Parahorics

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1 Introduction

This is an expository paper on the structure of parahoric subgroups. We will also treat the simpler case of parabolic subgroups to motivate the exposition. All of the difficult results on the determination of these subgroups up to conjugacy are stated without proof. A full treatment can be found in the papers of Chevalley [5], Borel and Tits [3], and Bruhat and Tits [4]. We will be concerned with the internal structure of these subgroups, once representatives for the distinct conjugacy classes have been chosen.

Let A be a complete discrete valuation ring with quotient field k. Let π be a uniformizing parameter in A, and assume that the residue field $\mathfrak{f} = A/\pi A$ is finite of characteristic p. Let $\mathbf{G}$ be a split reductive group of rank ℓ over A, which is absolutely simple and simply-connected. The additive group of k has a locally compact topology coming from the valuation, and the group $\mathbf{G}(k)$ inherits a locally compact topology for which the subgroup $\mathbf{G}(A)$ is compact. In fact, $\mathbf{G}(A)$ is a maximal compact subgroup of $\mathbf{G}(k)$.

Unlike the case of real Lie groups, the maximal compact subgroups of $\mathbf{G}(k)$ do not lie in a single conjugacy class. There are ℓ further conjugacy classes of maximal compact subgroups – the remaining maximal parahoric subgroups of $\mathbf{G}(k)$. Following Bruhat and Tits [4], we construct representatives for the conjugacy classes which stabilize the $\ell + 1$ distinct vertices in an alcove of the building of $\mathbf{G}$ over k. Taking the intersections of these maximal compact subgroups, we obtain representatives of the $2^{\ell+1} - 1$ conjugacy classes of general parahoric subgroups, which stabilize the distinct facets of this alcove.

Bruhat and Tits have shown that each parahoric subgroup P can be identified with the A-valued points of a smooth group scheme $\mathbf{G}_P$ over A with general fiber $\mathbf{G}$ over k [4]. Hence P has a descending filtration $P \triangleright P_1 \triangleright P_2 \triangleright \cdots$ where P_m is the subgroup which is the kernel of the reduction map $\mathbf{G}_P(A) \to \mathbf{G}_P(A/\pi^m A)$. Moy and Prasad [12] have defined a refinement of this filtration

$$P \triangleright P_{1/d} \triangleright P_{2/d} \triangleright \cdots .$$

The integer $d \geq 1$ is the denominator of the coordinates of the barycenter x of the facet fixed by P, and $d = 1$ for $P = \mathbf{G}(A)$. The quotient $(P/P_{1/d}) = \mathbf{L}(\mathfrak{f})$ is the group of points of a split reductive group $\mathbf{L}$ of rank ℓ over the residue field $\mathfrak{f}$, and for $a \geq 1$ the quotients $P_{a/d}/P_{(a+1)/d} = V_a(\mathfrak{f})$ are the group of points of unipotent vector groups V_a over $\mathfrak{f}$. These afford algebraic linear representations of $\mathbf{L}$ over $\mathfrak{f}$ which are periodic with period d: $V_a \cong V_{a+d}$. In the notation of Moy and Prasad, we have $P_{a/d} = \mathbf{G}(k)_{x,a/d}$.

By the internal structure of P we mean the structure of the reductive quotient $\mathbf{L}$ over $\mathfrak{f}$ and the algebraic representations V_a of $\mathbf{L}$ which arise from the subquotients of the Moy-Prasad filtration. Reeder and Yu have recently elucidated this structure using Vinberg's theory of torsion automorphisms [15]. As a result, when the residual characteristic p of $\mathfrak{f}$ is a good prime for $\mathbf{G}$ which does not divide d, they determine the parahorics P where V_a has semi-stable orbits for the action of $\mathbf{L}$, in the sense of geometric invariant theory, as well as the parahorics where the dual representation V_1^* has stable orbits. The existence of stable orbits on V_1^* allows them to construct interesting families of supercuspidal representations $\mathbf{G}(k)$, which are compactly induced from complex characters of $P_{1/d}$ which are trivial on $P_{2/d}$.

We begin this paper with a review of roots and affine roots [2, Ch VI]. We then review what is known about $\mathbf{Z}$ and $\mathbf{Z}/d\mathbf{Z}$ gradings of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. Most of this theory is due to Richardson [16] Vinberg [23]. After this preparation, we define representatives of the conjugacy classes of of parabolic and parahoric subgroups, and discuss their internal structure, when the simple group $\mathbf{G}$ is split over k. In the last three sections, we use results of Kneser and Tate in Galois cohomology to classify the isomorphism classes of simple, simply connected groups $\mathbf{G}$ which are split by a tamely ramified extension of k. Following Bruhat and Tits [4] we then associate to each such group over k an affine diagram ${}^e\mathscr{R}$ with an automorphism F, from which the internal structure of the parahoric subgroups of $\mathbf{G}(k)$ can be determined. We end with a tabulation of the possible pairs $({}^e\mathscr{R}, F)$ and the simple groups $\mathbf{G}$ to which they are associated.

I want to thank Mark Reeder and Jiu-Kang Yu for teaching me this material. I also want to thank JK for preparing the Dynkin diagrams in this paper.

2 Roots and affine roots

In this section, we will review the theory of root systems and affine root systems as presented in Bourbaki [2, Ch VI], giving the relation with the simply-connected, absolutely simple, split group $\mathbf{G}$ over A.

Let $\mathbf{S}$ be a maximal split torus in $\mathbf{G}$, let $X = \mathrm{Hom}(\mathbf{G}_m, \mathbf{S})$ be the cocharacter group of S and let $Y = \mathrm{Hom}(\mathbf{S}, \mathbf{G}_m)$ be its character group. There is a non-degenerate pairing

$$\langle x, y\rangle : X \times Y \to \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m) = \mathbf{Z}$$

Let R be the set of roots of $\mathbf{G}$. This is the finite subset of non-zero elements α in Y which occur in the representation of $\mathbf{S}$ on the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. We have the Cartan decomposition of the Lie algebra (cf. [7]), as a representation of the torus $\mathbf{S}$:

$$\mathfrak{g} = \mathfrak{s} \oplus \sum \mathfrak{g}_\alpha$$

where the Lie algebra $\mathfrak{s}$ of $\mathbf{S}$ gives ℓ copies of the trivial character, and each root space $\mathfrak{g}_\alpha$ is one dimensional, just as in the complex case [18].

Let $\mathbf{Z}(\mathbf{G}) \subset \mathbf{S}$ be the center of $\mathbf{G}$ and $\mathbf{S}^* = \mathbf{S}/\mathbf{Z}(\mathbf{G})$ be the corresponding maximal torus in the adjoint group $\mathbf{G}^* = \mathbf{G}/\mathbf{Z}(\mathbf{G})$. The character group Y^* of $\mathbf{S}^*$ is the $\mathbf{Z}$-submodule of Y spanned by the roots, and the cocharacter group X^* of $\mathbf{S}^*$ is the subgroup of η in $X \otimes \mathbf{Q}$ which take integral values on R. The quotient group Y/Y^* is the Cartier dual of the center $\mathbf{Z}(\mathbf{G})$, which is of multiplicative type, and we have a duality of finite abelian groups

$$X^*/X \times Y/Y^* \to \mathbf{Q}/\mathbf{Z}.$$

There is also a set of coroots $R^\vee$ inside of X, defined using the theory of SL_2, which is in bijection with the set of roots. Each root α gives a simple reflections s_α of X and a simple reflection $s_{\alpha^\vee}$ of Y, defined by

$$s_\alpha(x) = x - \langle \alpha, x\rangle \alpha^\vee \quad s_{\alpha^\vee}(y) = y - \langle \alpha^\vee, y\rangle \alpha$$

These reflections preserve the sets of co-roots and roots respectively. The Weyl group $W = W(R)$ is the subgroup of $\mathrm{Aut}(R)$ that they generate. The pairing $X \times Y \to \mathbf{Z}$ is W-invariant, and by inspection the group W acts trivially on the quotients X^*/X and Y/Y^*.

In Bourbaki, root systems are discussed independent of algebraic groups. Their notation is a bit different than ours. Here is the translation (cf. [2, Ch VI 1.9]):

$$Q(R) = Y^* \;\; P(R) = Y \;\; Q(R^\vee) = X \;\; P(R^\vee) = X^*.$$

Hence

$$P/Q = Y/Y^* \quad P^\vee/Q^\vee = X^*/X.$$

Since our group $\mathbf{G}$ is assumed to be absolutely simple and split, R is an irreducible (reduced) root system of type $A_n\ n \geq 1, B_n\ n \geq 2, C_n\ n \geq 3, D_n\ n \geq 4, G_2, F_4, E_6, E_7, or E_8$.

Let $\mathbf{B}$ be a fixed Borel subgroup of $\mathbf{G}$ containing $\mathbf{S}$. Then $\mathbf{B}$ determines a subset of positive roots R^+. The positive roots are those which occur in the representation of $\mathbf{S}$ on the Lie algebra $\mathfrak{b}$ of $\mathbf{B}$. The choice of $\mathbf{B}$ also determines a root basis $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, and every positive root has a basis expansion $\alpha = \sum m_i(\alpha).\alpha_i$ with all coefficients $m_i \geq 0$. The Weyl group is a finite Coxeter group, generated by the simple reflections s_{α_i}.

The finite group $\mathrm{Aut}(R)/W(R) = \mathrm{Aut}(R, \Delta)$ is isomorphic to the group of automorphisms of the Dynkin diagram of R, which is a symmetric group S_1, S_2, S_3, with the latter case occurring only when R is of type D_4. This group acts faithfully on the finite abelian quotient groups X^*/X and Y/Y^*, and the natural pairing $X^*/X \times Y/Y^* \to \mathbf{Q}/\mathbf{Z}$ is non-degenerate and $\mathrm{Aut}(R, \Delta)$-invariant. The full group $\mathrm{Aut}(R)$ is isomorphic to the semi-direct product $W.\mathrm{Aut}(R, \Delta)$, and the choice of a Borel gives the splitting.

The roots α give linear functionals on the real vector space $X \otimes \mathbf{R}$. A fundamental domain for the action of W on $X \otimes \mathbf{R}$ is given by the closed Weyl chamber where $\alpha_i(x) \geq 0$ for $i = 1, 2, \ldots, \ell$. We equip $X \otimes \mathbf{R}$ with the sup norm $|x| = Max\{|\alpha_i(x)|\}$ and let C be the compact spherical alcove which is the intersection of the closed Weyl chamber with the unit sphere $\{x : |x| = 1\}$. A facet F of C consists of the points x where a fixed proper subset of the α_i vanish, and the remaining subset Σ take positive values. The barycenter x of F is the point where the remaining subset Σ of the basic roots all satisfy $\alpha_i(x) = 1$. Since all roots take integral values on the barycenter, x is a cocharacter of the adjoint torus $\mathbf{S}^*$, and gives a homomorphism

$$x = \eta : \mathbf{G}_m \to \mathbf{S}^* \to \mathbf{G}^* \to \mathrm{Aut}(G).$$

For example, if x is the barycenter of the interior of the alcove C, then $\alpha_i(x) = 1$ for all simple roots, and $x = \eta$ is the co-character $\rho^\vee$ of $\mathbf{S}^*$ given by half the sum of the positive co-roots.

Let

$$\beta = \sum m_i \alpha_i$$

be the highest root. This is by definition the highest weight for $\mathbf{S}$ in the adjoint representation of $\mathbf{G}$. The multiplicities $m_i = m_i(\beta)$ in the basis expansion satisfy $m_i \geq 1$ for all i, and $\sum m_i = h - 1$, where h is the Coxeter number of $\mathbf{G}$. These integers, which are critical in what follows, are tabulated for the different root systems in [2]. If α is any positive root, then the multiplicity $m_i(\alpha)$ of α_i in the basis expansion of α satisfies $0 \leq m_i(\alpha) \leq m_i(\beta)$.

We now define the affine root system associated to the split group $\mathbf{G}$. Consider the set of affine linear functionals $\psi = \psi(\alpha, n) = \alpha + n$ on $X \otimes \mathbf{R}$, with α in R and n in $\mathbf{Z}$. We say that α is the gradient of the affine root $\psi(\alpha, n)$. The affine roots $\psi_i = \psi(\alpha_i, 0) = \alpha_i$ together with $\psi_0 = \psi(-\beta, 1) = 1 - \beta$ form a basis for the affine root system, and the reflections in the $\ell + 1$ affine hyperplanes where $\psi_i = 0$ generate an affine Coxeter group. Putting $m_0 = 1$, we have the relation

$$\sum m_i \psi_i = 1$$

The (closure of the) fundamental alcove in $X \otimes \mathbf{R}$ is the compact region C where the $\ell + 1$ basic affine roots satisfy $\psi_i(x) \geq 0$. Since the basic roots $\alpha_i = \psi_i$ take non-negative values on C, the same is true for all of the positive roots. Since the highest root β satisfies $1 - \beta(x) = \psi_0(x) \geq 0$ on C, all positive roots take values $0 \leq \alpha(x) \leq 1$ on C.

A facet F of C consists of the points where a proper subset of the basic affine roots take the value 0 and the non-empty complementary subset of basic affine roots take non-zero values. The barycenter x of the facet F is the point of F where the remaining basic affine roots in the complementary subset Σ of $\{0, 1, \ldots, \ell\}$ take the same (rational) value. This value is completely determined by the relation $\sum m_i \psi_i = 1$. It is equal to $1/d$ with $d = d(F)$ is the sum of the multiplicities m_i over the subset Σ.

For example, if the complement Σ consists of a single affine root ψ_j, then the facet is a vertex x of C and $\psi_j(x) = 1/m_j$. At the other extreme, if the facet F is the interior of C, so the complement Σ contains of all the basic affine roots, then the common value $\psi_i(x)$ at the barycenter is equal to $1/h$. In general, we have $1 \leq d \leq h$.

For a general barycenter x, the element $d.x$ of $X \otimes \mathbf{R}$ takes integral values on all of the roots, so is a cocharacter of the adjoint torus and gives a homomorphism

$$d.x = \eta : \mathbf{G}_m \to \mathbf{S}^* \to \mathbf{G}^* \to \mathrm{Aut}(G).$$

3 $\mathbf{Z}$ and $\mathbf{Z}/d\mathbf{Z}$-gradings

In this section, we consider the gradings of the Lie algebra which are induced by a homomorphism

$$\eta : \mathbf{G}_m \to \mathbf{S}^* \to \mathbf{G}^* \to \mathrm{Aut}(\mathbf{G})$$

as well as by the restriction of η to the subgroup μ_d of d-torsion in $\mathbf{G}_m$.

The $\mathbf{G}_m$ action gives a $\mathbf{Z}$-grading of the Lie algebra

$$\mathfrak{g} = \sum_{a \in \mathbf{Z}} \mathfrak{g}(a),$$

where $\mathfrak{g}(a)$ is the subspace where t in $\mathbf{G}_m$ acts by multiplication by t^a. The reductive subgroup $\mathbf{G}(0)$ of $\mathbf{G}$ which is fixed by $\mathbf{G}_m$ contains $\mathbf{S}$ as a maximal split torus. It acts linearly on each subspace $\mathfrak{g}(a)$. For example, $\mathfrak{g}(a) = \mathfrak{g}(0)$ is the adjoint representation of $\mathbf{G}(0)$.

When a is non-zero, the subspace $\mathfrak{g}(a)$ decomposes as a representation of $\mathbf{S}$ as the direct sum of one dimensional root spaces:

$$\mathfrak{g}(a) = \sum_{\langle \eta, \alpha \rangle = a} \mathfrak{g}_\alpha.$$

In particular, when $a \neq 0$, the eigenspace $\mathfrak{g}(a)$ consists entirely of nilpotent elements. When $a = 0$, and one must add the ℓ dimensional space $\mathfrak{s}$.

Proposition 3.1 *If $a \neq 0$, then every $\mathbf{G}(0)$ invariant polynomial on $\mathfrak{g}(a)$ is a constant.*

This was proved by Richardson[16] and Vinberg[23]. It follows from the fact that over an algebraically closed field, $\mathbf{G}_0$ has an open dense orbit on $\mathfrak{g}(a)$.

Now consider the restriction of the homomorphism η to μ_d. This gives rise to a $\mathbf{Z}/d\mathbf{Z}$-grading of $\mathfrak{g}$

$$\mathfrak{g} = \sum_{a \in \mathbf{Z}/d\mathbf{Z}} \mathfrak{g}(a)$$

where $\mathfrak{g}(a)$ is the subspace of $\mathfrak{g}$ where ζ in μ_d acts by multiplication by ζ^a [17]. The reductive subgroup $\mathbf{G}_0$ of $\mathbf{G}$ which is fixed by μ_d contains $\mathbf{S}$ as a maximal split torus and acts on each subspace $\mathfrak{g}(a)$. For example, $\mathfrak{g}(0)$ is the adjoint representation of $\mathbf{G}_0$.

When a is non-zero (modulo d), the subspace $\mathfrak{g}(a)$ decomposes as a representation of $\mathbf{S}$ as the direct sum of one dimensional root spaces:

$$\mathfrak{g}(a) = \sum_{\langle \eta, \alpha \rangle \equiv a} \mathfrak{g}_\alpha.$$

When $a = 0$, and one must add the ℓ dimensional space $\mathfrak{s}$.

These representations were studied over $\mathfrak{f} = \mathbf{C}$ by Kostant and Rallis when $d = 2$, [9], and by Vinberg [23] and his school [13] when $d > 2$. Levy [11] considered the general situation where the characteristic of $\mathfrak{f}$ is a good prime which does not divide d. Under these assumptions, one can use the existence of Cartan subspaces and the theory of complex reflection groups to prove the following generalization of Chevalley's theorem [6] (which is the case $d = 1$).

Proposition 3.2 *The $\mathbf{G}_0$-invariant polynomials on $\mathfrak{g}(a)$ form a polynomial ring, with $r(a)$ independent generators.*

As a corollary, the geometric quotient $\mathfrak{g}(a)/\mathbf{G}_0$ is affine space of dimension $r(a)$ over $\mathfrak{f}$. The dimensions $0 \leq r(a) \leq \ell$ are interesting invariants of the μ_d action; we have $r(a) > 0$ if and only if $\mathfrak{g}(a)$ contains semi-simple elements.

4 The filtration of parabolics and parahorics (split case)

We first define the parabolic subgroups of $\mathbf{G}$, which are algebraic subgroups $\mathbf{P}$ over A with $\mathbf{G}/\mathbf{P}$ projective. More precisely, we define the $2^\ell - 1$ parabolic subgroups which contain the fixed Borel $\mathbf{B}$, as these represent the distinct conjugacy classes. They correspond bijectively to the barycenters x of facets F of the compact spherical alcove C. We recall that C is the intersection of the closed Weyl chamber defined by the inequalities $\alpha_i(x) \geq 0$ with the unit sphere $\{x : |x| = 1\}$, and a barycenter x satisfies $\alpha_i(x) = 1$ for a non-empty subset Σ of Δ and $\alpha_j(x) = 0$ for the basic roots in $\Delta - \Sigma$. Hence all roots α take integral values on x and $x = \eta$ is a cocharacter of the adjoint torus. Hence η gives a homomorphism

$$x = \eta : \mathbf{G}_m \to \mathbf{S}^* \to \mathbf{G}^* \to \mathrm{Aut}(G)$$

and by results of the previous section, an integral grading $\mathfrak{g} = \sum_{a \in \mathbf{Z}} \mathfrak{g}(a)$.

For each root α, we let $\mathbf{U}_\alpha$ be the corresponding root group (isomorphic to $\mathbf{G}_a$) in $\mathbf{G}$. Then the parabolic $\mathbf{P} = \mathbf{P}_x$ is generated by the torus $\mathbf{S}$ and the root groups $\mathbf{U}_\alpha$, for those roots α which satisfy $\langle \eta, \alpha \rangle \geq 0$. We define a terminating descending filtration

$$\mathbf{P} \triangleright \mathbf{P}_1 \triangleright \mathbf{P}_2 \triangleright \cdots \mathbf{P}_m \triangleright 1$$

where for $a \geq 1$, $\mathbf{P}_a$ is the unipotent subgroup of $\mathbf{P}$ generated the root groups $\mathbf{U}_\alpha$ with $\langle \eta, \alpha \rangle \geq a$. The integer $m = \langle \eta, \beta \rangle$, where β is the highest root. The subgroup $\mathbf{P}_1$ is the unipotent radical of $\mathbf{P}$, and for $a \geq 1$ the quotient $V_a = \mathbf{P}_a/\mathbf{P}_{a+1}$ is a vector group, which is a direct sum of the root spaces $\mathfrak{g}_\alpha$ for those roots with $\langle \eta, \alpha \rangle = a$. Hence V_a is isomorphic to $\mathfrak{g}(a)$ as a representation of $\mathbf{S}$.

The group $\mathbf{P}/\mathbf{P}_1 = \mathbf{L}$ is the Levi quotient of $\mathbf{P}$, which is reductive of rank ℓ with maximal split torus $\mathbf{S}$. From the definition, it is easy to see that the elements of $\Delta - \Sigma$ give a root basis for $\mathbf{L}$. Hence $\mathbf{L}$ has the same root datum as the subgroup $\mathbf{G}_0$ coming from the grading, and the two reductive groups are isomorphic. One can then use the exponential map (cf. [7]) to prove the following

Proposition 4.1 *The representation V_a of $\mathbf{L}$ is isomorphic to the representation $\mathfrak{g}(a)$ of $\mathbf{G}_0$.*

It follows that $\mathbf{L}$ has no polynomial invariants on V_a, other than constants. The open orbit over a separably closed field shows that each V_a is a prehomogeneous vector space. Since the weights of V_a are all roots of $\mathbf{G}$, the representations which occur are very restricted. In particular, when $\mathbf{G}$ is simply laced, the abelianization V_1 of the unipotent radical $\mathbf{P}_1$ is the direct sum of $\#\Sigma$ irreducible minuscule representations of $\mathbf{L}$, with lowest weights the roots in Σ and distinct central characters.[1].

We now give the analogous definition and results for the parahoric subgroups which contain a fixed Iwahori (= the elements of $\mathbf{G}(A)$ which reduce to the fixed Borel in $\mathbf{G}(\mathfrak{f})$). Equivalently, we will define the parahorics subgroups which contain a fixed maximal pro-p-subgroup of $\mathbf{G}(k)$ (= the elements of $\mathbf{G}(A)$ which reduce to the unipotent radical of a fixed Borel in $\mathbf{G}(\mathfrak{f})$). Recall the closed alcove C in $X \otimes \mathbf{R}$ which is defined by the inequalities $\psi_i(x) \geq 0$. Let x be the barycenter of a facet F of C. We define the parahoric subgroup $P = P_x$ fixing the facet F, as well as the Moy-Prasad filtration $P_{x,a/d}$ associated to the barycenter x of F with denominator d, as follows. Fix a Chevalley structure on $\mathbf{G}$ over A, consisting of the split maximal torus $\mathbf{S}$ and an isomorphism $e_\alpha : \mathbf{G}_a \to \mathbf{U}_\alpha$ over A for every root α, where $\mathbf{U}_\alpha$ is the corresponding root group. Associated to each affine root $\psi(\alpha, n)$ we define the subgroup $U_\psi = e_\alpha(\pi^n A)$ of $\mathbf{U}_\alpha(k)$. Then $P = P_x$ is the subgroup of $\mathbf{G}(k)$ which is generated by $\mathbf{S}(A)$ and the subgroups U_ψ, for the affine roots which satisfy $\psi(x) \geq 0$.

For example, if x is the vertex of C where $\psi_0 = 1 - \beta$ takes the value 1, then $\alpha(x) = 0$ for all roots of $\mathbf{G}$, and $P = \mathbf{G}(A)$. At the other extreme, if x is the barycenter of the interior of C, then $0 < \alpha(x) < 1$ for all positive roots. Hence P is the subgroup of $\mathbf{G}(A)$ generated by $\mathbf{S}(A)$, $\mathbf{U}_\alpha(A)$ for positive roots, and $\mathbf{U}_\alpha(\pi A)$ for negative roots. This is just the Iwahori subgroup which reduces to the fixed Borel subgroup of $\mathbf{G}$ modulo π. Finally, if x is a barycenter where $\psi_0(x) = 1 - \beta(x) = 0$, then the parahoric P_x contains the unipotent subgroup $\mathbf{U}_{-\beta}(\pi^{-1}A)$. In particular, P is not contained in $\mathbf{G}(A)$.

To define the Moy-Prasad filtration $P_{a/d} = \mathbf{G}(k)_{x,a/d}$ where d is the denominator of x, we let S_n be the subgroup of the torus $\mathbf{S}(A)$ which is the kernel of the reduction map $\mathbf{S}(A) \to \mathbf{S}(A/\pi^n A)$. Then $P_{a/d}$ is generated by S_n with $n \geq a/d$ and the subgroups U_ψ, for the affine roots which satisfy $\psi(x) \geq a/d$. The quotient $\mathbf{L} = P/P_{1/d}$ is reductive and contains the maximal torus $\mathbf{S}$ over $A/\pi A = \mathfrak{f}$. A basis for its root system is given by the gradients of the basic affine roots ψ_i which vanish at x.

When $a \geq 1$ is not divisible by d, the subquotient $V_a = P_{a/d}/P_{(a+1)/d}$ is the direct sum of one dimensional root spaces $\mathfrak{g}_\alpha$ over $\mathfrak{f}$ where $\alpha(x) + n = a/d$ for some integer n. This determines its structure as a representation of $\mathbf{S}$, and constrains its structure as a representation of $\mathbf{L}$. When a is divisible by d, so $a/d = n$ is an integer, one has to add the ℓ trivial root spaces coming from the Lie algebra of S over $\mathfrak{f}$.

Lemma 4.2 *The root spaces which occur in the representation of* $\mathbf{S}$ *on the subquotient* V_a *of* $P = P_x$ *are precisely those which occur in the component* $\mathfrak{g}(a)$ *of the* $\mathbf{Z}/d\mathbf{Z}$*-grading of* $\mathfrak{g}$ *over* $\mathfrak{f}$*, which comes from the restriction to* μ_d *of the co-character* $\eta = d.x : \mathbf{G}_m \to \mathbf{S}^{ad}$*. Hence these representations of* $\mathbf{S}$ *are isomorphic.*

To prove this, we first observe that $\eta = d.x$ is a co-character of $\mathbf{S}^{ad}$, as it takes the value 0 or 1 on each basic root α_i and hence takes integral values on all the roots. To identify the root spaces which occur in $\mathfrak{g}(a)$, we note that whenever the identity $\alpha(x) + n = a/d$ holds for some integer n, then $\alpha(d.x) = \langle \eta, \alpha \rangle \equiv a$ modulo d. Hence the root spaces $\mathfrak{g}_\alpha$ which lie in V_a are precisely those which contribute to the eigenspace $\mathfrak{g}(a)$, and these $\mathbf{S}$ modules are isomorphic over $\mathfrak{f}$. The coincidence of the root spaces in the lemma led Reeder and Yu [15] to a proof of the following stronger result.

Proposition 4.3 *Let* x *be the barycenter of the facet* F *of* C *fixed by* P*, so that the basic affine roots take the value* 0 *or* $1/d$ *on* x*. Let* $\eta = d.x$ *be the associated cocharacter of* $\mathbf{S}^{ad}$*. Then the subgroup* $\mathbf{G}_0$ *of* $\mathbf{G}$ *over* $\mathfrak{f}$ *which is fixed by* $\eta(\mu_d)$ *is isomorphic to the reductive quotient* $\mathbf{L}$ *of* P*. Moreover, for every* $a \geq 1$ *the subquotient* $V_a = P_{a/d}/P_{(a+1)/d}$ *is isomorphic as a representation of* $\mathbf{L} \cong \mathbf{G}_0$ *to the submodule* $\mathfrak{g}(a)$ *in the associated* $\mathbf{Z}/d\mathbf{Z}$*- grading of* $\mathfrak{g}$ *over* $\mathfrak{f}$*.*

Since the weights in the representations $V_a = \mathfrak{g}(a)$ of $\mathbf{L}$ are all roots of $\mathbf{G}$, one can show that the representations themselves are very restricted. For example, assume that $\mathbf{G}$ is simply laced and that $d > 1$. Then the representation V_1 is the direct sum of $\#\Sigma$ irreducible minuscule representations of $\mathbf{L}$, which are distinguished by their central characters. The lowest weights for these representations, as characters of $\mathbf{S}$, are the gradients of the simple affine roots in Σ.

We now sketch their proof of the proposition, in the simple case where the local field k contains the d^{th} roots of unity. Let E be the totally ramified Galois extension $k(\Pi)$ of k, where $\Pi^d = \pi$. The Galois group $\mathrm{Gal}(E/k)$ is then isomorphic to $\mu_d(k)$ via its action on Π. The barycenter x becomes hyperspecial in the building of $\mathbf{G}$ over the extension E, where there are more affine roots $\psi = \alpha + n/d$ to consider. In particular, the latice $\mathfrak{g}(E)_{x,a/d}$ is the scalar multiple $\Pi^a\mathfrak{g}(E)_{x,0}$ and the reductive quotient of $\mathbf{G}(E)_{x,0}$ is isomorphic to $\mathbf{G}$ over $\mathfrak{f}$. An argument of Serre and Tate [20] shows that the Galois group $\mathrm{Gal}(E/k)$ acts algebraically on the reductive quotient, and a short computation shows that this action is the inverse of the adjoint action given by $\eta : \mu_d \to \mathbf{T} \to \mathrm{Aut}(\mathbf{G})$.

Since the reductive quotient of $\mathbf{G}$ over k is the subgroup of Galois invariants in the reductive quotient over E, this gives the isomorphism $\mathbf{L} \cong \mathbf{G}_0$ over $\mathfrak{f}$. Similarly, the quotient $g_{x,a/d}/g_{x,(a+1)/d}$ is the vector space of Galois invariants in $\Pi^a\mathfrak{g}(E)_{x,0}/\Pi^{a+1}\mathfrak{g}(E)_{x,0}$. It then follows from our calculation of the Galois action on $\mathfrak{g}$ and Π^a that this quotient space is isomorphic to the representation $\mathfrak{g}(a)$ in the $\mathbf{Z}/d\mathbf{Z}$- grading of $\mathfrak{g}$ over $\mathfrak{f}$.

5 A cohomological classification

In the previous section, we considered the internal structure of the parahoric subgroups of a split, simply-connected, simple group $\mathbf{G}$ over a local field k with finite residue field. We now turn to the general case of a simply-connected, simple group $\mathbf{G}$ over k, assuming only that $\mathbf{G}$ splits over a tamely ramified extension of k.

A remarkable fact, which was discovered by Bruhat and Tits [4], is that the geometry of the alcove in the building of $\mathbf{G}$ over k, as well as the internal structure of its parabolic subgroups, depends only on two invariants of a combinatorial nature. The first is a simple affine diagram ${}^e\mathscr{R}$, which is a connected affine Coxeter graph of the type listed in Bourbaki [2] [Ch VI, Thm 4] together with an orientation chosen for each multiple edge. The second is a conjugacy class F in the finite group $\mathrm{Aut}({}^e\mathscr{R})$ of automorphisms of this affine diagram. More precisely, the affine diagram ${}^e\mathscr{R}$ determines the structure of parahorics in $\mathbf{G}$ over the maximal unramified extension K of k, and the automorphism F of the affine diagram determines the descent of this structure to k. We will show how one can compute the affine invariants $({}^e\mathscr{R}, F)$ from the cohomological data describing the isomorphism class of $\mathbf{G}$ over k, and will describe which invariants occur for the different isomorphism classes.

In this section, we recall the cohomological classification of the simple, simply-connected groups $\mathbf{G}$ over k. Much of this theory works over a general field. The first invariant is the split group $\mathbf{G}_0$ over k which becomes isomorphic to $\mathbf{G}$ over the separable closure k^s. The simply-connected group $\mathbf{G}_0$ is determined up to isomorphism by its root system. Let $\mathbf{S}_0$ be a maximal split torus in $\mathbf{G}_0$, let X be the cocharacter group of $\mathbf{S}_0$ and Y be the character group. We recall that the set of roots R is the finite subset of Y of the non-trivial characters of $\mathbf{S}_0$ which occur in the adjoint representation. Since $\mathbf{G}_0$ is simple, the root system is reduced and irreducible of type $A_n\ n \geq 1, B_n\ n \geq 2, C_n\ n \geq 3, D_n\ n \geq 4, G_2, F_4, E_6, E_7, or E_8$.

Fix an isomorphism $f : \mathbf{G}_0 \to \mathbf{G}$ over the separable closure k^s. For an element σ in the Galois group of k^s over k, we have ${}^\sigma f = f \cdot a_\sigma$, where a_σ is a one cocycle with values in $\mathrm{Aut}(\mathbf{G}_0)(k^s)$. The class of a_σ in the pointed cohomology set $H^1(k, \mathrm{Aut}(\mathbf{G}_0))$ determines the isomorphism class of $\mathbf{G}$ over k. We break the description of this cohomology class into two parts.

The first is the image q_σ of a_σ, as a cocycle with values in the quotient group $\mathrm{Out}(\mathbf{G}_0)(k^s)$. If we fix a pinning $(B_0, T_0, \{X_i\})$ of $\mathbf{G}_0$ over k, the finite group of pinned automorphisms over k maps isomorphically onto the group $\mathrm{Out}(\mathbf{G}_0)(k^s)$. Since the group of pinned automorphisms is isomorphic to the constant étale group $\mathrm{Aut}(R, \Delta)$ the class of q_σ gives a homomorphism

$$q : \mathrm{Gal}(k^s/k) \to \mathrm{Aut}(R, \Delta)$$

up to conjugation. When viewed as a map to the pinned automorphisms, it determines the quasi-split inner form $\mathbf{G}_q$ of $\mathbf{G}$ over k. Since $\mathrm{Aut}(\mathbf{G}_0)$ is the semi-direct product of the adjoint quotient $\mathbf{G}_0/\mathbf{Z}(\mathbf{G}_0)$ with the subgroup of pinned automorphisms, standard twisting arguments in non-commutative cohomology [19] show that the fibre over q in the surjective map $H^1(k, \mathrm{Aut}(\mathbf{G}_0)) \to H^1(k, \mathrm{Out}(\mathbf{G}_0))$ can be identified with the $\mathrm{Out}(\mathbf{G}_q)(k)$ orbits on the set of classes c in $H^1(k, \mathbf{G}_q/\mathbf{Z}(\mathbf{G}_q))$, where $\mathbf{Z}(\mathbf{G}_q)$ is the center of the quasi-split group $\mathbf{G}_q$. The orbit of the class c is the second cohomological invariant.

All of this is true over an arbitrary field k. In our case, when k is local and $\mathbf{G}$ splits over a tamely ramified extension, the homomorphism q is tamely ramified. Moreover, Kneser's theorem [10] shows that the coboundary induces an isomorphism of pointed sets

$$H^1(k, \mathbf{G}_q/\mathbf{Z}(\mathbf{G}_q)) \to H^2(k, \mathbf{Z}(\mathbf{G}_q)).$$

Finally, we can compute the finite abelian group $H^2(k, \mathbf{Z}(\mathbf{G}_q))$ from the root system R of $\mathbf{G}$ using Tate duality [19]. Let M_q be the Cartier dual of $\mathbf{Z}(\mathbf{G}_q)$, which is a finite, commutative, étale group scheme over k. Over the separable closure, $M_q(k^s) = P/Q = Y/Y^*$ is the quotient of the weight lattice by the root lattice. The Galois group acts via the homomorphism q with image J_q in $\mathrm{Aut}(R, \Delta)$. Tate's theorem then gives canonical isomorphisms

$$H^2(k, \mathbf{Z}(\mathbf{G}_q)) = \mathrm{Hom}(H^0(k, M_q), \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}((Y/Y^*)^{J_q}, \mathbf{Q}/\mathbf{Z}) = (X^*/X)_{J_q}$$

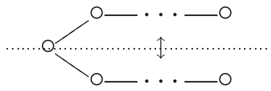
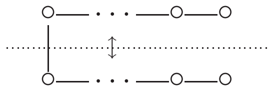
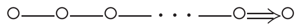
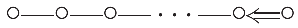
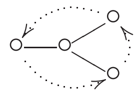
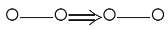
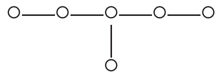
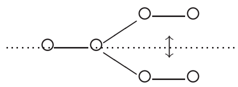
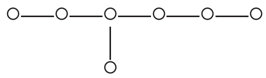
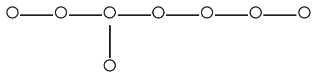
.

The group $\mathrm{Out}(\mathbf{G}_q)(k)$ is isomorphic to the centralizer of J_q in $\mathrm{Aut}(R, \Delta)$, and this centralizer acts naturally on the class c of $\mathbf{G}$ in $(X^*/X)_{J_q}$. Summarizing our results, we have proved the following.

Proposition 5.1 *The isomorphism class of the tamely ramified group $\mathbf{G}$ over the local field k is completely determined by the following data:*

- *The irreducible root system R*
- *The tamely ramified homomorphism $q : \mathrm{Gal}(k^s/k) \to \mathrm{Aut}(R, \Delta)$ with image J_q*
- *The orbit of the cohomology class c in the finite abelian group $(X^*/X)_{J_q}$ under the action of the centralizer of J_q in $\mathrm{Aut}(R, \Delta)$.*

Since the groups $\mathrm{Aut}(R, \Delta)$ and X^*/X are so small, we can tabulate all of the possibilities which occur. The image $J = J_q$ of the homomorphism q is a subgroup of S_1, S_2, S_3, so is determined by its order, which we tabulate as $\#J$.

R	$\#J$	$(X^*/X)_J$	Dynkin Diagram
A_{n-1}	1	$\mathbf{Z}/n\mathbf{Z}$	
A_{2n+1}	2	$\mathbf{Z}/2\mathbf{Z}$	
A_{2n}	2	1	
B_n	1	$\mathbf{Z}/2\mathbf{Z}$	
C_n	1	$\mathbf{Z}/2\mathbf{Z}$	
D_{2n}	1	$(\mathbf{Z}/2\mathbf{Z})^2$	
D_{2n+1}	1	$\mathbf{Z}/4\mathbf{Z}$	
D_n	2	$\mathbf{Z}/2\mathbf{Z}$	
D_4	$3, 6$	1	
G_2	1	1	
F_4	1	1	
E_6	1	$\mathbf{Z}/3\mathbf{Z}$	
E_6	2	1	
E_7	1	$\mathbf{Z}/2\mathbf{Z}$	
E_8	1	1	

To illustrate, we will describe the tamely ramified simply-connected groups $\mathbf{G}$ with root system of type E_6. There is the split group $\mathbf{G}_0$ of rank 6, and for each quadratic field extension L of k which is either unramified or tamely ramified there is the quasi-split group $\mathbf{G}_q$ of rank 4 which is split by L. Finally, when q is trivial, the group $\mathbf{G}_0$ has a inner form $\mathbf{G}$ of rank 2, corresponding to a non-trivial class c in $X^*/X = \mathbf{Z}/3\mathbf{Z}$. Both non-trivial classes in X^*/X lie in the same $\mathrm{Aut}(R, \Delta) = \mathbf{Z}/2\mathbf{Z}$ orbit. (As a consequence,when $\mathbf{G}$ has rank 2, the map $\mathrm{Aut}(\mathbf{G})(k) \to \mathrm{Out}(\mathbf{G})(k) = \mathbf{Z}/2\mathbf{Z}$ is not surjective.)

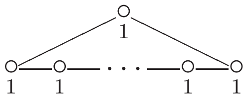
6 The affine diagram and its automorphism

We recall that K is the maximal unramified extension of k. Let E be the unique cyclic, tamely ramified extension of K which splits the group $\mathbf{G}$, and let e be the degree of E over K. Since $\mathbf{G}$ is split over E, an alcove in the building of $\mathbf{G}$ over E is a simplex whose facets correspond to the non-empty subsets of the nodes of the extended Dynkin diagram $\mathscr{R}$ associated to the based root system (R, Δ).

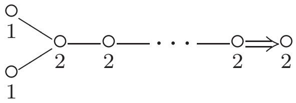
We can determine the structure of an alcove in the building of $\mathbf{G}$ over K using the techniques of tame descent. Since the cyclic group $\mathrm{Gal}(E/K)$ acts via an element of order e in the group $\mathrm{Aut}(R, \Delta)$ of pinned outer automorphisms of $\mathbf{G}$, the fixed affine root system has been computed in [14]. The alcove of $\mathbf{G}$ over K is again a simplex, whose facets correspond to the non-empty subsets of the nodes of the twisted affine diagram ${}^e\mathscr{R}$.

The affine diagrams which arise are tabulated below. Note that the removal of any non-empty subset of nodes (with the edges attached to those nodes) results in an ordinary Dynkin diagram. When $e = 1$ the positive integers attached to the nodes give the multiplicities of the simple roots α_i in the highest root β. When $e > 1$, they give e times the multiplicities of the simple roots in the highest short root, or twice the highest short root of the group fixed by the pinned outer automorphism. When $e = 1$ we write ${}^e\mathscr{R}$ simply as $\mathscr{R}$.

$\mathscr{A}_n$

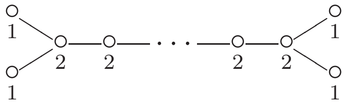


$\mathscr{B}_n$

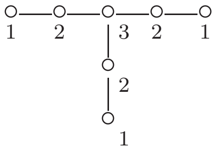


$\mathscr{C}_n$

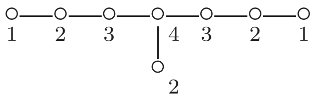
$\mathscr{D}_n$



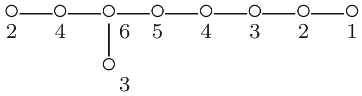
$\mathscr{E}_6$



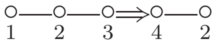
$\mathscr{E}_7$



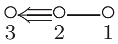
$\mathscr{E}_8$



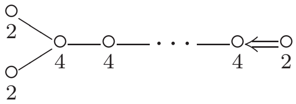
$\mathscr{F}_4$



$\mathscr{G}_2$



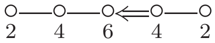
${}^2\mathscr{A}_{2n+1}$



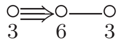
$^2\mathscr{A}_{2n}$

$^2\mathscr{D}_n$

$^2\mathscr{E}_6$



$^3\mathscr{D}_4$

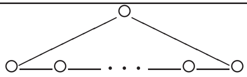
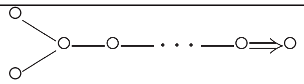
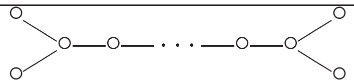
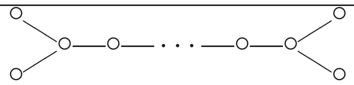
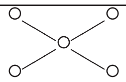
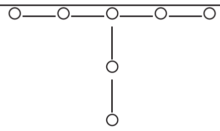
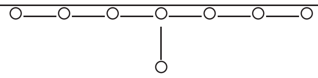
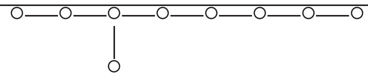
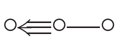
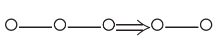
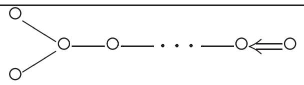
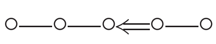
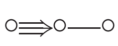


Finally, we obtain the alcove of $\mathbf{G}$ over k by unramified descent, which gives a homomorphism

$$\mathrm{Gal}(K/k) \to \mathrm{Aut}({}^e\mathscr{R}).$$

The image F of Frobenius is a well-defined conjugacy class in the finite group $\mathrm{Aut}({}^e\mathscr{R})$. The facets of the alcove over k correspond to non-empty F-stable subsets of the nodes of ${}^e\mathscr{R}$.

To enumerate the pairs $({}^e\mathscr{R}, F)$ which occur, we need to know the structure of the finite group of automorphisms for each affine diagram. When $e = 1$ the group $\mathrm{Aut}(\mathscr{R})$ is the semi-direct product $(X^*/X) \rtimes \mathrm{Aut}(R, \Delta)$, and when $e > 1$ the group $Aut({}^e\mathscr{R})$ is the abelian quotient $(X^*/X)_{\mathrm{Gal}(E/K)}$. These finite groups are tabulated below, where we use T_{2n} to denote the dihedral group of order $2n$ and S_n to denote the symmetric group on n letters. We give the semi-direct product decomposition when $e = 1$: the normal subgroup X^*/X acts simply-transitively on the nodes with multiplicity 1 and the canonical splitting is given by the subgroup fixing one such node.

${}^e\mathscr{R}$	$\mathrm{Aut}({}^e\mathscr{R})$	Affine Dynkin Diagram
$\mathscr{A}_{n-1}$	$T_{2n} = \mathbf{Z}/n\mathbf{Z} \rtimes S_2$	
$\mathscr{B}_n$	$\mathbf{Z}/2\mathbf{Z}$	
$\mathscr{C}_n$	$\mathbf{Z}/2\mathbf{Z}$	
$\mathscr{D}_{2n+1}$	$T_8 = (\mathbf{Z}/4\mathbf{Z}) \rtimes S_2$	
$\mathscr{D}_{2n}$	$T_8 = (\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_2$	
$\mathscr{D}_4$	$S_4 = (\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_3$	
$\mathscr{E}_6$	$S_3 = (\mathbf{Z}/3\mathbf{Z}) \rtimes S_2$	
$\mathscr{E}_7$	$\mathbf{Z}/2\mathbf{Z}$	
$\mathscr{E}_8$	1	
$\mathscr{G}_2$	1	
$\mathscr{F}_4$	1	
${}^2\mathscr{A}_{2n+1}$	$\mathbf{Z}/2\mathbf{Z}$	
${}^2\mathscr{A}_{2n}$	1	
${}^2\mathscr{D}_n$	$\mathbf{Z}/2\mathbf{Z}$	
${}^2\mathscr{E}_6$	1	
${}^3\mathscr{D}_4$	1	

7 From cohomological invariants to affine data

To identify the parahoric structure of $\mathbf{G}$, we need a recipe to pass from the cohomological invariants (R, q, c) to the affine data $({}^e\mathscr{R}, F)$. The recipe for the parahoric subgroups of $\mathbf{G}$ over k is then as follows. Suppose that there are $\ell + 1$ orbits of the automorphism F on the nodes of the affine diagram ${}^e\mathscr{R}$. Then the group $\mathbf{G}$ has rank ℓ over k, and the parahoric subgroups P of $\mathbf{G}(k)$ which contain a fixed Iwahori correspond bijectively to the non-empty F-stable subsets Σ of the nodes of ${}^e\mathscr{R}$. The integer $d = d(P)$ in the Moy-Prasad filtration

$$P \rhd P_{1/d} \rhd P_{2/d} \rhd \cdots$$

is the sum, taken over the subset Σ, of the integral labels of the respective nodes. The reductive quotient $\mathbf{L}(\mathfrak{f}) = (P/P_{1/d})$ is quasi-split over $\mathfrak{f}$. Over the separable closure, $\mathbf{L}$ contains the maximal split torus

$\mathbf{S}$ and its root basis is given by the complement of Σ in the basic affine roots. The action of F on this complement determines the descent of $\mathbf{L}$ to $\mathfrak{f}$.

The affine diagram ${}^e\mathscr{R}$ is obtained from the root system R and the restriction of the homomorphism q to the inertia subgroup of $\mathrm{Gal}(k^s/k)$. If the image I_q of (tame) inertia in $\mathrm{Aut}(R, \Delta)$ is cyclic of order e, then the affine diagram has type ${}^e\mathscr{R}$. Note that non-isomorphic quasi-split groups $\mathbf{G}_q$ can have the same affine diagram, as the latter only depends on the order e of the tame inertia subgroup, whereas the isomorphism class of the group depends on the tamely ramified extension of k which splits $\mathbf{G}$. Thus non-isomorphic groups can have the same internal parahoric structure.

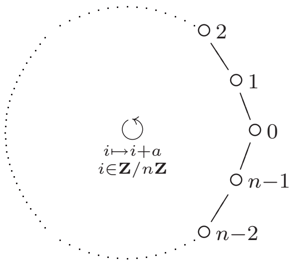
To determine the automorphism F of ${}^e\mathscr{R}$, we first assume that $e = 1$. Then the homomorphism q factors through $\mathrm{Gal}(K/k)$ and the image of Frobenius gives a conjugacy class s in the group $\mathrm{Aut}(R, \Delta)$.

If $s = 1$ the isomorphism class of $\mathbf{G}$ is determined by the $\mathrm{Aut}(R, \Delta)$ orbit of the cohomology class c in X^*/X. This gives a well-defined conjugacy class $F = c \times 1$ in the group $\mathrm{Aut}(\mathscr{R}) = (X^*/X) \rtimes \mathrm{Aut}(R, \Delta)$.

If $s \neq 1$ the group $\mathbf{G}$ is determined by the $\langle s \rangle$ orbit of the cohomology class c in the quotient group $(X^*/X)_{\langle s \rangle}$. If we lift c to a class c^* in X^*/X, then the product $F = c^* \times s$ gives a well-defined conjugacy class in $\mathrm{Aut}(\mathscr{R})$.

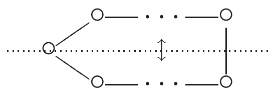
When $e > 1$, F is equal to the cohomology class c in $(X^*/X)_I = (X^*/X)_{J_q} = \mathrm{Aut}({}^e\mathscr{R})$.

With this recipe, we can easily identify the different isomorphism classes of groups associated to each pair $({}^e\mathscr{R}, F)$. We give the affine diagram with its automorphism marked, then describe the simple group over k to which it belongs. This description is also tabulated (with a bit more information) in the survey article of Tits [22]. When $F = 1$, the reductive quotients $\mathbf{L}$ of the parahoric subgroups of $\mathbf{G}$ are all split over the residue field $\mathfrak{f}$. Tits calls such a group $\mathbf{G}$ residually split.



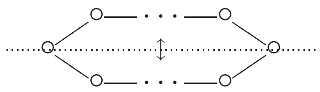
The affine diagram is $\mathscr{A}_{n-1}$. This is an n-gon, and F is a rotation by $\pm a$ units in T_{2n}. Write $a/n = b/m$ in lowest terms. There are m orbits on the nodes, and the rank of $\mathbf{G}$ over k is $m - 1$.

Let D be the division algebra of degree m^2 over k with invariant b/m. Then $\mathbf{G} = SL_{n/m}(D)$. This is the split form $\mathbf{G}_0 = \mathrm{SL}_n$ when $F = 1$ and hence $a \equiv 0$.



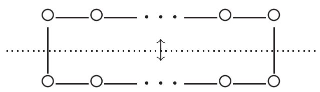
The affine diagram is $\mathscr{A}_{n-1}$ with $n = 2m+1$. This is an n-gon and F is a reflection fixing a vertex in T_{2n}. There are $m+1$ orbits on the nodes, and the rank of $\mathbf{G}$ over k is m.

Let L be the unramified quadratic extension of k and let W be a non-degenerate Hermitian space of odd rank $n = 2m+1$ over L. Then $\mathbf{G} = SU(W)$. This is the unramified quasi-split form $\mathbf{G}_q$.



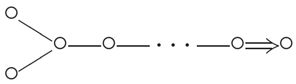
The affine diagram is $\mathscr{A}_{n-1}$ with $n = 2m$. This is an n-gon and F is a reflection fixing a vertex in T_{2n} (hence fixing the opposite vertex). There are $m+1$ orbits on the nodes, and the rank of $\mathbf{G}$ over k is m.

Let L be the unramified quadratic extension of k and let W be a non-degenerate Hermitian space of even rank $n = 2m$ over L which contains an isotropic subspace of dimension m. Then $\mathbf{G} = SU(W)$. This is the unramified quasi-split form $\mathbf{G}_q$.



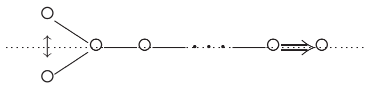
The affine diagram is $\mathscr{A}_{n-1}$ with $n = 2m$. This is an n-gon and F is a reflection through the midpoint of an edge in T_{2n}. There are m orbits on the nodes, and the rank of $\mathbf{G}$ over k is $m-1$.

Let L be the unramified quadratic extension of k and let W be a non-degenerate Hermitian space of even rank $n = 2m$ over L which does not contain an isotropic subspace of dimension m. Then $\mathbf{G} = SU(W)$.



The affine diagram is $\mathscr{B}_n$ and F is trivial. There are $n+1$ nodes and the rank of $\mathbf{G}$ over k is n.

Let W be a non-degenerate orthogonal space of odd dimension $2n+1$ over k (the bilinear pairing is even and the associated quadratic form Q is non-singular) which contains an isotropic subspace of dimension n. Then $\mathbf{G} = \mathrm{Spin}(W)$. This is the split form $\mathbf{G}_0$.

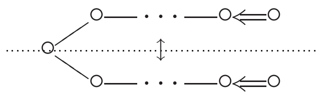


The affine diagram is $\mathscr{B}_n$ and F is the non-trivial involution. There are n orbits on the nodes and the rank of $\mathbf{G}$ over k is $n-1$.

Let W be a non-degenerate orthogonal space of odd dimension $2n+1$ over k (the bilinear pairing is even and the associated quadratic form Q is non-singular) which does not contain an isotropic subspace of dimension n. Then $\mathbf{G} = \mathrm{Spin}(W)$.

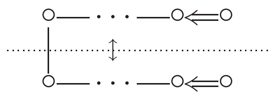
The affine diagram is $\mathscr{C}_n$ and F is trivial. There are $n+1$ nodes and the rank of $\mathbf{G}$ over k is n.

Let W be a non-degenerate symplectic space of even dimension $2n$ over k. Then $\mathbf{G} = Sp(W)$. This is the split form $\mathbf{G}_0$.



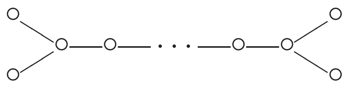
The affine diagram is $\mathscr{C}_n$ with $n = 2m$ and F is the non-trivial involution. There are $m+1$ orbits of the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be a non-degenerate Hermitian space over D of dimension n. Then $\mathbf{G} = U(W)$.



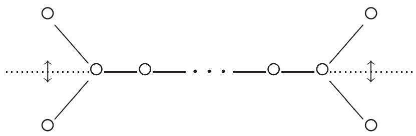
The affine diagram is $\mathscr{C}_n$ with $n = 2m+1$ and F is the non-trivial involution. There are $m+1$ orbits of the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be a non-degenerate Hermitian space over D of dimension n. Then $\mathbf{G} = U(W)$.



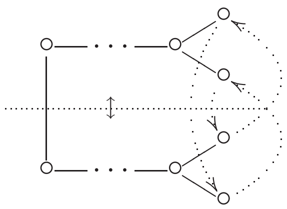
The affine diagram is $\mathscr{D}_n$ and F is trivial. There are $n+1$ nodes and the rank of $\mathbf{G}$ over k is n.

Let W be a non-degenerate orthogonal space of even dimension $2n$ which contains a maximal isotropic subspace of dimension n. Then $\mathbf{G} = \mathrm{Spin}(W)$ and the center of the Clifford algebra is the split étale quadratic extension $L = k + k$. This is the split form $\mathbf{G}_0$.



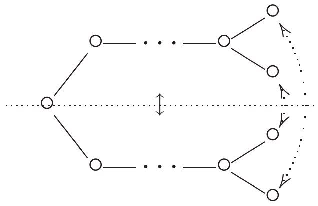
The affine diagram is $\mathscr{D}_n$ and F is the central element of order 2 in T_8. There are $n-1$ orbits on the nodes and the rank of $\mathbf{G}$ over k is $n-2$.

Let W be a non-degenerate even orthogonal space of even dimension $2n$ which does not contain an isotropic subspace of dimension $n-1$. Then $\mathbf{G} = \mathrm{Spin}(W)$ and the center of the Clifford algebra is the split étale quadratic extension $L = k + k$.



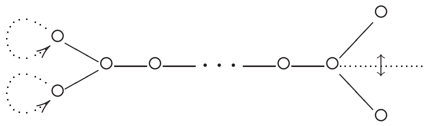
The affine diagram is $\mathscr{D}_n$ with $n = 2m+1$ and F is an element of order 4 in T_8. There are m orbits on the nodes and the rank of $\mathbf{G}$ over k is $m-1$.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra (in the sense of Tits [21] and Jacobson[8]) is the split étale quadratic algebra $L = k + k$. Then $\mathbf{G} = U(W)$.



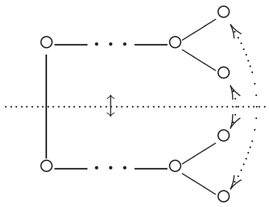
The affine diagram is $\mathscr{D}_n$ with $n = 2m$ and F is an element of order 2 in $P^\vee/Q^\vee$ which is not central in T_8. There are $m+1$ orbits on the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is the split étale quadratic algebra $L = k + k$. Then $\mathbf{G} = U(W)$.



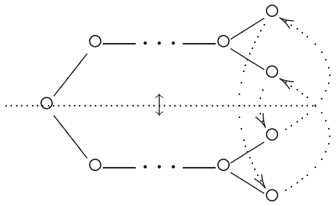
The affine diagram is $\mathscr{D}_n$ and F has order 2 in $\mathrm{Aut}(R, \Delta)$. There are n orbits on the nodes and the rank of $\mathbf{G}$ over k is $n-1$.

Let W be a non-degenerate orthogonal space of even dimension $2n$ where the center of the Clifford algebra is the unramified quadratic extension L of k. Then $\mathbf{G} = \mathrm{Spin}(W)$. This is the unramified quasi-split form $\mathbf{G}_q$.



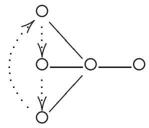
The affine diagram is $\mathscr{D}_n$ with $n = 2m+1$ and F is the remaining class of order 2 in T_8. There are $m+1$ orbits on the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is the unramified quadratic extension L of k. Then $\mathbf{G} = U(W)$.



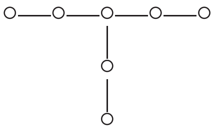
The affine diagram is $\mathscr{D}_n$ with $n = 2m$ and F has order 4 in T_8. There are m orbits on the nodes and the rank of $\mathbf{G}$ over k is $m - 1$.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is the unramified quadratic extension L of k. Then $\mathbf{G} = U(W)$.



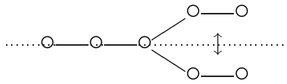
The affine diagram is $\mathscr{D}_4$ and F has order 3 in S_4. There are 3 orbits on the nodes and the rank is 2.

This is the unramified quasi-split inner form $\mathbf{G}_q$, split by a cubic extension.



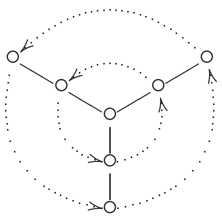
The affine diagram is $\mathscr{E}_6$ and F is trivial. There are 7 nodes and the rank is 6.

This is the split form $\mathbf{G}_0$.

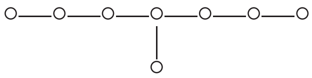


The affine diagram is $\mathscr{E}_6$ and F is a non-trivial involution in S_3. There are 5 orbits on the nodes and the rank is 4.

This is the unramified quasi-split form $\mathbf{G}_q$.

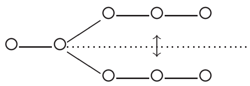


The affine diagram is $\mathscr{E}_6$ and F is an element of order 3 in S_3. There are 3 orbits on the nodes and the rank is 2.

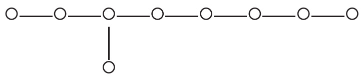


The affine diagram is $\mathscr{E}_7$ and F is trivial. There are 8 nodes and the rank is 7.

This is the split form $\mathbf{G}_0$.

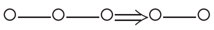


The affine diagram is $\mathscr{E}_7$ and F is the non-trivial involution. There are 5 orbits on the nodes and the rank is 4.



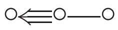
The affine diagram is $\mathscr{E}_8$ and F is trivial. There are 9 nodes and the rank is 8.

This is the split form $\mathbf{G}_0$.



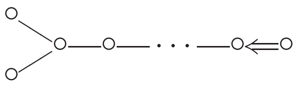
The affine diagram is $\mathscr{F}_4$ and F is trivial. There are 5 nodes and the rank is 4.

This is the split form $\mathbf{G}_0$.



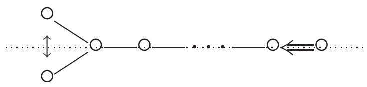
The affine diagram is $\mathscr{G}_2$ and F is trivial. There are 3 nodes and the rank is 2.

This is the split form $\mathbf{G}_0$.



The affine diagram is ${}^2\mathscr{A}_{2m+1}$ and F is trivial. There are $m+1$ nodes and the rank of $\mathbf{G}$ over k is m.

Let L be a tamely ramified quadratic extension of k and let W be a non-degenerate Hermitian space of even rank $n = 2m$ over L which contains an isotropic subspace of dimension m. Then $\mathbf{G} = SU(W)$. This is a ramified quasi-split form $\mathbf{G}_q$.



The affine diagram is ${}^2\mathscr{A}_{2m+1}$ and F is the non-trivial involution. There are m orbits on the nodes and the rank of $\mathbf{G}$ over k is $m-1$.

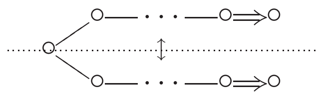
Let L be a tamely ramified quadratic extension of k and let W be a non-degenerate Hermitian space of even rank $n = 2m$ over L which does not contain an isotropic subspace of dimension m. Then $\mathbf{G} = SU(W)$.

The affine diagram is ${}^2\mathscr{A}_{2m}$ and F is trivial. There are $m+1$ nodes and the rank of $\mathbf{G}$ over k is m.

Let L be a tamely ramified quadratic extension of k and let W be a non-degenerate Hermitian space of odd rank $n = 2m+1$ over L. Then $\mathbf{G} = SU(W)$. This is a ramified quasi-split form $\mathbf{G}_q$.

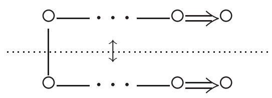
The affine diagram is ${}^2\mathscr{D}_n$ and F is trivial. There are n nodes and the rank of $\mathbf{G}$ over k is $n-1$.

Let W be a non-degenerate even orthogonal space of even dimension $2n$ where the center of the Clifford algebra is a tamely ramified quadratic extension L of k. Then $\mathbf{G} = \mathrm{Spin}(W)$. This is a ramified quasi-split form $\mathbf{G}_q$.



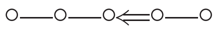
The affine diagram is ${}^2\mathscr{D}_n$ with $n = 2m$ and F is the non-trivial involution. There are m orbits on the nodes and the rank of $\mathbf{G}$ over k is $m-1$

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is a tamely ramified quadratic extension L of k. Then $\mathbf{G} = U(W)$



The affine diagram is ${}^2\mathscr{D}_n$ with $n = 2m+1$ and F is the non-trivial involution. There are $m+1$ orbits on the nodes and the rank of $\mathbf{G}$ over k is m.

Let D be the quaternion division algebra over k and let W be an anti-Hermitian space of dimension n over D, such that the center of the Clifford algebra is a tamely ramified quadratic extension L of k. Then $\mathbf{G} = U(W)$.



The affine diagram is ${}^2\mathscr{E}_6$ and F is trivial. There are 5 nodes and the rank is 4.

This is a tamely ramified quasi-split form $\mathbf{G}_q$.

$\circ \Rrightarrow \circ - \circ$

The affine diagram is ${}^3\mathscr{D}_4$ and F is trivial. There are 3 nodes and the rank is 2.

This is a tamely ramified quasi-split form $\mathbf{G}_q$.

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