On Minuscule Representations and the Principal $\mathbf{SL_2}$

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In this paper, we review the theory of minuscule coweights λ for a simple adjoint group G over $\mathbf{C}$, as presented by Deligne [D]. We then decompose the associated irreducible representation V_λ of the dual group $\hat{G}$, when restricted to a principal SL_2. This decomposition is given by the action of a Lefschetz SL_2 on the cohomology of the flag variety $X = G/P_\lambda$, where P_λ is the maximal parabolic subgroup of G associated to the coweight λ. We reinterpret a result of Vogan and Zuckerman [V-Z, Prop 6.19] to show that the cohomology of X is mirrored by the bigraded cohomology of the L-packet of discrete series with infinitesimal character ρ, for a real form G_0 of G with a Hermitian symmetric space.

We then focus our attention on those minuscule representations with a non-zero linear form $t : V \to \mathbf{C}$ fixed by the principal SL_2, such that the subgroup $\hat{H} \subset \hat{G}$ fixing t acts irreducibly on the subspace $V_0 = \ker(t)$. We classify them in §10; since $\hat{H}$ turns out to be reductive, we have a decomposition

$$V = \mathbf{C}e + V_0$$

where e is fixed by $\hat{H}$, and satisfies $t(e) \neq 0$. We study V as a representation of $\hat{H}$, and give an $\hat{H}$-algebra structure on V with identity e.

The rest of the paper studies representations π of G which are lifted from H, in the sense of Langlands. We show this lifting is detected by linear forms on π which are fixed by a certain subgroup L of G. The subgroup L descends to a subgroup $L_0 \to G_0$ over $\mathbf{R}$; both have Hermitian symmetric spaces $\mathcal{D}$ with $\dim_{\mathbf{C}}(\mathcal{D}_L) = \frac{1}{2}\dim_{\mathbf{C}}(\mathcal{D}_G)$. We hope this will provide cycle classes in the Shimura varieties associated to G_0, which will enable one to detect automorphic forms in cohomology which are lifted from H.

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1. Minuscule coweights

Let G be a simple algebraic group over $\mathbf{C}$, of adjoint type. Let $T \subset B \subset G$ be a maximal torus, contained in a Borel subgroup, and let Δ be the corresponding set of simple roots for T. Then Δ gives a $\mathbf{Z}$-basis for $\mathrm{Hom}(T, \mathbf{G}_m)$, so a coweight λ in $\mathrm{Hom}(\mathbf{G}_m, T)$ is completely determined by the integers $\langle \lambda, \alpha \rangle$, for α in Δ, which may be arbitrary. Let P_+ be the cone of dominant co-weights, where $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta$.

A coweight $\lambda : \mathbf{G}_m \to T$ gives a $\mathbf{Z}$-grading $\mathfrak{g}_\lambda$ of $\mathfrak{g} = \mathrm{Lie}(G)$, defined by

$$\mathfrak{g}_\lambda(i) = \{X \in \mathfrak{g} : \mathrm{Ad}\, \lambda(a)(X) = a^i \cdot X\}$$

We say λ is minuscule provided $\lambda \neq 0$ and the grading $\mathfrak{g}_\lambda$ satisfies $\mathfrak{g}_\lambda(i) = 0$ for $|i| \geq 2$. Thus

$$\mathfrak{g} = \mathfrak{g}_\lambda(-1) + \mathfrak{g}_\lambda(0) + \mathfrak{g}_\lambda(1). \tag{1.1}$$

The Weyl group $N_G(T)/T = W$ of T acts on the set of minuscule coweights, and the W-orbits are represented by the dominant minuscule coweights. These have been classified

Proposition 1.2 ([D, 1.2]). *The element λ is a dominant, minuscule coweight if and only if there is a single simple root α with $\langle \lambda, \alpha \rangle = 1$, the root α has multiplicity 1 in the highest root β, and all other simple roots α' satisfy $\langle \lambda, \alpha' \rangle = 0$.*

Thus, the W-orbits of minuscule coweights correspond bijectively to simple roots α with multiplicity 1 in the highest root β. If λ is minuscule and dominant, $\mathfrak{g}_\lambda(1)$ is the direct sum of the positive root spaces $\mathfrak{g}_\gamma$, where γ is a positive root containing α with multiplicity 1. Hence the dimension N of $\mathfrak{g}_\lambda(1)$ is given by the formula

$$N = \dim \mathfrak{g}_\lambda(1) = \langle \lambda, 2\rho \rangle, \tag{1.3}$$

where ρ is half the sum of the positive roots.

The subgroup $W_\lambda \subset W$ fixing λ is isomorphic to the Weyl group of T in the subalgebra $\mathfrak{g}_\lambda(0)$, which has root basis $\Delta - \{\alpha\}$. We now tabulate the W-orbits of minuscule coweights, by listing the simple α occurring with multiplicity 1 in β in the numeration of Bourbaki [B]. We also tabulate $N = \dim \mathfrak{g}_\lambda(1)$ and $(W : W_\lambda)$; a simple comparison show that $(W : W_\lambda) \geq N + 1$ in all cases; we will explain this inequality later.

Table 1.4

G	α	$(W : W_\lambda)$	N
A_ℓ	α_k	$\binom{\ell+1}{k}$	$k(\ell+1-k)$
	$1 \leq k \leq \ell$		
B_ℓ	α_1	2ℓ	$2\ell - 1$
C_ℓ	α_ℓ	2^ℓ	$\frac{\ell(\ell+1)}{2}$
D_ℓ	α_1	2ℓ	$2\ell - 2$
	$\alpha_{\ell-1}, \alpha_\ell$	$2^{\ell-1}$	$\frac{\ell(\ell-1)}{2}$
E_6	α_1, α_6	27	16
E_7	α_1	56	27

2. The real form G_0

We henceforth fix G and a dominant minuscule coweight λ. Let G_c be the compact real form for G, so $G = G_c(\mathbf{C})$ and $G_c(\mathbf{R})$ is a maximal compact subgroup of G. Let $g \mapsto \overline{g}$ be the corresponding conjugation of G.

Let $T_c \subset G_c$ be a maximal torus over $\mathbf{R}$. We have an identification of co-character groups

$$\mathrm{Hom}_{\mathrm{cont}}(S^1, T_c(\mathbf{R})) = \mathrm{Hom}_{\mathrm{alg}}(\mathbf{G}_m, T).$$

We view λ as a homomorphism $S^1 \to T_c(\mathbf{R})$, and define

$$\theta = \mathrm{ad}\ \lambda(-1) \qquad \text{in} \quad Inn(G). \tag{2.1}$$

Then θ is a Cartan involution, which gives another descent G_0 of G to $\mathbf{R}$. The group G_0 has real points

$$G_0(\mathbf{R}) = \{g \in G : \overline{g} = \theta(g)\},$$

and a maximal compact subgroup K of $G_0(\mathbf{R})$ is given by

$$\begin{aligned} K &= \{g \in G : g = \overline{g} \quad \text{and} \quad g = \theta(g)\} \\ &= G_0(\mathbf{R}) \cap G_c(\mathbf{R}). \end{aligned}$$

The corresponding decomposition of the complex Lie algebra $\mathfrak{g}$ under the action of K is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, with

$$\begin{cases} \mathfrak{k} = \mathrm{Lie}(K) \otimes \mathbf{C} = \mathfrak{g}_\lambda(0) \\ \mathfrak{p} = \mathfrak{g}_\lambda(-1) + \mathfrak{g}_\lambda(1). \end{cases} \tag{2.2}$$

The torus $\lambda(S^1)$ lies in the center of the connected component of K, and the element $\lambda(i)$ gives the symmetric space

$$\mathcal{D} = G_0(\mathbf{R})/K$$

a complex structure, with

$$N = \dim_{\mathbf{C}}(\mathcal{D}). \tag{2.3}$$

Proposition 2.4. ([D, 1.2]). *The real Lie groups $G_0(\mathbf{R})$ and K have the same number of connected components, which is either 1 or 2. Moreover, the following are all equivalent:*

1) *$G_0(\mathbf{R})$ has 2 connected components.*
2) *The symmetric space $\mathcal{D}$ is a tube domain.*
3) *The vertex of the Dynkin diagram of G corresponding to the simple root α is fixed by the opposition involution of the diagram.*
4) *The subgroup W_λ fixing λ has a nontrivial normalizer in W, consisting of those w with $w\lambda = \pm\lambda$.*

In fact, the subgroup $W_c \subset W$ which normalizes W_λ is precisely the normalizer of the compact torus $T_c(\mathbf{R})$ in $G_0(\mathbf{R})$. When $W_\lambda \neq W_c$, it is generated by W_λ and the longest element w_0, which satisfies $w_0\lambda = -\lambda$.

As an example, let $G = SO_3$ and

$$\lambda(t) = \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix}.$$

Then θ is conjugation by

$$\lambda(-1) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix},$$

and $G_0 = SO(1,2)$ has 2 connected components. We have $K \simeq O(2)$, $W_c = W$ has order 2 in this case, and $W_\lambda = 1$. The tube domain $\mathcal{D} = G_0(\mathbf{R})/K$ is isomorphic to the upper half plane.

3. The Weyl group (cf. [H])

The Weyl group W is a Coxeter group, with generating reflections s corresponding to the simple roots in Δ. Recall that ρ is half the sum of the positive roots and $W_\lambda \subset W$ is the subgroup fixing λ.

Proposition 3.1. *Each coset wW_λ of W_λ in W has a unique representative y of minimal length. The length $d(y)$ of the minimal representative is given by the formula:*

$$d(y) = \langle \lambda, \rho \rangle - \langle w\lambda, \rho \rangle,$$

where w is any element in the coset.

Proof. Let $R^{\pm}$ be the positive and negative roots, let $R^{\pm}_\lambda$ be the subsets of positive and negative roots which satisfy $\langle \lambda, \gamma \rangle = 0$. Then $R^+ - R^+_\lambda$ consists of the roots with $\langle \lambda, \gamma \rangle = 1$, and $R^- - R^-_\lambda$ consists of the roots with $\langle \lambda, \gamma \rangle = -1$. These sets are stable under the action of W_λ on R. On the other hand, if $w \in W_\lambda$ stabilizes R^+_λ (or R^-_λ) then $w = 1$, as W_λ is the Weyl group of the root system $R_\lambda = R^+_\lambda \cup R^-_\lambda$.

Since the length $d(y)$ of y in W is given by

$$d(y) = \#\{\gamma \text{ in } R^+ : y^{-1}(\gamma) \text{ is in } R^-\}, \tag{3.2}$$

the set

$$Y = \{y \in W : y(R^+_\lambda) \subset R^+\} \tag{3.3}$$

gives coset representatives for W_λ of minimal length. Moreover, for $y \in Y$ the set $y^{-1}(R^+)$ contains $d(y)$ elements of R^-_λ, and hence $N - d(y)$ elements of R^+_λ. Hence, if $wW_\lambda = yW_\lambda$, we find

$$\begin{aligned}
\langle w\lambda, \rho \rangle = \langle y\lambda, \rho \rangle &= \langle \lambda, y^{-1}\rho \rangle \\
&= \frac{1}{2}((N - d(y)) - d(y)) \\
&= \frac{1}{2}N - d(y).
\end{aligned}$$

Since

$$\langle \lambda, \rho \rangle = \frac{1}{2} N,$$

we obtain the desired formula.

As an example of Proposition 3.1, the minimal representative of W_λ is $y = 1$, with $d(y) = 0$, and the minimal representative of $s_\alpha W_\lambda$ is $y = s_\alpha$, with $d(y) = 1$. If w_0 is the longest element in the Weyl group, then $w_0(R^\pm) = R^\mp$, so $w_0^2 = 1$, and $w_0 \rho = -\rho$. Hence

$$\langle w_0 \lambda, \rho \rangle = \langle \lambda, w_0^{-1} \rho \rangle = -\langle \lambda, \rho \rangle = -N/2.$$

Consequently, the length of the minimal representative y of $w_0 W_\lambda$ is $d(y) = N$. This is the maximal value of d on W/W_λ, and we will soon see that d takes all integral values in the interval $[0, N]$.

Assume λ is fixed by the opposition involution $-w_0$, so $w_0 \lambda = -\lambda$. Then $\mathcal{D}$ is a tube domain, and W_λ has nontrivial normalizer $W_c = \langle W_\lambda, w_0 \rangle$ in W, by Proposition 2.4. The 2-group W_c/W_λ acts on the set W/W_λ by $wW_\lambda \mapsto ww_0 W_\lambda$, and this action has no fixed points. Hence we get a fixed point-free action $y \mapsto y*$ on the set Y, and find:

$$d(y) + d(y*) = N. \tag{3.4}$$

4. The flag variety

Associated to the dominant minuscule coweight λ is a maximal parabolic subgroup P, which contains B and has Lie algebra

$$\mathrm{Lie}(P) = \mathfrak{g}_\lambda(0) + \mathfrak{g}_\lambda(1). \tag{4.1}$$

The flag variety $X = G/P$ is projective, of complex dimension N.

The cohomology of X is all algebraic, so $H^{2n+1}(X) = 0$ for all $n \geq 0$. Let

$$f_X(t) = \sum_{n \geq 0} \dim H^{2n}(X) \cdot t^n \tag{4.2}$$

be the Poincaré polynomial of $H^*(X)$. Then we have the following consequence of Chevalley-Bruhat theory, which also gives a convenient method of computing the values of the function $d : W/W_\lambda \to \mathbf{Z}$.

Proposition 4.3. 1) *We have* $f_X(t) = \sum_Y t^{d(y)}$.

2) *If* $\underline{G}$ *is the split adjoint group over* $\mathbf{Z}$ *with the same root datum as* G, *and* $\underline{P}$ *is the standard parabolic corresponding to* λ, *then*

$$f_X(q) = \#\underline{G}(F)/\underline{P}(F)$$

for all finite fields F, *with* $q = \#F$.

3) *The Euler characteristic of* X *is given by*

$$\chi = f_X(1) = \#(W : W_\lambda).$$

Proof. We have the decomposition

$$G = \bigcup_Y ByP,$$

where we have chosen a lifting of y from W to $N_G(T)$. If U is the unipotent radical of B, then $B = UT$. Since y normalizes T,

$$UyP = ByP.$$

This gives a cell decomposition

$$X = \bigcup_Y Uy/P \cap y^{-1}Uy$$

where the cell corresponding to y is an affine space of dimension $d(y)$. This gives the first formula.

The formula for $f_X(q)$ follows from the Bruhat decomposition, which can be used to prove the Weil conjectures for X. Formula 3) for $f_X(1)$ follows immediately from 1).

For an example, let $G = PSp_{2n}$ be of type C_n. Then P is the Siegel parabolic subgroup, with Levi factor GL_n/μ_2. From the orders of $Sp_{2n}(q)$ and $GL_n(q)$, we find

$$\begin{aligned}\#\underline{G}(F)/\underline{P}(F) &= \frac{(q^2-1)(q^4-1)\dots(q^{2n}-1)}{(q-1)(q^2-1)\dots(q^n-1)}\\ &= (1+q)(1+q^2)\dots(1+q^n).\end{aligned}$$

Hence we find

(4.4) $$f_X(t) = (1+t)(1+t^2)\dots(1+t^n).$$

The fact that $X = G/P$ is a Kahler manifold imposes certain restrictions on its cohomology. For example, if ω is a basis of $H^2(X)$, then $\omega^k \neq 0$ in $H^{2k}(X)$ for all $0 \leq k \leq N$. Hence we find that

Corollary 4.5. *The function $d : W/W_\lambda \to \mathbf{Z}$ takes all integral values in $[0, N]$, and $(W : W_\lambda) \geq N + 1$.*

For $0 \leq k \leq N$, let

$$m(k) = \#\{y \in Y : d(y) = k\}.$$

We have seen that $m(0) = m(1) = 1$ in all cases. By Poincaré duality

$$m(k) = m(N-k). \tag{4.6}$$

Finally, the Lefschetz decomposition into primitive cohomology shows that:

$$m(k-1) \leq m(k) \tag{4.7}$$

whenever $2k \leq N$. Indeed, the representation of the Lefschetz SL_2 on $H^*(G/P)$ has weights $N - 2d(y)$ for the maximal torus $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

5. The representation V of the dual group $\hat{G}$

Let $\hat{G}$ be the Langlands dual group of G, which is simply-connected of the dual root type. This group comes (in its construction) with subgroups $\hat{T} \subset \hat{B} \subset \hat{G}$, and an identification of the positive roots for $\hat{B}$ in $\mathrm{Hom}(\hat{T}, \mathbf{G}_m)$ with the positive co-roots for B in $\mathrm{Hom}(\mathbf{G}_m, T)$ (cf. [G]). Hence, the dominant co-weights for T give dominant weights for $\hat{T}$, which are the highest weights for $\hat{B}$ on irreducible representations of $\hat{G}$.

Let V be the irreducible representation of $\hat{G}$, whose highest weight for $\hat{B}$ is the dominant, minuscule co-weight λ.

Proposition 5.1. *The weights of $\hat{T}$ on V consist of the elements in the W-orbit of λ. Each has multiplicity 1, so* $\dim V = (W : W_\lambda)$.

The central character χ of V is given by the image of λ in $\mathrm{Hom}(\hat{T}, \mathbf{G}_m)/\bigoplus_{\Delta} \mathbf{Z}\alpha^\vee$*, and is nontrivial.*

Proof. For μ and λ dominant, we write $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of positive co-roots. These are precisely the other dominant weights for $\hat{T}$ occurring in V_λ. When λ is minuscule, $\mu \leq \lambda$ implies $\mu = \lambda$, so only the W-orbit of λ occur as weights. Each has the same multiplicity as the highest weight, which is 1. Since $\mu = 0$ is dominant, λ is not in the span of the co-roots, and $\chi \neq 1$.

This result gives another proof of the inequality of Corollary 4.5: $(W : W_\lambda) \geq N + 1$. Indeed, let L be the unique line in V_λ fixed by $\hat{B}$. The fixer of L is the standard parabolic $\hat{P}$ dual to P. This gives an embedding of projective varieties:

$$\hat{G}/\hat{P} \hookrightarrow \mathbf{P}(V_\lambda).$$

Since $\hat{G}/\hat{P}$ has dimension N, and $\mathbf{P}(V_\lambda)$ has dimension $(W : W_\lambda) - 1$, this gives the desired inequality.

The real form G_0 defined in §2 has Langlands L-group

$$(5.2) \qquad {}^L G = \hat{G} x \mathrm{Gal}(\mathbf{C}/\mathbf{R}).$$

The action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on $\hat{G}$ exchanges the irreducible representation V with dominant weight λ with the dual representation V^* with dominant weight $-w_0\lambda$. Hence the sum $V + V^*$ always extends to a representation of ${}^LG$. The following is a simple consequence of Proposition 2.4.

Proposition 5.3. *The following are equivalent:*

1) We have $w_0\lambda = -\lambda$.

2) The symmetric space $\mathcal{D}$ is a tube domain.

3) The representation V is isomorphic to V^*.

4) The central character χ of V satisfies $\chi^2 = 1$.

5) The representation V of $\hat{G}$ extends to a representation of ${}^LG$.

The principal $SL_2 \to \hat{G}$

The group $\hat{G}$ also comes equipped with a principal $\varphi : SL_2 \to \hat{G}$ [G]. The co-character $\mathbf{G}_m \to \hat{T}$ given by the restriction of φ to the maximal torus $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ of SL_2 is equal to 2ρ in $\mathrm{Hom}(\mathbf{G}_m, \hat{T}) = \mathrm{Hom}(T, \mathbf{G}_m)$. From this, and Proposition 5.1, we conclude the following.

Proposition 6.1. *The restriction of the minuscule representation V to the principal SL_2 in $\hat{G}$ has weights*

$$\bigoplus_{W/W_\lambda} t^{\langle w\lambda, 2\rho\rangle}$$

for the maximal torus $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in SL_2.

On the other hand, by Proposition 3.1, we have

$$\langle w\lambda, 2\rho\rangle = \langle \lambda, 2\rho\rangle - 2d(y) = N - 2d(y) \tag{6.2}$$

where $d(y)$ is the length of the minimal representative y in the coset wW_λ. Hence the weights for the principal SL_2 acting on V are the integers

$$N - 2d(y) \qquad y \in Y \tag{6.3}$$

in the interval $[-N, N]$. Since these are also the weights of the Lefschetz SL_2 acting on the cohomology $H^*(G/P)$ by §4, we obtain the following.

Corollary 6.4. *The representation of the principal SL_2 of $\hat{G}$ on V is isomorphic to the representation of the Lefschetz SL_2 on the cohomology of the flag variety $X = G/P$.*

7. Examples

We now give several examples of the preceding theory, using the notation for roots and weights of [B].

If G is of type A_ℓ and $\alpha = \alpha_1$ we have $\lambda = e_1$. The flag variety G/P is projective space $\mathbf{P}^N$, with $N = \ell$, and the Poincaré polynomial is $1+t+t^2+\cdots+t^N$. The dual group $\hat{G}$ is SL_{N+1}, and V is the standard representation. The restriction of V to a principal SL_2 is irreducible, isomorphic to $S^N = \mathrm{Sym}^N(\mathbf{C}^2)$.

A similar result holds when G is of type B_ℓ, so $\alpha = \alpha_1$ and $\lambda = e_1$. Here G/P is a quadric of dimension $N = 2\ell - 1$, with $P(t) = 1 + t + \cdots + t^N$ as before. The dual group is $\hat{G} = \mathrm{Sp}_{2\ell}$, the representation V is the standard representation, and its restriction to the principal SL_2 is the irreducible representation S^N.

Next, suppose G is of type D_ℓ and $\alpha = \alpha_1$, so $\lambda = e_1$. Then G/P is a quadric of dimension $N = 2\ell - 2$, and we have $P(t) = 1 + t + \cdots + 2t^{\ell-1} + \cdots + t^N$. The dual group $\hat{G}$ is $\mathrm{Spin}_{2\ell}$, and V is the standard representation of the quotient $SO_{2\ell}$. Its restriction to the principal SL_2 is a direct sum $S^N + S^0$, where S^0 is the trivial representation.

A more interesting case is when G is of type C_ℓ, so $\alpha = \alpha_\ell$ and $\lambda = \frac{e_1+e_2+\cdots+e_\ell}{2}$. Here G/P is the Lagrangian Grassmanian of dimension $N = \frac{\ell(\ell+1)}{2}$, and $P(t) = (1+t)(1+t^2)\ldots(1+t^\ell)$ was calculated in (4.4). The dual group $\hat{G}$ is $\mathrm{Spin}_{2\ell+1}$, and V is the spin representation of dimension 2^ℓ. Its decomposition to a principal SL_2 is given by §6, and we find the following representations, for $\ell \leq 6$.

$$
\begin{array}{rl}
S^1 & \ell = 1 \\
S^3 & \ell = 2 \\
S^6 + S^0 & \ell = 3 \\
S^{10} + S^4 & \ell = 4 \\
S^{15} + S^9 + S^5 & \ell = 5 \\
S^{21} + S^{15} + S^{11} + S^9 + S^3 & \ell = 6
\end{array}
\tag{7.1}
$$

As the last example, suppose G is of type E_6. Then G/P has dimension 16 and Poincaré polynomial

$$\begin{aligned} P(t) = 1 + t + t^2 + t^3 + 2t^4 + 2t^5 + 2t^6 + 2t^7 + 3t^8 \\ + 2t^9 + 2t^{10} + 2t^{11} + 2t^{12} + t^{13} + t^{14} + t^{15} + t^{16}. \end{aligned}$$

The representation V has dimension 27, and its restriction to a principal SL_2 is the representation

$$S^{16} + S^8 + S^0. \tag{7.2}$$

Proposition 7.3. *The representation V of the principal SL_2 is irreducible, hence isomorphic to S^N, if and only if G is of type A_ℓ or B_ℓ and $\alpha = \alpha_1$.*

The representation V of the principal SL_2 is isomorphic to $S^N + S^0$ if and only if G is of type D_ℓ and $\alpha = \alpha_1$, or G is of type D_4 and $\alpha = \alpha_3$ or α_4, or G is of type C_3 and $\alpha = \alpha_3$.

Proof. The condition $V = S^N$ as a representation of SL_2 is equivalent to the equality

$$\dim V = (W : W_\lambda) = N + 1.$$

The condition $V = S^N + S^0$ as a representation of SL_2 is equivalent to the equality

$$\dim V = (W : W_\lambda) = N + 2.$$

One obtains all the above cases by a consideration of the columns in Table 1.4.

8. Discrete series and a mirror theorem

Let G_0 be the real form of G described in §2, and let $G_0(\mathbf{R})^+$ be the connected component of $G_0(\mathbf{R})$. The L-packet of discrete series representations π^+ of $G_0(\mathbf{R})^+$ with infinitesimal character the W-orbit of ρ is in canonical bijection with the coset space W_λ/W. Indeed, W_λ is the compact Weyl group of the simply-connected algebraic cover G_0^{sc} of G_0, and any discrete series for $G_0^{sc}(\mathbf{R})$ with infinitesimal character ρ has trivial central character, so descends to the quotient group $G_0(\mathbf{R})^+$. On the other hand, such discrete series for $G_0^{sc}(\mathbf{R})$ are parameterized by their Harish-Chandra parameters in $\mathrm{Hom}(T_c^{sc}(\mathbf{R}), S^1)/W_\lambda$, which lie in the W-orbit of ρ. The coset $W_\lambda\rho$ corresponds to the holomorphic discrete series, and the coset $W_\lambda w_0\rho = W_\lambda w_0^{-1}\rho$ corresponds to the anti-holomorphic discrete series.

Proposition 8.1. ([V-Z, Prop. 6.19]) *Assume that the discrete series π^+ of $G_0(\mathbf{R})^+$ has Harish-Chandra parameter $W_\lambda w^{-1}\rho$. Then π^+ has bigraded cohomology*

$$H^{p,q}(\mathfrak{g}, K^+; \pi^+) \simeq \mathbf{C}$$

for $p+q = N$ and $q = d(y)$, the length of the minimal representative of wW_λ. The cohomology of π vanishes in all other bidegrees (p', q').

Proof. The bigrading of the $(\mathfrak{g}, K^+)$ cohomology of any π^+ in the L-packet is discussed in [V-Z, (6.18)(a-c)]. The cohomology has dimension 1 for degree N, and dimension 0 otherwise, so we must have $p+q = N$.

On the other hand, Arthur (cf. [A, pgs. 62–63]) interprets the calculation of [V-Z, Prop 6.19] to obtain the formula:

$$-\frac{1}{2}(p-q) = \langle \lambda, w^{-1}\rho \rangle = \langle w\lambda, \rho \rangle.$$

Since $\frac{1}{2}(p+q) = \langle \lambda, \rho \rangle$, we find that $q = \langle \lambda, \rho \rangle - \langle w\lambda, \rho \rangle = d(y)$, by Proposition 3.1.

If $G_0(\mathbf{R}) \neq G_0(\mathbf{R})^+$, the discrete series π for $G_0(\mathbf{R})$ with infinitesimal character ρ correspond to the coset space W_c/W, where W_c is the (nontrivial) normalizer of W_λ in

W. We find that the bigraded cohomology of π with Harish-Chandra parameter $W_c w^{-1}\rho$ is the direct sum of two lines of type (p,q) and (q,p), with $p+q=N$ and $q=d(y)$.

The 2-group K/K^+ acts on $H^N(\mathfrak{g},K^+,\pi)$, switching the two lines. When $p=q=N/2$, there is a unique line in $H^{p,p}(\mathfrak{g},K^+,\pi)$ fixed by K/K^+.

A suggestive way to restate the calculation of the bigraded cohomology is the following.

Corollary 8.2. *The Hodge structure on the sum* $H^N(G_0)=\bigoplus_\pi H^{*,*}(\mathfrak{g},K^+,\pi)$ *over the L-packet of discrete series for* $G_0(\mathbf{R})$ *with infinitesimal character* ρ *mirrors the Hodge structure on* $H^*(G/P)$. That is:

$$\dim H^{q,q}(G/P)=\dim H^{N-q,q}(G_0)$$

Indeed, both dimensions are equal to the number of classes wW_λ in W/W_λ with $d(w,W_\lambda)=q$.

9. Discrete series for $SO(2, 2n)$

Assume that G is of type D_{n+1}, with $n \geq 2$, and that $\alpha = \alpha_1$. The group $G_0(\mathbf{R})$ is then isomorphic to $PSO(2, 2n) = SO(2, 2n)/\langle \pm 1 \rangle$, and $\mathcal{D}$ is a tube domain of complex dimension $N = 2n$. There are $n + 1$ discrete series representations π of $G_0(\mathbf{R})$ with infinitesimal character ρ. We will describe these as representations of $SO(2, 2n)$, with trivial central character, and will calculate their minimal K^+-types and Hodge cohomology.

Let V be a 2-dimensional real vector space, with a positive definite quadratic form, and write $-V$ for the same space, with the negative form. For $k = 0, 1, \ldots, n$ define the quadratic space

$$W_k = V_0 + V_1 + \cdots + (-V_k) + \cdots + V_n,$$

so $SO(W_k) \simeq SO(2, 2n)$, a maximal compact torus T_c in $SO(W_k)$ is given by $\prod_{i=0}^{n} SO(V_i)$, and a maximal compact, connected subgroup K^+ containing T_c is given by $SO(V_k) \times SO(V_k^{\perp})$. If e_i is a generator of $\mathrm{Hom}(SO(V_i), S^1)$, then the character group of T_c is $\bigoplus_{i=0}^{n} \mathbf{Z}e_i$, and the roots of T_c on $\mathfrak{g}$ are the elements

$$\gamma_{ij} = \pm e_i \pm e_j \qquad i \neq j.$$

The compact roots of T_c on k are those roots γ_{ij} with $i \neq k+1$ and $j \neq k+1$, so the $(k+1)$st coordinate of γ is zero.

A set of positive roots is given by

$$R^+ = \{e_i \pm e_j : i < j\}.$$

This has root basis

$$\Delta = \{e_0 - e_1, e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$$

and

$$\rho = (n, n-1, n-2, \ldots, 1, 0).$$

On the other hand, half the sum ρ_c of the compact positive roots is given by

$$\rho_c = (n-1, n-2, \dots, n-k, 0, n-k-1, k \dots, 1, 0).$$

At the two extremes, we find:

$$k = 0 \qquad \rho_c = (0, n-1, n-2, \dots, 1, 0)$$

$$k = n \qquad \rho_c = (n-1, n-2, \dots, 1, 0, 0).$$

The lowest K^+-type of a discrete series π^+ for $SO(2,2n)^+$ with Harish-Chandra parameter $\lambda = \rho$ is given by Schmid's formula:

$$\lambda + \rho - 2\rho_c = 2(\rho - \rho_c).$$

For the realizations $SO(2,2n) \simeq SO(W_k)$ above, we obtain $n+1$ discrete series π_k^+ with minimal $K^+ \simeq SO(2) \times SO(2n)$ type

$$\chi^{2(n-k)} \otimes (2, \underset{\substack{\uparrow \\ k \text{ times}}}{2}, 2, \dots, 2, 0, 0 \dots 0)$$

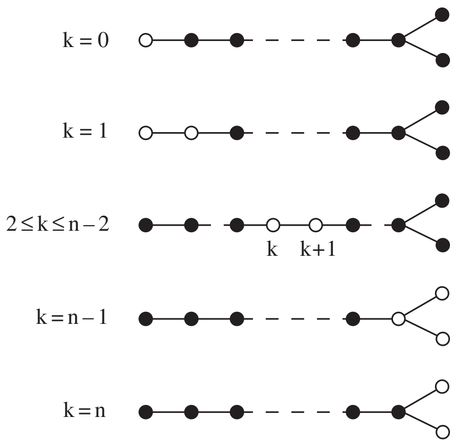
where χ is the fundamental character of $SO(2)$, giving the action on $\mathfrak{p}^+$. The irreducible representation of $SO(2n)$ with highest weight $2(e_1 + \cdots + e_k)$ appears with multiplicity 1 in $\mathrm{Sym}^2(\overset{k}{\wedge}\mathbf{C}^{2n})$, and the minimal K^+-type appears with multiplicity 1 in the representation $\overset{k}{\wedge}\mathfrak{p}_- \otimes \overset{2n-k}{\wedge}\mathfrak{p}_+$. Hence the Hodge type of π_k^+ is $(2n-k, k)$.

Each discrete series π_k of $SO(2,2n)$ with infinitesimal character ρ decomposes as $\pi_k = \pi_k^+ + \pi_k^-$ when restricted to $SO(2,2n)^+$, with π_k^+ as above, and π_k^- its conjugate by $G_0(\mathbf{R})/G_0(\mathbf{R})^+$. The minimal K^+-type of π_k^- is

$$\chi^{2(k-n)} \otimes (2, \underset{\substack{\uparrow \\ k \text{ times}}}{2}, 2, \dots, 2, 0, 0 \dots 0)$$

so π_k^- has Hodge type $(k, 2n-k)$, and π_k has Hodge type $(k, 2n-k) + (2n-k, k)$.

If we label the simple roots in the Dynkin diagram for G: white for non-compact roots, black for compact roots, then the discrete series π_k of $SO(2,2n)$ gives the labelled diagram



In the case $k = 0$, π_k is the sum of holomorphic and anti-holomorphic discrete series, and is an admissible representation of the subgroup $SO(2) \subset K^+$. In the case $k = n$, π_n is admissible for the subgroup $SO(2n) \subset K^+$, and has Hodge type $(n, n) + (n, n)$.

10. A classification theorem: $V = \mathbf{C}e + V_0$

We now return to the restriction of a minimal representation V of $\hat{G}$ to a principal SL_2 in $\hat{G}$. Since V will be fixed, we will replace the simply-connected group $\hat{G}$ by its quotient which acts *faithfully* on V, and will henceforth use the symbol $\hat{G}$ for this subgroup of $GL(V)$. The group G is therefore no longer necessarily of adjoint type. We have

$$X_\bullet(T) = \mathbf{Z}\lambda + \bigoplus_{\text{co}-\text{roots}} \mathbf{Z}\alpha^\vee \tag{10.1}$$

and $\ell\lambda$ lies in the sublattice $\oplus\mathbf{Z}\alpha^\vee$, with ℓ the order of the (cyclic) center of $\hat{G}$. Since $\langle \alpha^\vee, \rho\rangle$ is an integer for all co-roots, we find that ρ is in $X^\bullet(T)$ if and only if $\langle \lambda, \rho\rangle$ is an integer. By (1.3) this occurs precisely when the integer $N = \dim_{\mathbf{C}}(\mathcal{D})$ is even. Since the center $\langle \pm 1\rangle$ of a principal SL_2 in $\hat{G}$ acts on V by the character $(-1)^N$, we see that ρ is in $X^\bullet(T)$ precisely when principal homomorphism $SL_2 \to \hat{G}$ factors through the quotient group PGL_2.

Proposition 10.3. *Assume that there is a non-zero linear form $t : V \to \mathbf{C}$ which is fixed by the principal $SL_2 \to \hat{G}$, and that the subgroup $\hat{H}$ of $\hat{G}$ fixing t acts* **irreducibly** *on the hyperplane $V_0 = \ker(t)$.*

Then (up to the action of the outer automorphism group of the simply-connected cover of $\hat{G}$) the representation V is given by the following table

$\hat{G}$	V	$\hat{H}$
SL_{2n}/μ_2	$\overset{2}{\wedge}\mathbf{C}^{2n}$	Sp_{2n}/μ_2
SO_{2n}	$\mathbf{C}^{2n}$	SO_{2n-1}
E_6	$\mathbf{C}^{27}$	F_4
Spin_7	$\mathbf{C}^8$	G_2

Proof. By definition, $\hat{H}$ contains the image of the principal SL_2 (which is isomorphic to PGL_2). These subgroups of simple $\hat{G}$ have been classified by de Siebenthal [dS]. One has the chains:

$$SL_2 \to SO_{2n+1} \to SL_{2n+1}$$

$$SL_2 \to Sp_{2n} \to SL_{2n}$$

$$SL_2 \to SO_{2n-1} \to SO_{2n}$$

$$SL_2 \to F_4 \to E_6$$

$$SL_2 \to G_2 \to \mathrm{Spin}_7 \to SO_8$$

$$SL_2 \to G_2 \to SO_7 \to SL_7$$

It is then a simple matter to check, for any V, whether an $\hat{H}$ containing the principal SL_2 can act irreducibly on V_0.

Beyond the examples given in Proposition 10.3, we have one semi-simple example with the same properties:

$$(10.4) \qquad \hat{G} = SL_n^2/\Delta\mu_n \qquad V = \mathbf{C}^n \otimes (\mathbf{C}^n)^* \qquad \hat{H} = PGL_n.$$

In all cases, $\hat{H}$ is a group of adjoint type.

Proposition 10.5. *For the groups $\hat{G}$ in Proposition 10.3, the center is cyclic of order $\ell \geq 2$. The integer ℓ is the number of irreducible representations in the restriction of V to a principal SL_2.*

The $\hat{G}$-invariants in the symmetric algebra on V^* form a polynomial algebra, on one generator $d : V \to \mathbf{C}$ of degree ℓ. The group $\hat{G}$ has an open orbit on the projective space of lines in V, with connected stabilizer $\hat{H}$, consisting of the lines where $d(v) \neq 0$.

Proof. The first assertion is proved by an inspection of the following table. We derive the decomposition of V from §6.

Table 10.6

$\hat{G}$	$\ell =$ order of center	decomp. of V
SL_{2n}/μ_2	$n \geq 2$	$S^{4n-4} + S^{4n-8} + \cdots + S^4 + S^0$
SO_{2n}	2	$S^{2n-2} + S^0$
E_6	3	$S^{16} + S^8 + S^0$
Spin_7	2	$S^6 + S^0$
$SL_n^2/\Delta\mu_n$	$n \geq 2$	$S^{2n-2} + S^{2n-4} + \cdots + S^2 + S^0$

The calculation of $S^\bullet(V^*)^{\hat{G}}$ follows from [S-K], which also identifies the connected component of the stabilizer with $\hat{H}$. Note that the degree of any invariant is divisible by ℓ, as the center acts faithfully on V^*.

11. The representation V of $\hat{H}$

Recall that $\ell \geq 2$ is the order of the cyclic center of $\hat{G}$, tabulated in 10.6. Since the subgroup $\hat{H} \subset \hat{G}$ fixing the linear form $t: V \to \mathbf{C}$ is reductive, we have a splitting of $\hat{H}$-modules

$$V = \mathbf{C}e + V_0 \tag{11.1}$$

with $V_0 = \ker(t)$, and e a vector fixed by $\hat{H}$ satisfying $t(e) \neq 0$. Once t has been chosen, we may normalize e by insisting that

$$t(e) = \ell. \tag{11.2}$$

Proposition 11.3. *The representation V_0 of $\hat{H}$ is orthogonal. Its weights consist of the short roots of $\hat{H}$ and the zero weight. The zero weight space for $\hat{H}$ in V has dimension ℓ, and V is a polar representation of $\hat{H}$ of type $A_{\ell-1}$: the $\hat{H}$-invariants in the symmetric algebra of $V \simeq V^*$ form a polynomial algebra, with primitive generators in degrees $1, 2, 3, \ldots, \ell$.*

Proof. The fact that V_0 is orthogonal, and its weights, are obtained from a consideration of the table in Proposition 10.1. Since

$$\dim V = \ell + \#\{\text{short roots of } \hat{H}\},$$

this gives the dimension of the zero weight space.

Let $\hat{S} \subset \hat{H}$ be a maximal torus, with normalizer $\hat{N}$. The image of $\hat{N}/\hat{S}$ in $GL(V^{\hat{S}}) = GL_\ell$ is the symmetric group Σ_ℓ. The fact that V is polar follows from the tables in [D-K], which also gives an identification of algebras: $S^\bullet(V)^{\hat{H}} \simeq S^\bullet(V^{\hat{S}})^{\hat{N}/\hat{S}}$. The latter algebra is generated by the elementary symmetric functions, of degrees $1, 2, 3, \ldots, \ell$.

Note 11.4. The integer ℓ is also the number of distinct summands in the restriction of V to a principal SL_2. Since each summand is an orthogonal representation of SL_2, $\ell = \dim V^{\hat{S}_0}$, where $\hat{S}_0 \subset SL_2$ is a maximal torus. Hence $V^{\hat{S}_0} = V^{\hat{S}}$.

We will now define an $\hat{H}$-algebra structure on V, with identity element e, in a case by case manner. Although the multiplication law $V \otimes V \to V$ is not in general associative, it is power associative, and for $v \in V$ and $k \geq 0$ we can define v^k in V unambiguously. The primitive $\hat{H}$-invariants in $S^\bullet(V^*)$ can then be given by

$$v \mapsto t(v^k) \qquad 1 \leq k \leq \ell. \tag{11.5}$$

In (11.5), $t\colon V \to \mathbf{C}$ is the $\hat{H}$-invariant linear form, normalized by the condition that

$$t(e) = \ell.$$

We will also identify the $\hat{G}$-invariant ℓ-form $\det\colon V \to \mathbf{C}$, normalized by the condition that

$$\det(e) = 1.$$

The simplest case, when the algebra structure on V is associative, is when $\hat{H} = PGL_n$ and V is the adjoint representation (of GL_n) on $n \times n$ matrices. The algebra structure is matrix multiplication, e is the identity matrix, t is the trace, and det is the determinant (which is invariant under the larger group $\hat{G} = SL_n \times SL_n/\Delta\mu_n$ acting by $v \mapsto AvB^{-1}$).

Another algebra structure on V, with the same powers v^k, is given by the Jordan multiplication $A \circ B = \frac{1}{2}(AB + BA)$. This algebra is isomorphic to the Jordan algebra of Hermitian symmetric $n \times n$ matrices over the quadratic $\mathbf{C}$-algebra $\mathbf{C} + \mathbf{C}$, with involution $\overline{(z, w)} = (w, z)$.

The representation V has a similar Jordan algebra structure when $\hat{H} = PSp_{2n}$ and when $\hat{H} = F_4$. In the first case, V is the algebra of Hermitian symmetric $n \times n$ matrices over the complex quaternion algebra $M_2(\mathbf{C})$, and in the second V is the algebra of Hermitian symmetric 3×3 matrices over the complex octonion algebra.

When $\hat{H} = SO_{2n-1}$, the representation $V = \mathbf{C}e + V_0$ has a Jordan multiplication given by the quadratic form $\langle , \rangle$ on V. We normalize this bilinear paring to satisfy $\langle e, e \rangle = 2$, so $\det(v) = \frac{\langle v, v \rangle}{2}$ is the $\hat{G}$-invariant 2-form on V. The multiplication is defined, with e as identity, by giving the product of two vectors v, w in V_0 : $v \circ w = -\frac{1}{2}\langle v, w \rangle e$.

Finally, when $\hat{H} = G_2$, the representation V of dimension 8 has the structure of an octonion algebra, with $t(v) = v + \bar{v}$ and $\det(v) = v\bar{v}$. In all cases but this one $\hat{G}$ is the connected subgroup of $GL(V)$ preserving det, and $\hat{H}$ is the subgroup of $GL(V)$ preserving all the forms $t(v^k)$ for $1 \leq k \leq \ell$. In the octonionic case, the subgroup $SO_8 \subset GL_8$ preserves det, and the subgroup $SO_7 \subset SO_8$ preserves $t(v)$ and $t(v^2)$.

In general, $\det\colon V \to \mathbf{C}$ is a polynomial in the $\hat{H}$-invariants $t(v^k)$, given by the Newton formulae. The expression for $\ell! \cdot \det$ has integral coefficients; for example:

$$\begin{cases} 2\det(v) = t(v)^2 - t(v^2) & \ell = 2 \\ 6\det(v) = t(v)^3 - 3t(v^2)t(v) + 2t(v^3) & \ell = 3. \end{cases} \tag{11.6}$$

12. Representations of G lifted from H

We now describe the finite dimensional irreducible holomorphic representations π of G which are lifted from irreducible representations π' of H. This notion of lifting is due to Langlands: the parameter of π, which is a homomorphism $\varphi : \mathbf{C}^* \to \hat{G}$ up to conjugacy, should factor through a conjugate of $\hat{H}$.

We can parameterize the finite dimensional irreducible holomorphic representations π of G by their highest weights ω for B. The weight ω is a positive, integral combination of the fundamental weights ω_i of the simply-connected cover of G, so we may write (using the numeration of [B])

$$\omega = \sum_{i=1}^{\mathrm{rank}(G)} b_i\omega_i \qquad b_i \geq 0. \tag{12.1}$$

For ω to be a character of G, there are some congruences which must be satisfied by the coefficients b_i. (The group G is *not* simply connected, as its dual $\hat{G}$ acts faithfully on the minuscule representation V.)

Since

$$\mathrm{rank}(G) = \mathrm{rank}(H) + (\ell - 1), \tag{12.2}$$

there are $(\ell - 1)$ linear conditions on the coefficients b_i which are necessary and sufficient for π to be lifted from π' of H. These conditions refine the congruences, and we tabulate them below.

Table 12.3

G	H	$\omega = \sum b_i\omega_i$ of G	ω lifted from H
SL_{2n}/μ_n	Spin_{2n+1}	$\sum_{i=1}^{n-1} i(b_i - b_{2n-i}) \equiv 0(n)$	$b_i = b_{2n-i}$ $1 \leq i \leq n-1$
SO_{2n}	Sp_{2n-2}	$b_{n-1} - b_n \equiv 0(2)$	$b_{n-1} = b_n$
E_6/μ_3	F_4	$(b_1 - b_6) + 2(b_2 - b_5) \equiv 0(3)$	$b_1 = b_6$ $b_2 = b_5$
Sp_6/μ_2	G_2	$b_1 - b_3 \equiv 0(2)$	$b_1 = b_3$
$SL_n \times SL'_n/\Delta\mu_n$	SL_n	$\sum_{i=1}^{n-1} i(b_i - b'_{n-1}) \equiv 0(n)$	$b_i = b'_{n-1}$ $1 \leq i \leq n-1$

When $G = SL_{2n}/\mu_n$, SO_{2n}, or Sp_6/μ_2 there are more classical descriptions of ω in the weight spaces $\mathbf{R}^{2n}$, $\mathbf{R}^n$, and $\mathbf{R}^3$, respectively. We describe, in this language, which representations are lifted from H.

For $G = SL_{2n}/\mu_n$, a dominant weight ω is a vector $(a_1, a_2, \ldots, a_{2n})$ in $\mathbf{R}^{2n}$ with

$$a_1 \geq a_2 \geq \cdots \geq a_{2n}$$

$$a_i \quad \text{in} \quad 1/2\,\mathbf{Z} \qquad 1 \leq i \leq 2n$$

$$a_i \equiv a_j \pmod{\mathbf{Z}}$$

$$\sum a_i = 0.$$

The representations lifted from Spin_{2n+1} give dominant weights ω with

$$a_i + a_{2n+1-i} = 0 \qquad 1 \leq i \leq n.$$

In particular: $a_n \geq 0 \geq a_{n+1}$, as $a_n + a_{n+1} = 0$.

For $G = SO_{2n}$, a dominant weight ω is a vector $(a_1, \ldots, a_n)$ in $\mathbf{Z}^n$ with

$$a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n|.$$

The representations lifted from Sp_{2n-2} satisfy $a_n = 0$.

Finally, for $G = Sp_6/\mu_2$, a dominant weight is given classically as a vector (a_1, a_2, a_3) in $\mathbf{Z}^3$ with $a_1 \geq a_2 \geq a_3 \geq 0$ and $a_1 \equiv a_2 + a_3 \pmod 2$. The representations lifted from G_2 are those with $a_1 = a_2 + a_3$.

Define a connected, reductive subgroup L of G as follows.

$G = SL_{2n}/\mu_n$	$L = SL_n^2/\Delta\mu_n$	fixing a decomposition of the standard representation of $SL_{2n} : \mathbf{C}^{2n} = \mathbf{C}^n + \mathbf{C}^n$, and having determinant 1 on each factor
$G = SO_{2n}$	$L = SO_{n+1}$	fixing a non-degenerate subspace $\mathbf{C}^{n-1}$ in the standard representation $\mathbf{C}^{2n}$.
$G = E_6/\mu_3$	$L = SL_6/\mu_3$	fixing the highest and lowest root spaces in the adjoint representation
$G = Sp_6/\mu_2$	$L = SL_2^3/\Delta\mu_2$	fixing a decomposition of the standard representation of $Sp_6 : \mathbf{C}^6 = \mathbf{C}^2 + \mathbf{C}^2 + \mathbf{C}^2$ into three non-degenerate, orthogonal subspaces
$G = SL_n^2/\Delta\mu_n$	$L = PGL_n$	fixing the identity matrix in the representation on $M_n(\mathbf{C})$

Proposition 12.4. *The finite dimensional irreducible representation π of G is lifted from H if and only if the space $\mathrm{Hom}_L(\pi, \mathbf{C})$ of L-invariant linear forms on π is non-zero. In this case, the dimension of the space of L-invariant linear forms is given by the following table:*

Table 12.5

G	ω lifted from H	$\dim \mathrm{Hom}_L(\pi, \mathbf{C})$
SL_{2n}/μ_n	$b_1(\omega_1+\omega_{2n-1})+b_2(\omega_2+\omega_{2n-2})+\cdots$ $\cdots+b_{n-1}(\omega_{n-1}+\omega_{n+1})+b_n\omega_n$	b_n+1
SO_{2n}	$b_1\omega_1+b_2\omega_2+\cdots+b_{n-2}\omega_{n-2}+$ $b_{n-1}(\omega_{n-1}+\omega_n)$	$\prod_{1\le i<j\le n-2} \frac{b_1+b_2+\cdots+b_{j-1}+j-i}{j-i}$
E_6/μ_3	$b_1(\omega_1+\omega_6)+b_3(\omega_3+\omega_5)+$ $b_2\omega_2+b_4\omega_4$	$\frac{(b_2+1)(b_4+1)(b_2+b_4+2)}{2}$
Sp_6/μ_2	$b_1(\omega_1+\omega_3)+b_2\omega_2$	b_2+1
$SL_n^2/\Delta\mu_n$	$V\otimes V^*$	1

13. The proof of Proposition 12.4

The only easy case is when $G = SL_n^2/\Delta\mu_n$, so an irreducible π has the form $V \otimes V'$, where V and V' are irreducible representations of SL_n with inverse central characters. We have

$$\mathrm{Hom}_L(\pi, \mathbf{C}) = \mathrm{Hom}_{SL_n}(V \otimes V', \mathbf{C})$$

This space is non-zero if and only if $V' \simeq V^*$, when it has dimension 1 by Schur's lemma. These are exactly the π lifted from H.

When $G = Sp_6/\mu_2$ and $L = SL_2^3/\mu_2$, the space $\mathrm{Hom}_L(\pi, \mathbf{C})$ was considered in [G-S]. In the other cases, the subgroup L may be obtained as follows. Let $G_{\mathbf{R}}$ be the quasi-split inner form of G with non-trivial Galois action on the Dynkin diagram, and let $K_{\mathbf{R}}$ be a maximal compact subgroup of $G_{\mathbf{R}}$. We have

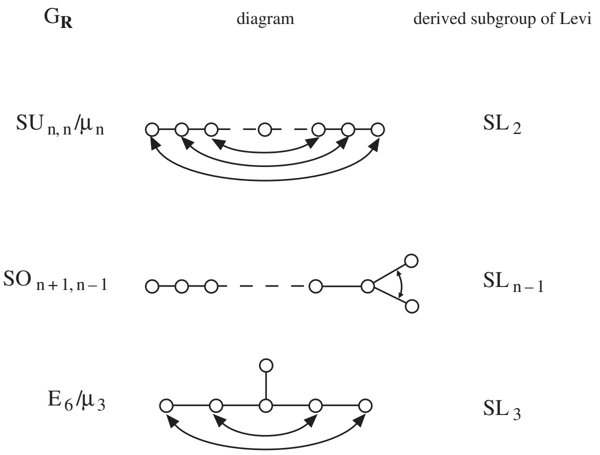
G	$G_{\mathbf{R}}$	$K_{\mathbf{R}}$
SL_{2n}/μ_n	$SU_{n,n}/\mu_n$	$S(U_n \times U_n)/\mu_n$
SO_{2n}	$SO_{n+1,n-1}$	$S(O_{n+1} \times O_{n-1})$
E_6/μ_3	${}^2E_{6,4}/\mu_3$	$(SU_2 \times SU_6/\mu_3)/\Delta\mu_2$

Note that in each case we have a homomorphism

$$L \hookrightarrow K = \text{complexification of } K_{\mathbf{R}}.$$

The image is a normal subgroup, and the connected component of the quotient is isomorphic to SO_2, SO_{n-1}, and SO_3, respectively.

There is a real parabolic $P_{\mathbf{R}}$ in $G_{\mathbf{R}}$ associated to the fixed vertices of the Galois action on the Dynkin diagram. The derived subgroup of a Levi factor of $P_{\mathbf{R}}$ is given as follows.



Let $B_{\mathbf{R}}$ be the Borel subgroup of $G_{\mathbf{R}}$ contained in $P_{\mathbf{R}}$, and let $T_{\mathbf{R}}$ be a Levi factor of $B_{\mathbf{R}}$.

In the Cartan-Helgalson theorem, one uses the Cartan decomposition $G_{\mathbf{R}} = K_{\mathbf{R}} \cdot B_{\mathbf{R}}$ to show that K has an open orbit on the complex flag variety G/B, with stabilizer the subgroup T^θ of T fixed by the Cartan involution. The representations π of G with $\mathrm{Hom}_K(\pi, \mathbf{C}) \neq 0$ are those whose highest weight χ is trivial on T^θ, in which case $\mathrm{Hom}_K(\pi, \mathbf{C})$ has dimension 1. This is proved in [G-W, 12.3], where the subgroup T^θ is also calculated.

Similarly, one shows that the subgroup L of K has an open orbit on the flag variety G/P, with stabilizer the connected component $(T^\theta)^0$ of T^θ, which is a torus. The representations π of G with $\mathrm{Hom}_L(\pi, \mathbf{C}) \neq 0$ are those whose highest weight χ is trivial on $(T^\theta)^0$. These one finds, after a brief calculation, are those lifted from H. The space $\mathrm{Hom}_L(\pi, \mathbf{C})$ is isomorphic, as a representation of K/L, to the irreducible representation of the Levi factor of P which has highest weight χ. This completes the proof.

14. The real form of L

We now descend the subgroup $L \to G$ defined before Proposition 12.4 to a subgroup $L_0 \to G_0$ over $\mathbf{R}$, by using minuscule co-weights. Let S be a maximal torus in L, and let $\lambda : \mathbf{G}_m \to T$ be a minuscule co-weight which occurs in the representation V of $\hat{G}$.

Proposition 14.1. *There is an inclusion $\alpha : L \to G$ mapping S into T, and a minuscule co-weight $\mu : \mathbf{G}_m \to S$ of L, such that the following diagram commutes*

$$\begin{array}{ccccc} \mathbf{G}_m & \underset{\mu}{\to} & S & \to & L \\ \| & & \downarrow \alpha & & \downarrow \alpha \\ \mathbf{G}_m & \underset{\lambda}{\to} & T & \to & G \end{array}$$

Proof. If $\alpha_0 : L \to G$ is any inclusion the image of S is contained in a maximal torus T_0 of G. Since T and T_0 are conjugate, we may conjugate α_0 to an inclusion $\alpha : L \to G$ mapping S into T.

The co-character group $X_\bullet(S)$ then injects into $X_\bullet(T)$. To finish the proof, we must identify the image, and show that it intersects the W-orbit of λ in a single W_L-orbit of minuscule co-weights for L. We check this case by case. For example, if $G = E_6/\mu_3$ and $L = SL_6/\mu_3$, the group $X_\bullet(T)$ is the dual $E_6^\vee$ of the E_6-root lattice, and $X_\bullet(S)$ is the subgroup orthogonal to a root β. One checks, using the tables in Bourbaki [B], that precisely 15 of the 27 elements in the orbit $W\lambda$ are orthogonal to each β, and that these give a single $W_\beta = W_{SL_6}$ orbit.

In each case, we tabulate the dimension of T/S, and the size of the W_L-orbit $W\lambda \cap X_\bullet(S) = W_L\mu$

Table 14.2

G	L	$\dim(T/S)$	$\#W\lambda$	$\#W_L\mu$	
SL_{2n}/μ_n	SL_n^2/μ_n	1	$2n^2-n$	n^2	
SO_{2n}	SO_{n+1}	$\frac{n+1}{2}$ n odd $\frac{n}{2}$ n even	$2n$	$n+1$ n	n odd n even
E_6/μ_3	SL_6/μ_3	1	27	15	
Sp_6/μ_2	SL_2^3/μ_2	0	8	8	
SL_n^2/μ_n	PGL_n	$n-1$	n^2	n	

Corollary 14.3. *If L_0 is the real form of L with Cartan involution $\theta = \mathrm{ad}\ \mu(-1)$, then L_0 embeds as a subgroup of G_0 over $\mathbf{R}$. The symmetric space $\mathcal{D}_L = L_0(\mathbf{R})/K_{L_0}$ has an invariant complex structure, and embeds analytically into $\mathcal{D}$. Moreover,*

$$\dim_{\mathbf{C}}\mathcal{D}_L = \frac{1}{2}\dim_{\mathbf{C}}\mathcal{D}.$$

The last inequality is checked, case by case. We tabulate G_0, L_0, $\dim\mathcal{D}$, and $\dim\mathcal{D}_L$ below

Table 14.4

G_0	L_0	$\dim\mathcal{D}$	$\dim\mathcal{D}_L$
$SU_{2,2n-2}/\mu_n$	$SU_{1,n-1}^2/\mu_n$	$4n-4$	$2n-2$
$SO_{2,2n-2}$	$SO_{2,n-1}$	$2n-2$	$n-1$
${}^2E_{6,2}/\mu_3$	$SU_{2,4}/\mu_3$	16	8
Sp_6/μ_2	SL_2^3/μ_2	6	3
$SU_{1,n-1}^2/\mu_m$	$PU_{1,n-1}$	$2n-2$	$n-1$

Since $\dim\, \mathcal{D}_L = \frac{1}{2} \dim\, \mathcal{D}$, this suggests the following problem. Let $G_{\mathbf{Q}}$ and $L_{\mathbf{Q}}$ be descents of G_0 and L_0 to $\mathbf{Q}$, with $L_{\mathbf{Q}} \hookrightarrow G_{\mathbf{Q}}$. This gives a morphism of Shimura varieties

$$S_L \to S_G$$

over $\mathbf{C}$, with $\dim(S_L) = \frac{1}{2} \dim\, S_G$. The algebraic cycles corresponding to S_L contribute to the middle cohomology $H^{\dim\, S_G}(S_G, \mathbf{C})$. Can these Hodge classes detect the automorphic forms lifted from H?

15. The group $\hat{G}$ in a Levi factor

Recall that the center μ_ℓ of $\hat{G}$ is cyclic. Let

$$\hat{J} = \mathbf{G}_m \times \hat{G}/\Delta\mu_\ell, \tag{15.1}$$

which is a group with connected center. We first observe that $\hat{J}$ is a Levi factor in a maximal parabolic subgroup $\hat{P}$ of a simple group of adjoint type $\hat{M}$. The minuscule representation V occurs as the action of $\hat{J}$ on the abelianization of the unipotent radical $\hat{U}$ of $\hat{P}$.

Recall that the maximal parabolic subgroups $\hat{P}$ of $\hat{M}$ are indexed, up to conjugacy, by the simple roots α. We tabulate $\hat{M}$, the simple root α corresponding to $\hat{P}$, and the representation $\hat{U}^{ab} = V$ below.

Table 15.2

$\hat{G}$	$\hat{M}$	α of $\hat{P}$	$V = U^{ab}$
SL_{2n}/μ_2	PSO_{4n}	α_{2n}	$\overset{2}{\wedge}\mathbf{C}^{2n}$
SO_{2n}	PSO_{2n+2}	α_1	$\mathbf{C}^{2n}$
E_6	E_7	α_7	$\mathbf{C}^{27}$
Spin_7	F_4	α_4	$\mathbf{C}^8$
SL_n^2/μ_n	PGL_{2n}	α_n	$\mathbf{C}^n \otimes \mathbf{C}^n$

Proposition 15.3. *The centralizer of $\hat{H}$ in $\hat{M}$ is SO_3, and $\hat{H} \times SO_3$ is a dual reductive pair in $\hat{M}$.*

This is checked case by case, and we list the pairs obtained below.

$$SO_3 \times PSp_{2n} \subset PSO_{4n}.$$

$$SO_3 \times SO_{2n-1} \subset PSO_{2n+2}$$

$$SO_3 \times F_4 \subset E_7$$

$$SO_3 \times G_2 \subset F_4$$

$$SO_3 \times PGL_n \subset PGL_{2n}.$$

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