Incoherent definite spaces and Shimura varieties

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1 Introduction

In [7] we considered the infinite set of quaternion algebras $B(v)$ over a totally real field k whose ramification locus has distance one from a finite set Σ of places of k which has odd cardinality. These algebras are indexed by the places v of k; the algebra $B(v)$ is ramified at the set $\Sigma \cup \{v\}$ when v is not contained in Σ, and is ramified at the set $\Sigma - \{v\}$ if v is contained in Σ. When the set Σ contains all the real places of k, each neighboring algebra $B(v)$ gives local information on a single Shimura curve S which is defined over k. For example, if v is a real place, the algebra $B(v)$ is indefinite at v and definite at all other real places, and Σ-arithmetic subgroups of the multiplicative group of $B(v)$ give an analytic description of the points of S over the quadratic extension $K_v = \mathbb{C}$ of the completion $k_v = \mathbb{R}$ (See [16] [17] for the construction of a canonical model for S over $k \hookrightarrow \mathbb{C}$ and [4] for the analytic description of S at other real places). If $v = p$ is a finite place where the curve has good reduction, the algebra $B(p)$ is definite at all real places and Σ-arithmetic subgroups of its multiplicative group give an analytic description of the points of S over the unramified quadratic extension K_p of the completion k_p which have supersingular reduction modulo p (cf. [1, §11]).

In this note, we generalize the notion of an odd set Σ of places of k, containing all the real places (which can be viewed as an incoherent definite quaternion data over k) to incoherent definite orthogonal and Hermitian data over k. The former is a collection of local orthogonal spaces V_v over the completions k_v of fixed rank $n \geq 3$ and determinant d in k^*/k^{*2}. We insist that the spaces V_v are positive definite for all real places of k (so the determinant d is totally positive in k^*), that the Hasse-Witt invariants of V_v are equal to $+1$ for almost all places v, and that the product of the Hasse-Witt invariants is equal to -1. There is no global orthogonal space over k with these completions at all places, for then the product of the Hasse-Witt invariants would be equal to +1 by Hilbert's reciprocity law. However, for each place v of k there is a global orthogonal space $V(v)$ of rank n and discriminant d which is locally isomorphic to V_u at all places $u \neq v$, and **not** locally isomorphic to V_v at v. The last condition determines the local orthogonal space when v is finite; for a real place we must also insist that the signature is equal to $(n-2, 2)$. The global space $V(v)$ is determined up to isomorphism by its localizations, by the Hasse-Minkowski theorem. We call the global orthogonal spaces $V(v)$, indexed by the places v of k, the neighbors of the incoherent definite data $\{V_v\}$.

An odd set Σ of places of k which contains all the real places gives incoherent definite orthogonal data of dimension $n = 3$ and determinant $d \equiv 1$. Indeed, for each place v, let B_v be the local quaternion algebra over k_v which is split if v is not in Σ and is ramified if v is in Σ. Let V_v be the local orthogonal space given by the norm form on elements of trace zero in B_v. The Hasse-Witt invariant of V_v is the product of the Hasse invariant of the quaternion algebra B_v with the class $(-1,-1)_v$ in the local Brauer group, so almost all of the Hasse-Witt invariants are equal to $+1$ and the product of Hasse-Witt invariants is equal to -1. The nearby quaternion algebras $B(v)$ defined above give the neighboring global orthogonal spaces $V(v)$.

There is a similar definition of an incoherent definite Hermitian data and its neighbors, where we fix the rank $n \geq 1$ of the space and a totally complex quadratic extension field K of k. In this case, we only define a neighboring Hermitian space $V(v)$ at the places v of k which are not split in K. After working through the definitions, we show how incoherent definite orthogonal data of dimension $n \geq 3$ determines a Shimura variety S of orthogonal type and dimension $n-2$, with field of definition k.

We show how incoherent definite Hermitian data of dimension $n \geq 1$ determines a Shimura variety S of unitary type and dimension $n-1$, with field of definition K. The special orthogonal groups $\mathrm{SO}(V(v))$ of the neighbors of orthogonal data at real places v can be used to describe S over the complex quadratic extension K_v of the completion k_v. Similarly, the unitary groups $U(V(v))$ of the neighbors of Hermitian data at real places v can be used to describe S over the complex completion K_v. Finally, at primes p of k where the orthogonal Shimura variety S has good reduction, we propose to use the neighboring orthogonal space $W(p)$ to describe the points over the unramified quadratic extension K_p of the completion k_p which have superspecial reduction modulo p. Similarly at the primes p of k which are inert in K and where the unitary Shimura variety S has good reduction, we propose to use the neighboring Hermitian space $V(p)$ to describe the points over the completion K_p which have superspecial reduction modulo p.

This paper represents my attempt to approach some of the work of Kudla and Rapoport on incoherent Eisenstein series and the arithmetic intersection of cycles on Shimura varieties. It is an expanded version of a letter that I wrote to Deligne in 2009.

2 Local and global orthogonal spaces

In this section, we review the classification of orthogonal spaces over number fields and their completions. These results are due to Minkowski, Witt, and Hasse. Some good references are [10] [14] [15, Ch IV].

We begin with a review of some invariants of an orthogonal space over a general field k, of characteristic not equal to 2. An orthogonal space (V, q) is a finite dimensional vector space V over k together with a quadratic form $q : V \to k$. The bilinear form

$$\langle v, w \rangle = q(v+w) - q(v) - q(w)$$

is symmetric, and we will always assume that it is non-degenerate (so gives an isomorphism from V to its dual). The first invariant of the space (V, q) is the rank, defined as the dimension of V.

We can choose a basis $\{e_1.e_2, \dots e_n\}$ of V whose elements are pairwise orthogonal so that

$$q(\sum x_i e_i) = \sum a_i x_i^2.$$

The coefficients a_i are all non-zero elements in k and their product

$$d = a_1 a_2 \cdots a_n$$

is well-defined in k^*/k^{*2}, independent of the orthogonal basis chosen. The class of d in k^*/k^{*2} is called the determinant of the quadratic space (V, q), and is the second invariant we will use. If the rank is equal to 1, the determinant d determines the orthogonal space up to isomorphism – the space is isometric to k with the form dx^2.

For two classes a_i and a_j in $H^1(k, \mu_2) = k^*/k^{*2}$, we let (a_i, a_j) denote their cup product in a $H^2(k, (\mu_2)^{\otimes 2}) = H^2(k, \mu_2) = Br_2(k)$. Here $Br(k) = H^2(k, \mathbb{G}_m)$ is the Brauer group of the field

k. The Witt invariant of the quadratic space (V, q) is defined as the sum in the Brauer group

$$w = \sum_{i<j} (a_i, a_j).$$

An important theorem, due to Witt, shows that this class is again independent of the orthogonal basis chosen. This is the third invariant of (V, q) we will need.

If the rank of (V, q) is equal to 2, the determinant d and the Witt invariant determine the orthogonal space up to isomorphism. In this case, the two dimensional k vector space V has the structure of a free module of rank one over the étale algebra $K = k[t]/(t^2 + d)$ (in two different ways). Indeed, any element in the special orthogonal group of V which is not a scalar satisfies a quadratic polynomial whose roots generate the quadratic extension K. Picking one of the roots gives the action of K on V (the other root gives the conjugate action). Then V is a one dimensional K-vector space with basis e, $q(e) = a$ is non-zero in k, and we have $q(\alpha.e) = a.N(\alpha)$ for all α in K^*, where $N : K^* \to k^*$ is the norm.The Witt invariant is determined by the class of a in $Br_2(K/k) = k^*/N(K^*)$. Using the basis 1 and t for K over k, and writing $\alpha = x + yt$ we see that the quadratic form on V is given by $ax^2 + ady^2$ and that V has Witt invariant $w = (a, ad) = (a, -d)$.

We call the choice of a homomorphism $i : K \to \mathrm{End}(V)$ of k-algebras such that $q(i(\alpha).v) = N(\alpha)q(v)$ for all $\alpha \in K^*$ and $v \in V$ an **orientation** of the two dimensional space V. The orientations on V form a principal homogenous space for the group $O(V)/\mathrm{SO}(V) = \mu_2$, and the group preserving the quadratic form on V and a fixed orientation is $\mathrm{SO}(V)$. When K is a field, we can also give an orientation of V by choosing one of the two isomorphisms $(\mathrm{Res}_{K/k}\,\mathbb{G}_m)_{N=1} \cong \mathrm{SO}(V)$ of non-split one dimensional tori over k. Indeed, the K^*-action on V is determined by the action of the elements of norm 1, so by the elements in $T(k)$. Finally, the choice of an orientation determines, and is determined by the choice of an isotropic line in the split orthogonal space $V \otimes K$ (where there are two isotropic lines). From an orientation, we obtain an isotropic vector $e \otimes t + t(e) \otimes 1$ from any non-zero vector $e \in V$, and replacing e by the vector $\alpha.e$ gives us an isotropic vector on the same line. Conversely, the choice of an isotropic line in $V \otimes K$ gives an isometric action of the group K^* on this space, where $\alpha \in K^*$ acts by multiplication by α on the chosen isotropic line and by multiplication by α^{-1} on the other isotropic line. When α has norm 1, this action preserves the space V and gives an isomorphism $(\mathrm{Res}_{K/k}\,\mathbb{G}_m)_{N=1} \cong \mathrm{SO}(V)$.

As a final example in in low rank, let $B = k + ki + kj + k(ij)$ be a quaternion algebra over k, with products $i^2 = a, j^2 = b, ij = -ji$. Let V be the orthogonal space of dimension three over k, given by the norm form on elements of trace zero in B. Then

$$q(xi + yj + z(ij)) = -ax^2 - by^2 + abz^2.$$

The determinant of the $d(V, q)$ is $\equiv 1$ in k^*/k^{*2}, and the Witt invariant of (V, q) is given by the sum $(a, b) + (-1, -1)$ in $Br_2(k)$.

When $k = k_v$ is a local field which is not isomorphic to $\mathbb{C}$, the group $Br_2(k)$ is isomorphic to $\langle \pm 1 \rangle$ and the cup-product of two classes in k^*/k^{*2} is given by the Hilbert symbol $(a_i, a_j)_v = \pm 1$. In this case, the Witt invariant is often referred to as the Hasse-Witt invariant, and is written as a product of signs:

$$\epsilon_v(V, q) = \prod_{i<j} (a_i, a_j)_v = \pm 1.$$

When $k_v = \mathbb{C}$ we put $\epsilon_v = +1$.

When $k_v = \mathbb{R}$, there is a further invariant of (V, q) – the signature (r, s). Here r is the dimension of a maximal positive definite subspace of V and s is the dimension of a maximal negative definite subspace of V. We have $r + s = \dim(V)$ and there is an orthogonal basis with

$$q(\sum x_i e_i) = x_1^2 + x_2^2 + \ldots + x_r^2 - x_{r+1}^2 - x_{r+2}^2 + \ldots - x_n^2.$$

We have

$$d_v \equiv (-1)^s \quad \epsilon_v = (-1)^{s(s-1)/2}.$$

Over the complex numbers, the dimension is a complete invariant of (V, q) and over the real numbers, the signature is a complete isomorphism invariant of (V, q). Over a non-Archimedean field, the dimension, determinant, and Hasse-Witt invariant are complete isomorphism invariants of (V, q). The latter two invariants may be chosen arbitrarily in k^*/k^{*2} and $\langle\pm 1\rangle$, provided that $\dim(V) \geq 3$. In particular, for a fixed dimension $n \geq 3$ and determinant d there are precisely two isomorphism classes of orthogonal spaces over a non-Archimedean field, which are distinguished by the value of their Hasse-Witt invariant.

With these local invariants in hand, we can now describe the isomorphism classes of global orthogonal spaces (V, q) over a number field k, with fixed dimension $n \geq 3$ and determinant d in k^*/k^{*2}. At each real place v choose a signature (r_v, s_v) with $d_v \equiv (-1)^s$, and put $\epsilon_v = (-1)^{s(s-1)/2}$. At each finite place v, choose ϵ_v in $\langle\pm 1\rangle$, with $\epsilon_v = +1$ for almost all v. The choice of local invariants gives us, for each place v, a orthogonal space over k_v which is unique up to isomorphism. A necessary and sufficient condition for the existence of a global orthogonal space (V, q) over k of dimension of n and determinant d, with these localizations is given by Hilbert's reciprocity law: $\prod_v \epsilon_v = +1$. In this case, the theorem of Hasse and Minkowski states that the global space (V, q) is uniquely determined up to isomorphism over k by its localizations.

We may summarize these results as follows.

Theorem 2.1 *Let k be a number field, let $n \geq 3$ be an integer and let d be a class in k^*/k^{*2}. For each place v of k, let (V_v, q_v) be a local orthogonal space over k_v of dimension n and determinant d. Assume that the local Hasse-Witt invariants $\epsilon_v = \epsilon_v(V_v, q_v)$ are equal to $+1$ for almost all places v.*

Then a necessary and sufficient condition for the existence of a global orthogonal space (V, q) over k with these localizations is given by:

$$\prod_v \epsilon_v = +1.$$

If a global orthogonal space (V, q) exists with these localizations, it is unique up to isomorphism.

3 Local and global Hermitian spaces

Let k be a number field, and let K be a quadratic field extension of k. In this section, we review the classification of Hermitian spaces over k and over the completions k_v. These results are due to

Landherr, and a reference is [14]. We begin with the theory over a general field k and a separable quadratic field extension K. Let τ be the non-trivial involution of K over k.

A Hermitian space (V, ϕ) is a finite dimensional vector space over K together with a Hermitian symmetric form $\phi : V \times V \to K$. We assume that ϕ is K-linear in the first variable and satisfies $\phi(w, v) = \tau(\phi(v, w))$. Then ϕ is K-anti-linear in the second variable and $q(v) = \phi(v, v)$ takes values in k. We also assume that ϕ is non-degenerate, so gives an isomorphism from V to its dual. If $A = (\phi(e_i, e_j))$ is the matrix of the values of ϕ on a basis for V, then A is Hermitian symmetric and $d = \det(A)$ is an element of k^*. Let $N(K^*)$ be the subgroup of k^* consisting of those elements which are norms from K^*. Then the value of d in the quotient group $k^*/N(K^*)$ is independent of the basis chosen, and is called the Hermitian determinant of (V, ϕ). A one dimensional Hermitian space is determined up to isomorphism by its Hermitian determinant – the space is isomorphic to K with form $\phi(z, w) = d.zw^\tau$.

One can also define Hermitian spaces when K is the étale quadratic algebra $k + k$ with involution $\tau(a, b) = (b, a)$. In this case, there is a unique non-degenerate space, up to isomorphism. The free K-module V of rank n has the form to $V = W + W'$, where he W is a vector space of dimension n over k and W' is its dual space. The pairing ϕ is defined by

$$\phi(w + w', u + u') = (\langle w, u' \rangle, \langle u, w' \rangle)$$

In this case, all elements of k^* are norms and $d \equiv 1$.

If $k = k_v$ is a non-Archimedean local field, the dimension and the Hermitian determinant are complete isomorphism invariants of (V, ϕ). When $K = K_v$ is a field, the group $k^*/N(K^*)$ has order two and there are precisely two Hermitian spaces of each dimension $n \geq 1$. If K_v is unramified over k_v, the subgroup of norms are just the elements of k^* with even valuation. If K_v is not a field, there is a unique Hermitian space of each dimension. If $k_v = \mathbb{R}$ and $K_v = \mathbb{C}$, there is a further invariant of a Hermitian space, its signature (r, s). Here r is the dimension of a maximal positive definite subspace and s is the dimension of a maximal negative definite subspace. We have $r + s = \dim(V)$ and $d \equiv (-1)^s$. The signature is a complete isomorphism invariant in the real case.

Finally, let k be a number field and let K be a quadratic field extension of k. We want to describe all Hermitian spaces of a fixed dimension n over K, up to isomorphism. At each real place v of k which is complex in K, choose a signature (r_v, s_v) with $r_v + s_v = n$ and let $\epsilon_v = (-1)^{s_v}$. At each finite place v of k which is not split in K, choose a class $\epsilon_v = \pm 1$ in $k_v^*/N(K_v^*) = \langle \pm 1 \rangle$ with $\epsilon_v = +1$ for almost all v. This choice gives us local Hermitian spaces of dimension n for all places v, and the condition that these are the localizations of a fixed global space over k is that $\prod_v \epsilon_v = +1$. When a global space exists, it is uniquely determined up to isomorphism by its local invariants. We summarize this result as follows.

Theorem 3.1 *Let k be a number field and let K be a quadratic field extension of k. Fix a dimension $n \geq 1$. For each place v of k let (V_v, ϕ_v) be a local Hermitian space over K_v of dimension n. Assume that the Hermitian determinant d_v is a norm from K_v^* for almost all places v, or equivalently that the Hilbert symbol $(d_v, \mathrm{disc}_{K/k})_v = +1$ for almost all v.*

Then a necessary and sufficient condition for the existence of a global Hermitian space (V, ϕ) over K

with these localizations is given by:

$$\prod_v (d_v, \mathrm{disc}_{K/k})_v = +1$$

If a global Hermitian space (V, ϕ) exists with these localizations, it is unique up to isomorphism.

4 Incoherent definite orthogonal and Hermitian data

We henceforth assume that k is a totally real number field, and in the Hermitian case, that K is a totally complex quadratic field extension of k. Let k^*_+ denote the elements in k^* which are positive at all real places.

Incoherent definite orthogonal data consists of a dimension $n \geq 3$, a determinant d in k^*_+/k^{*2}, and for each place v of k a local orthogonal space V_v of dimension n and determinant d over k_v. We assume that V_v is positive definite for all real places v, that the Hasse-Witt invariant $\epsilon_v = \epsilon_v(V_v)$ is equal to $+1$ for almost all v, and that $\prod_v \epsilon_v = -1$.

Incoherent definite Hermitian data consists of a dimension $n \geq 1$ and for each place v of k a local Hermitian space V_v over $K_v = K \otimes k_v$ of dimension n. We assume that V_v is positive definite for all real places v, that the Hermitian determinant d_v of V_v is a norm from K_v^* for almost all of v, and that the number of places where d_v is not a norm is odd. Equivalently, we assume that the Hilbert symbol $(d_v, \mathrm{disc}_{K/k})_v$ is equal to $+1$ for almost all v and that $\prod_v (d_v, \mathrm{disc}_{K/k})_v = -1$.

We will use the notation $V = \{V_v\}$ for incoherent definite data, either orthogonal or Hermitian

Given incoherent definite orthogonal data V over k, we construct for each place v of k a global orthogonal space $V(v)$ of dimension n and determinant d over k which we call the neighbor at v. This space has localizations V_u for all places $u \neq v$. At the place v we insist that

$$\epsilon_v(V(v)) = -\epsilon_v(V_v)$$

This determines the localization of $V(v)$ at a finite place v. At a real place, we insist further that the signature of $V(v)$ at v is equal to $(n-2, 2)$. Since the product of local epsilon factors is now equal to $+1$, there is a global space $V(v)$ with these localizations, which is unique up to isomorphism. Finally, for each place v of k we let $G(v) = \mathrm{SO}(V(v))$ be the special orthogonal group of the neighboring space over k.

Given incoherent definite Hermitian data V over k relative to the quadratic extension K, we construct for each place v of k which is not split in K a global Hermitian space $V(v)$ of dimension n over K which we call the neighbor at v. This space has localizations V_u for all places $u \neq v$. At the place v we insist that

$$(d(V(v)), \mathrm{disc}_{K/k})_v = -(d(V_v), \mathrm{disc}_{K/k})_v$$

This determines the localization of $V(v)$ at a finite place v. At a real place, we insist further that the signature of $V(v)$ at the place v is equal to $(n-1, 1)$. Since the product of local Hilbert symbols is now equal to $+1$, there is a global Hermitian space $V(v)$ with these localizations, which is unique up to isomorphism. Finally, for each place v of k which is not split in K, we let $G(v) = U(V(v))$ be the unitary group of the neighboring space.

5 Incoherent data in codimension one

Let k be a totally real field and let $V = \{V_v\}$ be incoherent definite orthogonal data of dimension $n \geq 4$ and determinant d, and let a be any class in k_+^*/k^{*2}. Then a is represented by all of the local quadratic spaces V_v. Let U_v denote the orthogonal complement of the corresponding one dimensional subspace of V_v, so that

$$V_v = U_v \oplus \langle a \rangle.$$

Then $\{U_v\}$ is incoherent definite orthogonal data of dimension $n-1$ and determinant ad. Clearly the subspace U_v is positive definite at all real places v. But we also have the formula for the Hasse-Witt invariants

$$\epsilon_v(U_v) = \epsilon_v(V_v) \cdot (da, a)_v.$$

with $(da, a)_v = +1$ for almost all v and $\prod_v (da, a)v = +1$. Hence $\epsilon_v(U)_v = +1$ for almost all v and $\prod_v \epsilon_v(U_v) = -1$. It follows from a calculation of Hasse-Witt invariants that the neighboring spaces $U(v)$ all have the property that

$$U(v) \oplus \langle a \rangle \simeq V(v).$$

Conversely, if we are given incoherent definite data $U \subset V = \{U_v \subset V_v$ of codimension 1, with determinants $d(U)$ and $d(V)$, let $a = d(V)/d(U)$ in k_+^*/k^{*2}. Then $U_v^\perp = \langle a \rangle$ is coherent definite orthogonal data of dimension 1. Hence we have shown that

Theorem 5.1 *For incoherent definite orthogonal data $V = \{V_v\}$ of dimension $n \geq 4$, the classes of incoherent definite orthogonal data $U = \{U_v\}$ of codimension one in V correspond bijectively to classes a in k_+^*/k^{*2}.*

For each class a, we write $V \simeq U \oplus \langle a \rangle$. There is a similar result when the dimension of V is 3, but one has to restrict to classes a in k_+^*/k^{*2} which are locally representable at the finite set of places v where the space V_v is anisotropic. Also, incoherent definite orthogonal data $U = \{U_v\}$ of dimension 2 and determinant d is better viewed as incoherent definite Hermitian data of dimension 1 over the imaginary quadratic extension $E = k(\sqrt{-d})$ of k. In the Hermitian case, we have the following.

Theorem 5.2 *Let K be an imaginary quadratic extension of the totally real field k. For incoherent definite Hermitian data $V = \{V_v\}$ of dimension $n \geq 2$ over K, the classes of incoherent definite Hermitian data $U = \{U_v\}$ of codimension one in V correspond bijectively to classes a in $k_+^*/N(K^*)$.*

Indeed, the subspaces $U_v^\perp = \langle a_v \rangle$ give coherent definite Hermitian data of dimension 1. Since $\prod_v (a_v, \operatorname{disc} K/k) = +1$, the class of (a_v) is trivial in the quotient group $\mathbb{A}_k^*/N(\mathbb{A}_K^*)k^*$, which has order 2. Hence the local classes a_v can be written as a product of a global class a and a norm from K_v^*, and a is well-defined in the quotient group $k_+^*/N(K^*)$, and $V \simeq U \oplus \langle a \rangle$.

6 Orthogonal lattices

In this section, k is a local non-Archimedian field, with ring of integers A, uniformizing parameter π, and finite residue field $F = A/\pi A$ of order q. We will assume further that the residue characteristic of F is odd.

Let $V = (V, q)$ be a non-degenerate orthogonal space over k of dimension n, with associated bilinear form $\langle v, w\rangle = q(v + w) - q(v) - q(w)$. A lattice $L \subset V$ is by definition a free A-module of rank n which spans V over k, such that the bilinear form $\langle , \rangle$ takes integral values on L. The dual lattice $L^\vee$ is the free A-module of vectors in V with integral pairing against all elements of L. Then L is a submodule of $L^\vee$ and the quotient $L^\vee/L$ has finite length, by the non-vanishing of the determinant d of V.

We say that the lattice L is non-degenerate if $L = L^\vee$. A necessary condition for the existence of a non-degenerate lattice in V is that the determinant of V has even valuation, so d lies in the subgroup A^*k^{*2}/k^{*2} of k^*/k^{*2}. In this case, we say that the determinant of V is a unit. This condition is clearly sufficient for the existence of a non-degenerate lattice in V when the dimension $n = 1$, but is not sufficient when $n \geq 2$.

To see this in the two dimensional case, let D be a unit in A^* which is not a square, so $K = k(\sqrt{D})$ is the unramified quadratic extension of k. Let $V = K$ with quadratic form $q(v) = N(v)$. Then $q(x, y) = x^2 - Dy^2$, the determinant of V is equivalent to the unit $-D$ in k^*/k^{*2} and $\epsilon(V) = +1$. In this case V contains the non-degenerate lattice $A_K = A + \sqrt{D}A$ given by the ring of integers in K. On the other hand, if $V' = K$ with quadratic form $q(v) = \pi.N(v)$, then the determinant of V' is still equivalent to the unit $-D$ but now $\epsilon(V') = -1$. In this case, there are no non-degenerate lattices in V', but $L = A_K$ is a lattice with dual lattice $L^\vee = (\pi)^{-1}A_K$. The quadratic space $L^\vee/L$ has rank 2 over the residue field F and is isomorphic to the quadratic extension field $A_K/\pi A_K$ of order q^2 with its norm form. More generally, we have the following.

Theorem 6.1 *Let V be a quadratic space over k of dimension n whose determinant d is equivalent to a unit.*

If $\epsilon(V) = +1$ then V contains a non-degenerate lattice L. All such lattices are conjugate under the action of the group $\mathrm{SO}(V)$.

If $\epsilon(V) = -1$ then there are no non-degenerate lattices in V. However, V contains lattices L with the property that $L^\vee/L$ is a two dimensional orthogonal space over the residue field F, isomorphic to the quadratic extension field of F with its norm form. All such lattices are conjugate under the action of the group $\mathrm{SO}(V)$.

This is well-known – the lattices L in the theorem are called maximal lattices by Eichler, as they are maximal with the property that the bilinear form takes values in A. We will sketch a proof which involves embedding two dimensional spaces W in V, which will be useful later in this paper. The theorem is easily proved (using the construction above) when the dimension n of V is ≤ 2, so we will assume that $n \geq 3$.

Theorem 6.2 *Assume that the dimension of V satisfies $n \geq 2$ and that the determinant d of V is equivalent to a unit. Let D be a unit in A^* which is not a square, let K be the unramified quadratic extension $k(\sqrt{D})$ and let W be an orthogonal space of dimension 2 and determinant $-D$, which satisfies*

$$\epsilon(W) = \epsilon(V).$$

Then W embeds isometrically as a subspace of V, and all embeddings are conjugate under $\mathrm{SO}(V)$. The orthogonal complement U of W in V has invariants

$$\dim(U) = n - 2 \quad d(U) \equiv -dD \quad \epsilon(U) = +1$$

Note that the isomorphism class of the orthogonal complement U is independent of the isomorphism class of original space V of dimension n and determinant d. This is similar to what happens when $k = \mathbb{R}$, for the signatures we specified. The space V has signature $(n, 0)$ or $(n-2, 2)$, so has $d \equiv 1$. If we take $K = \mathbb{C}$, so $D \equiv -1$, and insist that the two dimensional space W has determinant $-D$ and the same Hasse-Witt invariant as V, then W has signature $(2, 0)$ in the first case and signature $(0, 2)$ in the second case. Hence W embeds in V and in both cases the orthogonal complement U has signature $(n-2, 0)$.

To prove Theorem 5.2, we begin with the case when the dimension of V is equal to 3. If a two dimensional space W of determinant $-D$ embeds into V, its orthogonal complement U would be one dimensional, of determinant $-dD$. But the quadratic form q on V represents the unit $-dD$, as it represents all elements of k^* except perhaps $-d$, and the discriminant D of K is not equivalent to 1. Taking the orthogonal complement of this line spanned by a vector v with $q(v) = -dD$ gives a two dimensional subspace W of V which has determinant $-D$. The quadratic form on W is given by $ax^2 - aDy^2$ and on V is given by $ax^2 - aDy^2 - Ddz^2$. Since $-D$ and $-dD$ are both units and the residue characteristic is odd, $(-D, -dD) = +1$ and the Hasse-Witt invariants of both W and V are both equal to $(a, -D)$, This proves our claim on the embedding when the dimension of V is three. The orthogonal group of V acts transitively on these two dimensional subspaces, by Witt's extension theorem. Since the stabilizer contains a reflection, the special orthogonal group also acts transitively.

One can also prove the special case when the dimension of V is 3 and the determinant $d \equiv 1$ using the theory of local quaternion algebras. Indeed, in this case the space V is given by the elements of trace zero in a quaternion algebra B over k, and the quadratic form is the norm form. The unramified quadratic extension K of k embeds as a subfield of B, and its orthogonal complement W is the desired two dimensional subspace of V. We obtain an orientation of the plane W in V by choosing one of the two vectors on the line $W^{\perp}$ with $q(v) = -D$, which also gives W a $K = k.1 + k.v$-linear structure.

Next consider the case when the dimension of V is equal to four. The quadratic space V represents every non-zero class a and the three dimensional orthogonal complement of the line spanned by a vector v with $q(v) = a$ has determinant ad. To obtain an embedding of W, we need to check that this three dimensional space represents $-aD$. But any three dimensional quadratic space of determinant ad represents all classes not equivalent to $-ad$, Hence we have an embedding of both orthogonal spaces of dimension 2 and determinant $-D$, except in the case when the determinant of V is equivalent to D. In that case, we can construct a four dimensional space V of determinant D from a two dimensional space W of determinant $-D$ by taking the direct sum of W with a hyperbolic plane. The quadratic form on V has the form $ax^2 - aDy^2 + z^2 - t^2$, so the Hasse-Witt invariant of both W and V are both equal to $(a, -D)$.

When the dimension of the space$V$ is at least five, then the quadratic form on V represents any non-zero class a and the orthogonal complement of the line spanned by a vector v with $q(v) = a$ represents $-aD$. Hence we may embed both quadratic spaces W of dimension 2 and determinant $-D$ in V. The special orthogonal group of V acts transitively on the subspaces W with $\epsilon(W) = \epsilon(V)$, by Witt's extension theorem. Hence the orthogonal complement U of W is well-defined up to isomorphism over k. Since $V = W \oplus U$ we find

$$d(U) \equiv -dD` \quad \epsilon(U) = +1$$

as claimed.

We can now prove Theorem 5.1 by an induction on the dimension of V. If $\epsilon(V) = +1$, we embed the two dimensional subspace W with determinant $-D$ and $\epsilon(W) = +1$. We have seen that there is a non-degenerate lattice in W, and by induction there is a non-degenerate lattice in its orthogonal complement U. The direct sum of these two lattices gives a non-degenerate lattice in V. If $\epsilon(V) = -1$ we embed the two dimensional subspace W with determinant $-D$ and $\epsilon(W) = -1$. By induction there is a non-degenerate lattice in the orthogonal complement U, and we have seen that there is a lattice L in W with $L^\vee/L$ isomorphic to the quadratic extension of F with its norm form. The direct sum of these lattices gives the desired lattice in V.

The fact that the group $\mathrm{SO}(V)$ acts transitively on the lattices described in Theorem 5.1 is proved in [16] and [5] They are the maximal lattices in V, where the bilinear form is integral. In the case when $\epsilon = +1$ and the lattice L is non-degenerate, the stabilizer of L is a hyperspecial maximal compact subgroup of the locally compact group $\mathrm{SO}(V)(k)$. When $n = 2m + 1$ is odd, its reductive quotient is the split group SO_{2m+1} over F. When $n = 2m$ is even and $(-1)^m d \equiv 1$ its reductive quotient is the split group SO_{2m} over F. When $n = 2m$ is even and $(-1)^m d \equiv D$ its reductive quotient is the quasi-split group $\mathrm{SO}^\epsilon_{2m}$ over F.

In the case when $\epsilon = -1$, let $T = \mathrm{SO}^\epsilon_2$ be the one dimensional torus over F which is split by the quadratic extension of F. Then the stabilizer of L is a parahoric subgroup of $\mathrm{SO}(V)(k)$ whose reduction is as follows. When $n = 2m + 1$ is odd, the reductive quotient is the group $T \times \mathrm{SO}_{2m-1}$ over F. When $n = 2m$ is even and $(-1)^m d \equiv 1$, the reductive quotient is the group $T \times \mathrm{SO}^\epsilon_{2m-2}$ over F. When $n = 2m$ is even and $(-1)^m d \equiv D$, the reductive quotient is the group $T \times \mathrm{SO}_{2m-2}$ over F. When $n \geq 3$ the parahoric subgroup $\mathrm{SO}(L)$ is not hyperspecial. It is maximal except when $2m = 4$ and $d \equiv D$, when it is an Iwahori subgoup of $\mathrm{SO}(V) \cong (\mathrm{Res}_{K/k} \mathrm{SL}_2)/\mu_2$. See [5] for details.

7 Hermitian lattices

In this section, we again assume that k is a local non-Archimedian field, with ring of integers A, uniformizing parameter π, and finite residue field $F = A/\pi A$ of order q. Let $K = K(\sqrt{D})$ be the unramified quadratic extension of k, where D is a unit in A^* which is not a square, and let τ be the non-trivial involution of K which fixes k. We make no assumption on the residue characteristic of F.

Let V be a non-degenerate Hermitian space over K of dimension n, with associated Hermitian symmetric form $\phi(v, w)$. A lattice $L \subset V$ is by definition a free A_K-module of rank n which spans V over K, such that the Hermitian form ϕ takes integral values on L. The dual lattice $L^\vee$ is the free A_K-module of vectors in V with integral pairing against all elements of L. Then L is an A_K-submodule of $L^\vee$ and the quotient $L^\vee/L$ has finite length, by the non-vanishing of the Hermitian determinant d of V.

We say that the lattice L is non-degenerate if $L = L^\vee$. A necessary condition for the existence of a non-degenerate lattice in V is that the determinant has even valuation, so d lies in the subgroup $N(K^*)$ of index 2 of k^*. In this case, we say that the determinant of V is a norm. When the dimension of V over K is equal to 1 and the determinant is not a norm, then the Hermitian space is isomorphic to K with form $\phi(x, y) = \pi.x.y^\tau$. In this case, there are no non-degenerate lattices in V. However the lattice $L = A_K$ has dual lattice $L^\vee = \pi^{-1}A_K$, and the quotient module $L^\vee/L$ is isomorphic to the quadratic extension field of F with its standard Hermitian form.

Theorem 7.1 *Let V be a non-degenerate Hermitian space of dimension $n \geq 1$ over the unramified quadratic extension $K = k(\sqrt{D})$ of k.*

If the Hermitian determinant d of V is a norm, then V contains a non-degenerate lattice L. All such lattices are conjugate under the action of the group $U(V)$.

If the Hermitian determinant d of V is not a norm, then there are no non-degenerate lattices in V. However, V contains lattices L with the property that $L^\vee/L$ is a one dimensional Hermitian space over the quadratic extension of the residue field F. All such lattices are conjugate under the action of the group $U(V)$.

The proof is similar to the orthogonal case, but simpler as it involves the embedding of a suitable one dimensional subspace W in V. Let W be a Hermitian space of dimension 1 with the property that

$$d(W) \equiv d(V)$$

in $k^*/N(K^*)$. It is then a simple matter to show that W embeds isometrically as a subspace of V, and that all such subspaces are conjugate under the action of $U(V)$. The orthogonal complement of W in V has Hermitian determinant which is a norm, so by induction contains non-degenerate lattice. The direct sum of that lattice with the lattice constructed above (in dimension 1) gives the desired lattice in V.

The fact that the unitary group of V acts transitively on lattices of this type follows from the theory of Iwahori and Bruhat-Tits on parahoric subgroups. Indeed, when the determinant of V is a norm, the stabilizer of a non-degenerate lattice L is a hyperspecial maximal compact subgroup of the locally compact group $U(V)$ whose reductive quotient is the group U_n over F. When the determinant of V is not a norm, the stabilizer is again a maximal compact subgroup, but is not hyperspecial once $n \geq 2$. Its reductive quotient is the group $U_1 \times U_{n-1} = T \times U_{n-1}$ over F.

8 The homogeneous spaces X, Y, and Z

We continue with the notation of the previous two sections: k is a non-Archimedian local field with ring of integers A and finite residue field $A/\pi A$.In the orthogonal case, we assume that the characteristic of $A/\pi A$ is odd. Let $K = k(\sqrt{D})$ be the unramified quadratic extension of k, and let A_K be the ring of integers of K.

Our aim is to generalize the following construction, where $k = \mathbb{R}$ and $K = \mathbb{C}$. If V is a real orthogonal space of dimension $n \geq 3$ and signature $(n-2, 2)$, the set $X = \mathscr{D}^{\pm}$ of oriented negative definite two planes W in V is a complex manifold of dimension $n-2$ which admits a transitive action of the group $G = \mathrm{SO}(V)$. Indeed, an oriented negative definite two plane determines a homomorphism $h : T \to \mathrm{SO}(V)$ over $\mathbb{R}$, where T is the one dimensional torus $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m)_{N=1} \cong \mathrm{SO}(W)$, and the centralizer of h is the compact subgroup

$$H = \mathrm{SO}(W) \times \mathrm{SO}(W^{\perp}).$$

The conjugacy class of such homomorphisms is isomorphic to the quotient space $X = G/H$. This quotient is isomorphic to two copies of the symmetric space of the connected real group $\mathrm{SO}(n-2, 2)^{+}$, as, the subgroup H has index two in its normalizer $S(O(W) \times O(W^{\perp}))$, which is a maximal compact subgroup of G. Hence every coset for the normalizer breaks into two cosets for the subgroup H. The space X has a Hermitian symmetric structure, as the tangent space to the identity coset of G/H is the orthogonal representation $\mathrm{Hom}(W, W^{\perp}) \cong W \otimes W^{\perp}$ of H, which has a complex Hermitian structure from the chosen orientation of W. The involution of X which reverses the orientation of W switches the two components and is anti-holomorphic. Finally, the complex manifold X embeds as an open G-orbit (with two components) in the complex quadric $Q(\mathbb{C})$ defined by the vanishing of the quadratic form on the projective space of $V \otimes \mathbb{C}$, via the Borel embedding.

In the non-Archimedean case, we start with an orthogonal space V over k of dimension $n \geq 3$, with unit determinant d and Hasse-Witt invariant $\epsilon = -1$. Let X be the set of oriented two planes W in V with determinant $-D$ and satisfying $\epsilon(W) = -1$. We have seen that such oriented two planes exist, and are permuted transitively by $G = \mathrm{SO}(V)$. An oriented two plane W determines a homomorphism $h : T \to \mathrm{SO}(V)$ over k, where T is the one dimensional torus $(\mathrm{Res}_{K/k}\, \mathbb{G}_m)_{N=1} \cong \mathrm{SO}(W)$, and the centralizer of h is the closed subgroup

$$H = \mathrm{SO}(W) \times \mathrm{SO}(W^{\perp}).$$

The conjugacy class X of such homomorphisms h, which can be identified with the quotient space $X \cong G/H$ once an oriented plane W has been chosen, has the structure of a homogeneous analytic manifold over k. The tangent space at the oriented two plane W is the k-vector space $\mathrm{Hom}(W, W^{\perp}) = W^{*} \otimes W^{\perp} \cong W \otimes W^{\perp}$, which affords an orthogonal representation of the stabilizer. Since the subspace W has K-linear structure given by the orientation, the tangent plane at each point is a K-vector space and X is a K-analytic manifold of dimension $n-2$.

As in the real case, there is a fixed point free involution of X, which acts K-anti-linearly on the tangent spaces and commutes with the action of $\mathrm{SO}(V)$. This is given by the action of normalizer of h in G, which contains the centralizer with index 2 and maps each homomorphism $h : T \to \mathrm{SO}(V)$ to its composition with the inverse homomorphism on T. Finally, the analytic manifold X embeds as an open G-orbit in the K-points of the smooth quadric Q of dimension $n-2$ defined by the vanishing of the quadratic form q on the projective space of $V \otimes K$. The embedding sends the oriented two plane $W = K.e$ in V spanned by the orthogonal vectors e and $f = \sqrt{D}.e$ to the line in $V \otimes K$ spanned by the vector $e \otimes 1 + f \otimes (\sqrt{D})^{-1}$. Since $q(f) = N(\sqrt{D})q(e) = -Dq(e)$ this line is isotropic, and defines a point on $Q(K)$.

Unlike the real case, the subgroup H need not be compact. It is compact only when the factor $\mathrm{SO}(W^\perp)$ is compact; since $\epsilon(W^\perp) = +1$ this can only occur when the dimension of V is three or four. To obtain a compact subgroup of H, we need to choose a non-degenerate lattice L in the subspace $W^\perp$. Then writing $W = K.e$ with valuation $q(e) = \pi$ we obtain a canonical lattice $A_K.e$ in W. The product

$$J = \mathrm{SO}(A_K.e) \times \mathrm{SO}(L)$$

is then a hyperspecial maximal compact subgroup of the closed subgroup H.

We define Y as the set of pairs (W, L), where W is an oriented two planes with determinant $-D$ and Hasse-Witt invariant $\epsilon(W) = -1$ and L is a non-degenerate lattice in $W^\perp$. The group $G = \mathrm{SO}(V)$ acts transitively on Y, which can be identified with the quotient space $Y \cong G/J$ once a fixed pair (W, L) has been chosen. Again, there is a fixed point free involution of Y given by the normalizer of J in G, which contains J with index 2. This changes the orientation of the plane W and preserves the lattice L.

We obtain a covering map $Y \to X$, which is G-equivariant and usually of infinite degree, by forgetting the choice of non-degenerate lattice L. This covering is what is needed to canonically reduce the structure on the tangent bundle of X to get an A_K-analytic manifold structure on Y. The tangent space at a point of Y, corresponding to the oriented two plane W and the non-degenerate lattice L in $W^\perp$, is the free A_K module of rank $n-2$ given by $\mathrm{Hom}_A(A_K.e, L) \cong (A_K.e)^* \otimes L$. The covering

$$Y \to X \hookrightarrow Q(K)$$

is probably related to the period mapping of Rapoport and Zink, restricted to K-rational points.

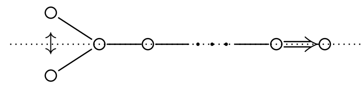
Finally, we let Z be the set of maximal lattices Λ in V, together with an orientation of the orthogonal space Λ^*/Λ over $A/\pi A$. Such an orientation is equivalent to giving the two dimensional orthogonal space Λ^*/Λ the structure of a one dimensional vector space over the quadratic extension field $A_K/\pi A_K$. There is a map

$$F : Y \to Z$$

which associates to the pair (W, L) the maximal lattice $\Lambda = A_K.e + L$. The orientation of W induces an orientation of Λ^*/Λ. The group $G = \mathrm{SO}(V)$ again acts transitively on Z and the map F is G-equivariant. Let $\mathrm{SO}(\Lambda)$ be the stabilizer of the lattice Λ. This is an open compact subgroup of G, which contains the parahoric subgroup N stabilizing Λ together with its orientation with index 2. The reductive quotient of N is isomorphic to a connected group of the type $\mathrm{SO}_2 \times \mathrm{SO}_{n-2}$ over the residue field. We can identify the space Z with the quotient space G/N, once an oriented maximal lattice has been chosen. We note that Z has an involution, defined by changing the orientation, and that the map $F : Y \to Z$ commutes with the respective involutions.

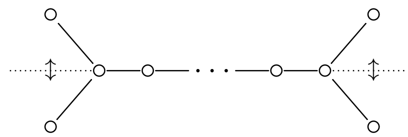
The parahoric subgroups of the simply-connected cover $\mathrm{Spin}(V)$ of $\mathrm{SO}(V)$ can be read off the affine diagram, together with its Frobenius automorphism. In the case when the dimension $n = 2m + 1 \geq 7$

is odd, the determinant is a unit, and $\epsilon(V) = -1$, we have the diagram



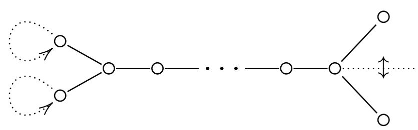
The maximal parahoric N corresponds to the two vertices interchanged by Frobenius at the left end of the diagram.

In the case when the dimension $n = 2m \geq 8$ is even, the determinant is $\equiv (-1)^m$ and $\epsilon(V) = -1$ we have the diagram



and the parahoric N corresponds to two vertices interchanged by Frobenius at the left end of the diagram.

In the case when the dimension $n = 2m$ is even and the determinant is $\equiv (-1)^m D$ we have the diagram



When $\epsilon(V) = -1$ the parahoric N corresponds to the two vertices interchanged by Frobenius at the right end of the diagram.

A similar theory works in the Hermitian case. First consider the situation over the real numbers. If V is a complex Hermitian space of dimension n and signature $(n-1, 1)$, then the set X of negative lines W in V is a complex manifold of dimension $n-1$ which admits a transitive action of the group $G = U(V)$. Indeed, such a line determines a homomorphism $h : T \to U(V)$ over $\mathbb{R}$, and the centralizer of h is the compact subgroup

$$H = U(W) \times U(W^\perp).$$

The quotient $X \cong G/H$ is isomorphic to the symmetric space of $U(n-1, 1)$. It has a Hermitian symmetric structure, as the tangent space to the identity is the representation $W \otimes W^\perp$ of H, of dimension $n-1$ over $\mathbb{C}$. Finally, the complex manifold embeds as an open G-orbit in the projective space of V over $\mathbb{C}$.

In the non-Archimedean case, we start with a Hermitian space V over K of dimension $n \geq 1$ whose Hermitian determinant is not a norm. Let X be the set of lines W in V with Hermitian determinant equal to the Hermitian determinant of V. We have seen that such lines exist and are permuted transitively by $G = U(V)$. Such a line determines a homomorphism $h : T \to U(V)$, where T is the one dimensional torus $(\mathrm{Res}_{K/k}\, \mathbb{G}_m)_{N=1} \cong U(W)$, and the centralizer of h is the closed subgroup

$$H = U(W) \times U(W^\perp).$$

The conjugacy class X of such homomorphisms h is isomorphic to the quotient space G/H and has the structure of a homogeneous analytic manifold of dimension $n-1$ over K. The tangent space at the subspace W is the representation $\mathrm{Hom}(W, W^\perp)$ of H. Finally, the analytic manifold X embeds as an open G-orbit in the K-valued points of the projective space of V.

The subgroup H can only be compact when the dimension of V is one or two. To obtain a suitable compact subgroup of H, we need to choose a non-degenerate lattice L in the subspace $W^\perp$. Writing $W = K.e$ with $\phi(e, e) = \pi$, we obtain a canonical lattice $A_K.e$ in $W = K.e$. The product

$$J = U(A_K.e) \times U(L)$$

is a hyperspecial maximal compact subgroup of H and the quotient $Y = G/J$ is an A_K-analytic homogeneous space of dimension $n-1$. The homogeneous space Y indexes the lines W in V with Hermitian determinant equal to the determinant of V together with a non-degenerate lattice L in $W^\perp$. The covering

$$Y \to X \hookrightarrow \mathbb{P}^{n-1}(K)$$

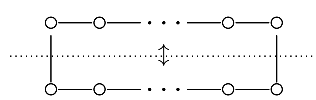
obtained by forgetting the choice of lattice L is G-equivariant and is usually of infinite degree. It is probably related to the period map of Rapoport and Zink, restricted to K-valued points.

Finally, let Z be the set of maximal lattices Λ in V. There is a map

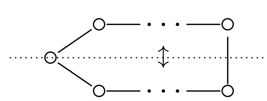
$$F : Y \to Z$$

which associates to the line W the maximal lattice $\Lambda = A_K.e + L$. The The group $G = U(V)$ again acts transitively on Z and the map F is G-equivariant. Let $N = U(\Lambda)$ be the stabilizer of the lattice Λ, which is a parahoric subgroup of G whose reductive quotient is isomorphic to $U(1) \times U(n-1)$ over the residue field. We can identify the space Z with the quotient space G/N, once a maximal lattice has been chosen.

Again the parahoric subgroups of $U(V)$ can be read off the affine diagram of $SU(V)$ with its automorphism given by Frobenius. When the dimension $n = 2m \geq 4$ of V is even and the determinant is not a norm, we have the diagram



and the parahoric N corresponds to the two vertices interchanged by Frobenius at either end of the diagram. When the dimension $n = 2m + 1 \geq 5$ of V is odd, we have the diagram



When the determinant of V is not a norm, the parahoric N corresponds to the two vertices interchanged by Frobenius at the right of the diagram.

9 The fibers of the map $F : Y \to Z$

If we fix a point $z \in Z$ we would like a description of the inverse image $F^{-1}(z)$ in Y.

In the orthogonal case, fix a point $y \in Y$ with $F(y) = z$. This expresses the lattice Λ as a direct sum $\Lambda = A_K.e + L$. To parametrize the other points in $F^{-1}(z)$, we need to determine all other such decompositions of the lattice Λ, or equivalently the rank two A-submodules $M = Av + Aw$ of Λ which are isometric to $A_K.e$. The key observation is that $\pi\Lambda^*/\pi\Lambda$ is the radical of the quadratic space $\Lambda/\pi\Lambda$, and that a rank 2 submodule M isometric to $A_K.e$ must have this reduction modulo π. Consequently M has a basis of the form

$$v = \alpha.e + \lambda$$

$$w = \beta.e + \mu$$

where α and β are units of A_K which are independent in the $A/\pi A$ vector space $A_K/\pi A_K$ and λ and μ are elements of the sublattice πL. We then find inner products

$$\langle v, v\rangle = 2N(\alpha)\pi + O(\pi^2) \qquad \langle w, w\rangle = 2N(\beta)\pi + O(\pi^2) \qquad \langle v, w\rangle = Tr(\alpha.\overline{\beta})\pi + O(\pi^2)$$

and from this it follows that $M = Av + Aw$ is a lattice (oriented, and isometric to $A_K.e$) in a two dimensional orthogonal space over k with determinant $-D$ and Hasse-Witt invariant -1. Taking the orthogonal complement $M^\perp$ of M in Λ, we obtain another point y^* in $F^{-1}(z)$. We still need to check that the lattice $M^\perp$ is non-degenerate. This argument is a bit simpler in the Hermitian case, and we defer it until that discussion.

We can parametrize the data determining the decomposition $\Lambda = M + M^\perp$ by the element

$$\alpha^{-1} \otimes \lambda + \beta^{-1} \otimes \mu \in A_K \otimes \pi L.$$

Picking a basis for L over A, this is the set $(\pi_K)^{n-2}$ of A_K-valued points in the formal scheme given by a polydisc of dimension $n-2$ and radius 1 over A. In summary, we have shown the following.

Theorem 9.1 *Assume that V is an orthogonal space over k of dimension $n \geq 3$ over k with unit determinant and Hasse-Witt invariant -1. Let Y be the homogeneous space parametrizing pairs (W, L) consisting of an oriented two plane W in V with determinant $-D$ and Hasse-Witt invariant -1, together with a non-degenerate lattice L in $W^\perp$. Let Z be the homogeneous space of maximal lattices Λ in V together with an orientation of the lattice Λ^*/Λ over the residue field of k.*

Let $F : Y \to Z$ be the $\mathrm{SO}(V)$-equivariant map taking the pair $y = (W, L$ to the non-degenerate lattice $z = \Lambda = A_K.e + L$ with orientation coming from the orientation of W. Then the inverse image $F^{-1}(z)$ in Y can be parametrized by the points of the polydisc $A_K \otimes \pi L \cong (\pi A_K)^{n-2}$.

We have a filtration of the parahoric N by normal subgroups N_m of finite index which act trivially on the quotient lattice $\Lambda/\pi^m\Lambda$. This gives a filtration of the quotient set

$$N/J > N_1J/J > N_2J/J > \ldots$$

If we identify $F^{-1}(z) = N/J$ with the points of $(\pi_K)^{n-2}$, then presumably the subset N_mJ/J corresponds to the points in $(\pi_K^{m+1})^{n-2}$. Indeed, the quotient groups $N_mJ/N_{m+1}J$ are isomorphic to $(A_K/\pi A_K)^{n-2}$ for all m.

In the simplest case, when $n = 3$ and $d \equiv 1$ the space V is given by the elements of trace zero in the quaternion division algebra B over k. Let R be the unique maximal order in B. Then A_K embeds as a subring of R, and we have the orthogonal decomposition

$$R = A_K + A_K.\Pi$$

where Π is a uniformizing parameter of R, which normalizes A_K and whose square is a uniformizing element of A. The special orthogonal group $\mathrm{SO}(V) = B^*/k^*$ is compact, and contains the parahoric subgroup $N = R^*/A^*$ with index 2. Hence the space Z consists of two points. The subgroup $H = K^*/k^* = A_K^*/A^*$ is compact in this special case, so $H = J$ and $X = Y = B^*/A_K^*$ is isomorphic to two copies of the disc p_{A_K}.

In the Hermitian case, a point $y \in Y$ corresponds to a line W in the Hermitian space V and a non-degenerate lattice L in the orthogonal complement. The direct sum $\Lambda = A_K.e + L$ is a maximal Hermitian lattice in V with $\Lambda^*/\Lambda = \pi^{-1}A_K.e/A_K.e$, and the point $z = F(y)$ in Z remembers only the lattice Λ, and not the decomposition. To find all points in $F^{-1}(z)$, we need o determine the rank 1 sublattices $M = A_K.v$ in Λ which are isometric to $A_K.e$. Since $M/\pi M = A_K.e/\pi A_K.e$ is the radical of $\Lambda/\pi\Lambda$, the vector v of Λ must have the form

$$v = \alpha.e + \lambda$$

where α is a unit in A_K and λ lies the the sublattice $\pi.L$. Let $M = A_K.v$ be the one dimensional lattice spanned by v, and let $M^\perp$ denote its orthogonal complement in the lattice Λ. We need to check that the lattice $M^\perp$ of rank $n-1$ is non-degenerate.

Since α is a unit, there is no loss of generality in assuming that the basis for M has the form $v = e + \mu$ with $\mu = \pi\nu \in \pi.L$. Let $\{f_1, f_2, \ldots, f_{n-1}\}$ be an orthonormal basis for L over A_K, and write $\nu = \sum \beta_i.f_i$. Then the vectors $-\overline{\beta_i}.e + f_i$ in Λ give a basis for $M^\perp$ over A_K, and the Gram matrix of their inner products has determinant $\equiv 1$ modulo π. This establishes the non-degeneracy of the lattice $M^\perp$ (a slightly more complicated version of this argument also works in the orthogonal case).

We can parametrize the decomposition $\Lambda = M + M^\perp$ by the element

$$\alpha^{-1} \otimes \lambda \in \pi L.$$

Picking a basis for L over A_K, this is the set $(\pi_K)^{n-1}$. In summary, we have shown the following.

Theorem 9.2 *Assume that V is a Hermitian space of dimension $n \geq 2$ over K whose Hermitian determinant is not a norm. Let Y be the homogeneous space parametrizing pairs (W, L) consisting of a line W in V whose determinant is not a norm together with a non-degenerate lattice L in $W^\perp$. Let Z be the homogenous space parametrizing the non-degenerate lattices Λ in V.*

Let $F : Y \to Z$ be the $U(V)$-equivariant map which takes the pair $y = (W, L)$ to the lattice $z = \Lambda = A_K.e + L$. Then the inverse image $F^{-1}(z)$ in Y can be parametrized by the points of the polydisc $\pi L \cong (\pi A_K)^{n-1}$.

Again, have a filtration of the parahoric N by normal subgroups N_m of finite index which act trivially on the quotient lattice $\Lambda/\pi^m\Lambda$. This gives a filtration of the quotient set

$$N/J > N_1J/J > N_2J/J > \ldots$$

If we identify $F^{-1}(z) = N/J$ with the points of $(\pi_K)^{n-1}$, then presumably the subset $N_m J/J$ corresponds to the points in $(\pi_K^{m+1})^{n-1}$. Indeed, the quotient groups $N_m J/N_{m+1} J$ are isomorphic to $(A_K/\pi A_K)^{n-1}$ for all m.

10 Shimura varieties associated to incoherent definite data

Let k be a totally real number field with ring of integers A, and fix incoherent definite orthogonal data $\{V_v\}$ over k of dimension $n \geq 3$ and determinant d. For each place v of k let $V(v)$ be the neighboring global orthogonal space, and let $G(v)$ be the special orthogonal group of $V(v)$ over k. Let $\mathbb{A}$ be the ring of adeles of k let $\mathbb{A}^f$ be the subring of finite adeles, and for a finite place p of k let $A^{f,p}$ denote the subring of finite adeles away from p.

To define the orthogonal Shimura variety S associated to this data, we first choose a real place v of k, and begin by defining S over the complex quadratic extension $K_v \cong \mathbb{C}$ of the completion $k_v = \mathbb{R}$. Let L be an integral A lattice in the neighboring orthogonal space $V(v)$, which has signature in $(n-2, 2)$ at v and signature $(n, 0)$ at all other real places w of k. We note that $L \otimes A_p$ is non-degenerate for almost all primes p. Then $M = \prod \mathrm{SO}(L_p)$ is an open compact subgroup of $G(v)(\mathbb{A}^f)$, and $M_p = \mathrm{SO}(L_p)$ is a hyperspecial maximal compact subgroup of $\mathrm{SO}(V_p)$ for almost all primes p.

Let $T = \mathrm{Res}_{K_v/k_v} \mathbb{G}_m/\mathbb{G}_m$ be the one dimensional non-split torus over $k_v = \mathbb{R}$, and let $h : T \to G(v)_{\mathbb{R}}$ be the homomorphism described in the previous section, associated to special oriented two dimensional subspaces of $V(v)$ over k_v. Then the conjugacy class X_v of h has the structure of two conjugate copies of the Hermitian symmetric space of $\mathrm{SO}(n-2, 2)$. We define

$$S_M(K_v) = G(v)(k) \backslash X_v \times G(v)(\mathbb{A}^f)/M$$

This double coset space is the disjoint union of a finite number of connected components each of the form $\Gamma \backslash X_v$, where Γ is an arithmetic subgroup of the algebraic group $G(v)(k)$. As such, it is a complex analytic orbifold, which has an algebraic structure (unique if M is small enough so that the subgroups Γ have no non-trivial torsion). The complex algebraic varieties S_M form a projective system, for $M' \subset M$, and one defines S as the projective limit. This complex scheme has a right action of the group $G(v)(\mathbb{A}^f)$, and S_M is the quotient of S by the open compact subgroup M.

The theory of canonical models shows that S descends canonically to k, viewed as a subfield of $k_v = \mathbb{R}$ in $K_v = \mathbb{C}$. We note that the descent to $k_v = \mathbb{R}$ is given by the anti-holomorphic involution of complex conjugation on X_v, and the action of the group $G(v)(\mathbb{A}^f)$ on S is defined over k on the canonical model. This is the Shimura variety S associated to the incoherent definite orthogonal data $\{V_v\}$. The irreducible components of S are rational over the maximal abelian extension F of k with exponent 2, and are permuted simply-transitively by the quotient $\mathbb{A}^*/k^*\mathbb{A}^{*2}$ of $G(v)(\mathbb{A}^f)$, which is isomorphic to the Galois group of F over k by the reciprocity homomorphism of global class field theory. To see that this Galois group is a quotient of $G(v)(\mathbb{A}^f)$, note that the spinor norm maps $G(v)(\mathbb{A}^f)$ onto the group $(\mathbb{A}^f)^*/(\mathbb{A}^f)^{*2}$. The quotient of this group by the group k^*_+ of totally positive elements in k^* is the group $\mathbb{A}^*/k^*\mathbb{A}^{*2}$.

The reason that we use incoherent data to define S, rather than just the orthogonal group $G(v)$ with its real conjugacy class X_v, is that the latter depends on the choice of a real place of k, and only defines

S over a subfield of $\mathbb{C}$. However, for any other real place w of k, we have an isomorphism of adelic groups $G(w)(\mathbb{A}^f) \cong G(v)(\mathbb{A}^f)$ and an isomorphism complex varieties

$$S_M(K_w) \cong G(w)(k)\backslash X_w \times G(w)(\mathbb{A}^f)/M$$

When $n = 3$, so S_M is a Shimura curve, this isomorphism is due to Doi and Naganuma. In the general case, it follows from results of Borovoi on the conjugation of Shimura varieties. Therefore it is more symmetrical to use the incoherent definite orthogonal data to define S. We will speculate on the use of the neighboring orthogonal spaces $V(p)$ for finite places of k in the next section.

A similar definition works for the unitary Shimura varieties associated to incoherent definite Hermitian data $\{V_v\}$ of dimension $n \geq 1$ over the imaginary quadratic extension K of the totally real field k. We choose a real place v of k and let $G(v)$ be the unitary group of the neighboring Hermitian space $V(v)$ over k. We begin by defining the Shimura variety S over the complex quadratic extension $K_v = \mathbb{C}$ of $k_v = \mathbb{R}$. Let L be a Hermitian A_K lattice in the neighboring space $V(v)$, which has signature $(n-1, 1)$ at v and signature $(n, 0)$ at all the other infinite places w. $M = \prod U(L_p)$ is an open compact subgroup of $G(v)(\mathbb{A}^f)$ and $M_p = U(L_p)$ is a hyperspecial maximal compact subgroup of $U(V_p)$ for almost all primes p.

Let $T = \mathrm{Res}_{K_v/k_v} \mathbb{G}_m/\mathbb{G}_m$ be the one dimensional non-split torus over $k_v = \mathbb{R}$, and let $h : T \to G(v)_{\mathbb{R}}$ be the homomorphism described in the previous section, associated to special one dimensional subspaces of $V(v)$ over K_v. Then the conjugacy class X_v of h is isomorphic to the Hermitian symmetric space of $U(n-1, 1)$. We define

$$S_M(K_v) = G(v)(k)\backslash X_v \times G(v)(\mathbb{A}^f)/M$$

Again, this is a complex orbifold which has an algebraic structure, and the associated complex varieties S_M form a projective system with limit S. The limit has a right action of the group $G(v)(\mathbb{A}^f)$, and S_M is the quotient of S by the open compact subgroup M.

The theory of canonical models shows that S descends canonically to a pro-scheme over K, viewed as a subfield of $K_v = \mathbb{C}$, and the action of the group $G(v)(\mathbb{A}^f)$ on S is defined over K on the canonical model. This is the Shimura variety S associated to the incoherent definite Hermitian data $\{V_v\}$. The irreducible components of S are rational over the maximal abelian extension F of K which is "dihedral" over k. They are permuted simply-transitively by the quotient $\mathbb{A}_K^*/(\mathbb{A}^*.K^*.\prod_v \mathbb{C}_v^*)$ of $G(v)(\mathbb{A}^f)$, which is isomorphic the Galois group of F over K by the reciprocity homomorphism of global class field theory.

The reason that we use incoherent data to define S, rather than just the unitary group $G(v)$ with its real conjugacy class X_v, is that the latter depends on the choice of a real place of k, and only defines S over a subfield of $\mathbb{C}$. However, for any other real place w of k, we have an isomorphism of adelic groups $G(w)(\mathbb{A}^f) \cong G(v)(\mathbb{A}^f)$ and an isomorphism complex varieties

$$S_M(K_w) \cong G(w)(k)\backslash X_w \times G(w)(\mathbb{A}^f)/M$$

This follows from results of Borovoi on the conjugation of Shimura varieties. Therefore it is more symmetrical to use the incoherent definite Hermitian data to define S.

11 A conjecture on the superspecial locus

Let k be a totally real number field, with ring of integers A. Let $V = \{V_v\}$ be definite incoherent orthogonal data of dimension n and determinant d for k. At each real place v we have used the neighboring orthogonal space $V(v)$, its special orthogonal group $G(v) = \mathrm{SO}(V(v))$ over k, and the Hermitian symmetric space $X_v = G(v)(k_v)/H(v)(k_v)$ to describe the points of the Shimura variety S_M over the quadratic extension $K_v \cong \mathbb{C}$ of the corresponding completion $k_v = \mathbb{R}$. It is reasonable to ask if something similar occurs at the finite places p of k. Namely, can we use the special orthogonal group $G(p) = \mathrm{SO}(V(p))$ of the neighboring space at p to say something about the K_p points of S, where K_p is the unramified quadratic extension of the completion k_p?

In this section, we will attempt to do so, under the assumptions that the residual characteristic is not equal to 2 and that the A-lattice L which we have used to define the open compact subgroup $M = M_p \times M^p$ of $G(v)(\mathbb{A}^f)$ is non-degenerate at the prime p. By this we mean that $L \otimes A_p$ is a non-degenerate lattice in the orthogonal space V_p. This assumption is true for almost all primes p. It implies that d is a unit at p, that $\epsilon(V_p) = +1$, and that M_p is a hyperspecial maximal compact subgroup of $G(v)(k_p)$. In this case, it is known that the Shimura variety S_M, associated to the incoherent data and the choice of M, is defined over k and has a model over A_p with good reduction modulo p. The neighboring orthogonal space $V(p)$ is positive definite at all real places of k. At p its determinant is a unit and $\epsilon_p(V(p)) = -1$.

Let $K_p = k_p(\sqrt{D})$ be the unramified quadratic extension of k_p and let W_p be an two dimensional orthogonal space over k_p of determinant $-D$ with $\epsilon(W_p) = -1$ and let T be the one dimensional torus $(\mathrm{Res}_{K_p/k_p}\mathbb{G}_m)_{N=1} = \mathrm{SO}(W_p)$. Recall that we have proved the existence and conjugacy of isometric embeddings $W_p \to V(p)$ over k_p. Choosing an orientation of W_p gives a homomorphism $h : T \to G(p)$ over k_p as in the real case. Writing $V(p) = W_p \oplus W_p^{\perp}$ as an orthogonal direct sum, and choosing a non-degenerate lattice L_p in $W_p^{\perp}$, we obtain a hyperspecial maximal compact subgroup $J_p = \mathrm{SO}(W_p) \times \mathrm{SO}(L_p)$ of the centralizer of h. We then defined the A_{K_p}-analytic manifold $Y_p \cong G(p)(k_p)/J_p$, which parametrizes the pairs (W_p, L_p) of oriented planes of this type with the choice of a non-degenerate lattice in the orthogonal complement.

We also defined a map $F : Y_p \to Z_p$ to the space of maximal lattices Λ_p in $V(p)$, with an orientation. Fix a vector e in W_p with $q(e) = \pi$ and let $A_{K_p}.e$ be the corresponding lattice in W_p. Then $\Lambda_p = A_{K_p} + L_p$ is an oriented lattice in $V(p)$, whose stabilizer is the parahoric subgroup N_p with reductive quotient of type $\mathrm{SO}_2 \times \mathrm{SO}_{n-2}$,

We would like an interpretation of the double coset spaces

$$D = G(p)(k)\backslash(Y_p \times G(p)(\mathbb{A}^{f,p})/M^p) \cong G(p)(k)\backslash G(p)(\mathbb{A}^f)/(J_p \times M^p).$$

$$T = G(p)(k)\backslash(Z_p \times G(p)(\mathbb{A}^{f,p})/M^p) \cong G(p)(k)\backslash G(p)(\mathbb{A}^f)/(N_p \times M^p).$$

The double coset space T is finite, and at each point t of T, we have a finite arithmetic subgroup $\Gamma(t)$ of $G(p)(k)$, which is defined by the intersection

$$\Gamma(t) = G(p)(k) \cap t.(N_p \times M^p).t^{-1}$$

The intersection takes place inside the group $G(p)(\mathbb{A}^f)$. This gives T the structure of a stack of dimension 0.

The double coset space D has the structure of a A_{K_p}-analytic orbifold. Via the map $F : Y_p \to Z_p$,we obtain a map $F_D : D \to T$.The fiber of the map $F : Y_p \to Z_p$ over a point $z \in Z_p$ is isomorphic to the polydisc $(\pi A_K)^{n-2} \cong N_p/J_p$. Hence the fiber of the map F_D over the point $t \in T$ is isomorphic to the quotient of the polydisc of dimension $n-2$ over A_K by the finite group $\Gamma(t)$. Finally, there is an involution of the set T given by the action of the normalizer of N_p, and this is compatible with the involution of D given by the normalizer of J_p.

Conjecture 1 *The finite set T parametrizes the set of superspecial points on the orthogonal Shimura variety S_M modulo p. These points are all rational over the quadratic extension $E = A_{K_p}/\pi A_{K_p}$ of the residue field $F = A_p/\pi A_p$ of k_p, and the action of the Galois group of E/F is given by the involution defined by the normalizer of N_p in $G(p)(k_p)$.*

The A_{K_p}-analytic orbifold D parametrizes the set of points of the orthogonal Shimura variety S_M over A_{K_p} which have superspecial reduction modulo p. The reduction map is given by F_D and the action of the Galois group of A_{K_p}/A is given by the normalizer of J_p in $G(p)(k_p)$.

In the Hermitian case, we expect a similar conjecture to hold, mutatis mutandis. Here we assume that k is totally real, that K is a totally imaginary quadratic extension of k, and that p is a finite prime of k which is inert in K. We start with incoherent Hermitian data $\{V_v\}$ and assume that the A_K-lattice in $V(v)$ that we have used to define the open compact subgroup $M = M_p \times M^p$ is non-degenerate at p. This implies that the Hermitian determinant d is a norm locally at the prime p and that M_p is a hyperspecial compact subgroup of $U(v)(k_p)$. In this case, the unitary Shimura variety S_M, associated to the incoherent data and the choice of M, is defined over K and has a model over A_{K_p} with good reduction modulo p. The neighboring orthogonal space $V(p)$ is positive definite at all complex places of K. At the prime p its Hermitian determinant is not a norm. Let $G(p) = U(V(p))$ be the corresponding unitary group.

Let W_p be a one dimensional Hermitian space over K_p whose determinant is not a norm and let T be the one dimensional torus $(\mathrm{Res}_{K_p/k_p}\,\mathbb{G}_m)_{N=1} = U(W_p)$. Recall that we have proved the existence and conjugacy of isometric embeddings $W_p \to V(p)$ over K_p. This gives a homomorphism $h : T \to U(V(p)$ over k_p. Writing $V(p) = W_p \oplus W_p^{\perp}$ as an Hermitian direct sum, and choosing a non-degenerate lattice L_p in $W_p^{\perp}$, we obtain a hyperspecial maximal compact subgroup $J_p = U(W_p) \times U(L_p)$ of the centralizer of h. We then defined the A_{K_p}-analytic manifold $Y_p \cong G(p)(k_p)/J_p$, which parametrizes the pairs (W_p, L_p) of lines of this type together with the choice of a non-degenerate lattice in the orthogonal complement.

We also defined a map $F : Y_p \to Z_p$ to the space of maximal lattices Λ_p in $V(p)$. Fix a vector e in W_p with $\phi(e, e) = \pi$ and let $A_{K_p}.e$ be the corresponding lattice in W_p. Then $\Lambda_p = A_{K_p} + L_p$ is a maximal lattice in $V(p)$. Its stabilizer is the maximal parahoric subgroup N_p with reductive quotient $U_1 \times U_{n-1}$.

As in the orthogonal case, we would like an interpretation of the double coset spaces

$$D = G(p)(k)\backslash(Y_p \times G(p)(\mathbb{A}^{f,p})/M^p) \cong G(p)(k)\backslash G(p)(\mathbb{A}^f)/(J_p \times M^p).$$

$$T = G(p)(k)\backslash(Z_p \times G(p)(\mathbb{A}^{f.p})/M^p) \cong G(p)(k)\backslash G(p)(\mathbb{A}^f)/(N_p \times M^p).$$

This double coset space D has the structure of a A_{K_p}-analytic orbifold. Via the map $F : Y_p \to Z_p$,we obtain a map $F_D : D \to T$. The double coset space T is finite, and at each point t of T, we have a finite arithmetic subgroup $\Gamma(t)$ of $G(p)(k)$, which is defined by the intersection

$$\Gamma(t) = G(p)(k) \cap t.(N_p \times M^p).t^{-1}$$

The intersection takes place inside the group $G(p)(\mathbb{A}^f)$. This gives T the structure of a stack of dimension 0.

The fiber of the map $F : Y_p \to Z_p$ over a point $z \in Z_p$ is isomorphic to the polydisc $(\pi A_K)^{n-1} \cong N_p/J_p$. Hence the fiber of the map F_D over the point $t \in T$ is isomorphic to the quotient of the polydisc of dimension $n-1$ over A_K by the finite group $\Gamma(t)$.

Conjecture 2 *The finite set T parametrizes the set of superspecial points on the unitary Shimura variety S_M modulo p. These points are all rational over $A_{K_p}/\pi A_{K_p}$*

The A_{K_p}-analytic orbifold D parametrizes the set of points of the unitary Shimura variety S_M over A_{K_p} which have superspecial reduction modulo p, and the reduction map is given by F_D.

The only problem with these conjectures is that we do not have a general definition of the superspecial points on S_M modulo p. If $k = \mathbb{Q}$ and the points of a Shimura variety modulo p correspond to the moduli of abelian varieties of dimension g, with a polarization, endomorphism ring, and level structure, then the superspecial points on S_M modulo p correspond to those abelian varieties which are isomorphic to the product of g supersingular elliptic curves. Unfortunately, the Shimura variety S_M does not parametrize a PEL family of abelian varieties, even in the Hermitian case, as the variation of Hodge structures which is defined over $K_v = \mathbb{C}$ has weight zero. But there should be a finite set of points modulo p, all defined over the quadratic extension of the residue field, which are stable under the action of the Hecke operators away from p.

It might be possible to establish the conjecture when $k = \mathbb{Q}$, using the moduli spaces associated to the groups $CSpin(V)$ and $GU(V)$ in the orthogonal and Hermitian cases respectively. Here one has a reasonable definition of superspecial points, and in all the cases which have been studied, they are parametrized by elements of a finite stack closely related to the stack T defined above. In the orthogonal case, when the dimension $n = 3$, this is the set of supersingular points on Shimura curves over $\mathbb{Q}$ [8] [1]. When the dimension $n = 4$ and the determinant is not a square, these are the superspecial points on Hilbert modular surfaces [12], and when the dimension $n = 5$ these are the superspecial points on Siegel space of genus 2 [13]. In the Hermitian case, for general n see [18].

As a simple example, in the case of $GU(2,1)$, the Shimura variety has dimension 2 and is defined over an imaginary quadratic field. The supersingular locus at an inert prime p has dimension 1. Its components are Deligne-Lusztig curves, which in this case are isomorphic to the Fermat curve of exponent $p+1$. Each curve has p^3+1 points over the field of p^2 elements, and there are $p+1$ components that pass through each $F(p^2)$ rational point. In this case, the superspecial points are precisely the supersingular points defined over the field of p^2 elements. The stabilizer of a pair consisting of one such point and a component passing through is given by a double coset space which involves the Iwahori subgroup, so the stabilizer of a superspecial point is the non-hyperspecial maximal compact subgroup N_p, which contains an Iwahori subgroup with index $p+1$.

More generally, when $k = \mathbb{Q}$ one expects that the points on the orthogonal Shimura variety S_M over the finite field of p^2 elements will parametrize a family of orthogonal motives of rank n and weight 0. The crystalline cohomology of such a motive will be a free $\mathbb{Z}_{p^2}$ module H of rank n with an orthogonal pairing $\langle , \rangle$ and a semi-linear map F which satisfies $\langle Fx, Fy\rangle = \langle x, y\rangle$. We say the point is supersingular if the $\mathbb{Z}_p$-submodule $\Lambda = \{x \in H : Fx = x\}$ has rank n, and is superspecial if the lattice Λ is maximal in $V = \Lambda \otimes \mathbb{Q}_p$. For the moduli of $K3$ surfaces, the superspecial condition says that the surface is supersingular and has Artin invariant $\sigma_0 = 1$. The lifting of superspecial points to characteristic zero should be determined by a decomposition of the lattice $\Lambda \otimes A_{K_p}$ into a suitable plane and its orthogonal complement. Indeed, the deRham cohomology group of a lift to A_{K_p} is identified with the crystalline cohomology group H, and the Hodge filtration of deRham cohomology should determine the modulus of the lifting.

Finally, note that when M_p is hyperspecial, the components of the Shimura variety S_M are all rational over the quadratic extension K_p of the localization k_p. This is certainly necessary for the existence of any K_p rational points, let alone points reducing to the superspecial locus modulo p! Indeed in the orthogonal case, the components of S_M are rational over a finite abelian extension L of k of exponent 2. This extension depends on the image of the spinor norm from the open compact subgroup M of $G(v)(\mathbb{A}^f)$. When M_p is hyperspecial, the image of the spinor norm contains the unit classes in k_p^*/k_p^{*2} and the extension L/k is unramified at the place p. Hence the decomposition group of p in the Galois group $\mathrm{Gal}(L/k)$ is cyclic, so has order 1 or 2, and the components of S_M are all rational over the unramified quadratic extension K_p of k_p. A similar result holds in the Hermitian case, as a prime p which is inert in the quadratic extension K splits completely in the ring class extension of K where the components of S_M are rational.

12 Local systems

Fix a real place v of k. Since the group $G(v) = \mathrm{SO}(V(v))$ has a natural orthogonal linear representation $V(v)$ over k and the center of G is finite, when M is small enough we obtain an orthogonal local system $\mathscr{F}_v$ of k-vector spaces of dimension n on the complex points of corresponding Shimura variety $S_M(K_v)$:

$$\mathscr{F}_v = G(v)(k)\backslash X_v \times V(v) \times G(v)(\mathbb{A}^f)/M$$

The local system $\mathscr{F}_v \otimes k_v$ underlies a polarized variation of real Hodge structures of rank n and weight zero on S_M over K_v [9], with non-vanishing Hodge numbers are $h^{1,-1} = 1$, $h^{0,0} = n-2$, and $h^{-1,1} = 1$.

If λ is a finite place of k the local system $\mathscr{F}_v \otimes k_\lambda$ comes from a lisse k_λ-adic sheaf $\mathscr{F}_\lambda$ of rank n on S_M over k [11]. Indeed, the theory of canonical models gives a family of coverings of S_M over k with group M_λ, and we can push this torseur out using the representation $V(v) \otimes k_\lambda$ of $G(v)(k_\lambda)$. It follows from the description of the points of S_M and its coverings over the other real completions k_w that the orthogonal lisse sheat $\mathscr{F}_\lambda$ on S_M over k is independent of the choice of real place v which was used to define it.

When $k = \mathbb{Q}$ we expect that the local system $\mathscr{F}_\infty$ corresponds to the Betti cohomology over $\mathbb{C}$, and the lisse ℓ-adic sheaf $\mathscr{F}_\ell$ to the p-adic étale cohomology over $\mathbb{Q}$ of a family of motives (of rank n and weight 0) parametrized by S_M.

Now let p be a finite place of k, and consider the k-orthogonal bundle

$$\mathscr{F}_p = G(p)(k) \backslash (Z_p \times V(p) \times G(p)(\mathbb{A}^{f,p})/M^p)$$

over the finite stack

$$T = G(p)(k) \backslash (Z_p \times G(p)(\mathbb{A}^{f,p})/M^p).$$

In the case when $k = \mathbb{Q}$ and we have a conjectural family of motives parametrized by S_M, the fibers of this bundle should be the cycle classes in the crystalline cohomology of the corresponding superspecial motives, defined over the field with p^2 elements.

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