$.*

The idea is to prove this statement for $A = A_{\max}$, and then use induction on the index of A in $A_{\max}$. Since arguments are repetitive, we will do the proof for $A = A_{\max}$ and the induction step at the same time.

Lemma 16.6. *Let B be a ring containing A with index p or contained in A with index p. Then $c_B(f) = 0$.*

Proof. If $A = A_{\max}$, then there are no rings containing it, and every ring B between A and A_1 has depth 0. Otherwise B_{-1} would exist and would contain $A_{\max}$ with index p, which is a contradiction.

(*induction step*) A ring B containing A with index p can have p-depth one. In that case, $B = \mathbb{Z} + pB_0$ where B_0 has p-depth 0. Since the index of B_0 in $A_{\max}$ is smaller then the index of A, the induction hypothesis implies that $c_B(f) = 0$. Similarly, any ring B between A_1 and A has p-depth at most 2. Hence, either B has p-depth 0 and $c_B(f) = 0$ by the assumption, or $B = \mathbb{Z} + p^i B_0$ for some $1 \le i \le 2$ in which case $c_B(f) = 0$ by the induction assumption since the index of B_0 in $A_{\max}$ is less then that of A. □

Now Proposition 15.6 (acting by the Hecke operator χ_1 on $c_A(f)$) and Lemma 16.6 imply that

$$c_{A_1}(f) = 0. \tag{16.7}$$

Lemma 16.8.

$$\sum_{A_1 \subset C} c_C(f) = 0$$

where the sum is taken over all rings C such that $C/A_1 = \mathbb{Z}/p^2\mathbb{Z}$.

Proof. Since $A/A_1 = (\mathbb{Z}/p\mathbb{Z})^2$, C is not contained in A. Thus, if $A = A_{\max}$, there are no such rings C.

(*induction step*) Let $B = A \cap C$. Then $A_1 \subset B \subset A$. By Proposition 5.5, the p-depth of B is at most 2. Since $B \subset_p C$, the p-depth of C could be at most 3, which happens only if B has p-depth 2. Thus, B has p-depth less then two, then C has p-depth less then 3, and the induction hypothesis implies that $c_C(f) = 0$, just as in Lemma 16.6. Now assume that B has p-depth 2. Then the ring B_{-1} exists, and we can apply the operator χ_1 to $c_{B_{-1}}(f)$. By the induction hypothesis the formula reduces to

$$p^{k-1} \sum_{B_{-1} \subset_p C} c_C(f) + p^{-k} \sum_{B \subset_p C} c_C(f).$$

The rings C in the first sum have p-depths less then 3. Therefore the coefficients $c_C(f)$ vanish by the induction hypothesis, again just as in Lemma 16.6. The second sum therefore also vanishes, and the lemma follows (the sum also includes the term $c_A(f)$ which is 0). □

Lemma 16.9.

$$\sum_{A_2 \subset B \subset A_1} c_B(f) = 0$$

Proof. Consider the action of the Hecke operator χ_2 on $c_A(f)$. Using the formula given in Proposition 15.5, one obtains the desired result from (16.7), Lemma 16.6 and Lemma 16.8. □

Now Proposition 15.6 (acting by the Hecke operator χ_1 on $c_{A_1}(f)$) and Lemma 16.9 imply that

$$c_{A_2}(f) = 0. \tag{16.10}$$

In view of 16.7 and 16.10, Proposition 16.5 is proved completely. Moreover, Propositions 16.5 and 16.1 imply the following corollary.

Corollary 16.11. *If the Fourier coefficents of f vanish for all rings of p-depth 0 in a $\sim_p$ equivalence class, then they vanish for all rings in the $\sim_p$ equivalence class.*

We are now ready to prove that cuspidal Hecke eigenforms are determined by their Gorenstein coefficients and Hecke eigenvalues.

Theorem 16.12. *Let f be a Hecke eigenform. If $c_A(f) = 0$ for all Gorenstein rings A, then all the Fourier coefficients of f vanish. In particular, if f is a non-zero cuspidal Hecke eigenform, then f has a non-zero Gorenstein coefficient.*

Proof. Suppose that there exists a totally real cubic ring A such that $c_A(f) \neq 0$. Let $E = A \otimes \mathbb{Q}$ which is a cubic étale algebra over $\mathbb{Q}$.

Pick an ordering of primes $p_1, p_2, \dots$ and let X_k be the set of cubic rings in E with trivial p_l-depth for all $l > k$. Clearly, X_0 is the set of all Gorenstein rings in E and every cubic ring in E is contained in some X_k for a sufficiently large k. Furthermore, X_k is a union of $\sim_{p_k}$ conjugacy classes, and the set of all elements in X_k of p_k-depth 0 is precisely X_{k-1}. Using induction on k, Corollary 16.11 implies that the Fourier coefficients of f vanish for all cubic rings in E. This is a contradiction, and the first statement of the theorem is proved. The second statement follows from this and Proposition 8.4. □

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