Heegner points and representation theory

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Our aim in this paper is to present a framework in which the results of Waldspurger and Gross-Zagier can be viewed simultaneously. This framework may also be useful in understanding recent work of Zhang, Xue, Cornut, Vatsal, and Darmon. It involves a blending of techniques from representation theory and automorphic forms with those from the arithmetic of modular curves. I hope readers from one field will be encouraged to pursue the other.

I want to thank Peter Green for his careful reading of an earlier version of this paper.

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1. Heegner points on $X_0(N)$

I first encountered Heegner points in 1978, when I was trying to construct points of infinite order on the elliptic curves A(p) I had introduced in my thesis [G0, page 79]. Barry Mazur gave me a lecture on Bryan Birch's work, and on his amazing computations. I had missed Birch's lectures at Harvard on the subject, as I was in Oxford in 1973-4, bemoaning the fact that no one was there to supervise graduate work in number theory.

By 1978, Birch had found the key definitions and had formulated the central conjectures, relating Heegner points to the arithmetic of elliptic curves (cf. [B], [B-S]). These concerned certain divisors of degree zero on the modular curves $X_0(N)$, and their images on elliptic factors of the Jacobian. I will review them here; a reference for this material is [G1].

A (non-cuspidal) point on the curve $X_0(N)$, over a field k of characteristic prime to N, is given by a pair (E, F) of elliptic curves over k and a cyclic N-isogeny $\phi : E \to F$, also defined over k. We represent the point x by the diagram $(E \underset{\phi}{\to} F)$; two diagrams represent the same point if they are isomorphic over a separable closure of k.

The ring $\mathrm{End}(x)$ associated to the point x is the subring of pairs (α, β) in $\mathrm{End}(E)\times\mathrm{End}(F)$ which are defined over k and give a commutative square

$$\begin{array}{ccc} E & \underset{\phi}{\to} & F \\ \alpha\downarrow & & \downarrow\beta \\ E & \underset{\phi}{\to} & F. \end{array}$$

When $\mathrm{char}(k) = 0$, the ring $\mathrm{End}(x)$ is isomorphic to either $\mathbf{Z}$ or an order $\mathcal{O}$ in an imaginary quadratic field K. In the latter case, we say the point x has complex multiplication.

Assume $\mathrm{End}(x) = \mathcal{O}$, and let $\mathcal{O}_K$ be the full ring of integers of K. The conductor c of $\mathcal{O}$ is defined as the index of $\mathcal{O}$ in $\mathcal{O}_K$. Then $\mathcal{O} = \mathbf{Z} + c\mathcal{O}_K$, and the discriminants of $\mathcal{O}_K$ and $\mathcal{O}$ are d_K and $D = d_K c^2$, respectively. We say x is a Heegner point if $\mathrm{End}(x) = \mathcal{O}$, and the conductor c of $\mathcal{O}$ is *relatively prime to* N. This forces an equality: $\mathrm{End}(x) = \mathrm{End}(E) = \mathrm{End}(F)$.

Heegner points exist (for all conductors c prime to N) precisely when all prime factors p of N are either split or ramified in K, and all factors with $\mathrm{ord}_p(N) \geq 2$ are split in K. The points of conductor c are defined over the ring class field k of conductor c over K, which is an abelian extension of K with Galois group isomorphic to $\mathrm{Pic}(\mathcal{O})$.

Let x be a Heegner point of conductor c, and let ∞ be the standard cusp on $X_0(N)$, given by the cyclic isogeny $(\mathbf{G}_m/q^{\mathbf{Z}} \underset{N}{\rightarrow} \mathbf{G}_m/q^{N\mathbf{Z}})$ of Tate curves over $\mathbf{Q}$. Consider the divisor $(x) - (\infty)$ of degree 0, and let

$$a \equiv (x) - (\infty)$$

be its class in the k-rational points of the Jacobian $J_0(N)$.

The finite dimensional rational vector space

$$W = J_0(N)(k) \otimes \mathbf{Q}$$

is a semi-simple module for the Hecke algebra $\mathbf{T}$, generated over $\mathbf{Q}$ by the operators T_m, for m prime to N, and the involutions w_n, for n dividing N. The Galois group $\mathrm{Gal}(k/K)$ also acts on W, and commutes with $\mathbf{T}$. Note that the complex characters of the commutative $\mathbf{Q}$-algebra $\mathbf{T}[\mathrm{Gal}(k/K)]$ are indexed by pairs (f, χ), where f is a cuspidal eigenform of weight 2 for $\Gamma_0(N)$ and χ is a ring class character of conductor dividing c.

Assume that f is a new form, of level N, and that χ is primitive, of conductor c. Let $a(f, \chi)$ be the projection of the class a in W to the (f, χ)-eigenspace in $W \otimes \mathbf{C}$. The central question on Heegner divisors is to determine *when* $a(f, \chi)$ *is non-zero.*

If a prime p divides both d_K and N, then $(p) = \wp^2$ is ramified in K and $\mathrm{ord}_p(N) = 1$. In this case, $\chi(\wp) = \pm 1$ and $a_p(f) = \pm 1$. When $a_p(f) \cdot \chi(\wp) = 1$, the class $a(f, \chi)$ is zero, for simple reasons [G1]. In what follows, we will assume that $a_p(f) \cdot \chi(\wp) = -1$, for all primes p dividing both d_K and N.

2. Rankin L-series and a height formula

Associated to the pair (f, χ), one has the Rankin L-function

$$L(f, \chi, s) = \prod_p L_p(f, \chi, s),$$

defined by an Euler product in the half plane $\mathrm{Re}(s) > 3/2$. The Euler factors have degree ≤ 4 in p^{-s}, and are given explicitly in [G1]. Rankin's method shows that the product

$$\Lambda(f, \chi, s) = ((2\pi)^{-s}\Gamma(s))^2 L(f, \chi, s)$$

has an analytic continuation to the entire plane, and satisfies the functional equation:

$$\Lambda(f, \chi, 2-s) = -A^{1-s} \cdot \Lambda(f, \chi, s)$$

with $A = (ND)^2/gcd(N, D)$. The sign in this functional equation is -1, as the local signs ϵ_p at finite primes p are all $+1$, and the local sign ϵ_∞ at the real prime is -1 [G1]. Hence $L(f, \chi, s)$ vanishes to odd order at the central critical point $s = 1$.

Birch considered the projection $a(f, \chi)$ in the special case of *rational* characters (f, χ) of the $\mathbf{Q}$-algebra $\mathbf{T}[\mathrm{Gal}(k/K)]$. In this case, f corresponds to an elliptic curve factor A of $J_0(N)$ over $\mathbf{Q}$, and χ to a factorization $D = d_1 d_2$ into two fundamental discriminants. If A_1 and A_2 are the corresponding quadratic twists of A over $\mathbf{Q}$, then

$$L(f, \chi, s) = L(A_1, s) L(A_2, s).$$

He discovered that $a(f, \chi)$ was non-zero precisely when the order of vanishing of $L(f, \chi, s)$ at $s = 1$ was equal to one. In this case, he obtained a wealth of computational evidence in support of a new limit formula, relating the first derivative of the Rankin L-series to the height of the projected Heegner point on the elliptic curve A, over the biquadratic field $\mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$.

The extension of Birch's conjecture to *all* complex characters (f, χ) of the algebra suggested a similar formula for $L'(f, \chi, 1)$, of the type

$$L'(f, \chi, 1) = \frac{(f, f)}{\sqrt{D}} \;\hat{h}(a(f, \chi)).$$

Here (f, f) is a normalized Petersson inner product, and $\hat{h}$ is the canonical height on $J_0(N)$ over K. Conjecturing the formula in this generality allows one to unwind the various projection operations, and obtain an equivalent identity involving the height paring of a Heegner divisor with a Hecke translate of its Galois conjugate. This is a simpler identity to prove, as one can use Néron's theory of local heights on the original modular curve, where the Heegner points and Hecke operators have a modular interpretation. Another advantage in this formulation is that the Rankin L-series is easier to study than the product of two Hecke L-series, owing to its integral representation.

Working along these lines, Zagier and I obtained such a formula, for D square-free and relatively prime to N, in 1982 [G-Z]. Zhang has recently established a similar formula in great generality [Z1].

3. Starting from the L-function

In the above formulation, one starts with Heegner divisor classes $a \equiv (x) - (\infty)$ on $J_0(N)$. The Rankin L- function $L(f, \chi, s)$ is introduced in order to study the projection $a(f, \chi)$.

To generalize beyond the modular curves $X_0(N)$, we will *reverse* matters and start with the Rankin L-function $L(f, \chi, s)$. The form f corresponds to an automorphic, cuspidal representation π of the adelic group $GL_2(\mathbf{A_Q})$, with trivial central character and π_∞ in the discrete series of weight 2. The character χ corresponds to a Hecke character of $GL_1(\mathbf{A}_K)$, with trivial restriction to $GL_1(\mathbf{A_Q})$, and $\chi_\infty = 1$. The tensor product $\pi \otimes \chi$ gives an automorphic cuspidal representation of a group G of unitary similitudes over $\mathbf{Q}$, which has L-function $L(f, \chi, s)$ via a four-dimensional, symplectic representation of the L-group of G.

The key idea is to use the local signs in the functional equation of $L(f, \chi, s)$ to define an arithmetic object — either an inner form G' of G, or a Shimura curve $M(G_S)$. We then use the representation theory of G' to study the central critical value $L(f, \chi, 1)$, following

Waldspurger [W], and the arithmetic geometry of the Shimura curve and its special points to study the central critical derivative $L'(f,\chi,1)$. Using the local and global representation theory of GL_2 and its inner forms, we can formulate both cases in a similar manner.

4. Local representation theory

We begin with the local theory. Let k be a local field, and let E be an étale quadratic extension of k. Then E is either a field, or is isomorphic to the split k-algebra $k[x]/(x^2-x)\simeq k+k$. In the latter case, there are two orthogonal idempotents e_1 and e_2 in E, with $e_1+e_2=1$. Let $e\mapsto \overline{e}$ be the nontrivial involution of E fixing k; in the split case $\overline{e}_1=e_2$. By local class field theory, there is a unique character $\alpha: k^*\to\langle\pm1\rangle$ whose kernel is the norm group $\mathbf{N}E^*=\{e\overline{e}: e\in E^*\}$ in k^*.

Let π be an irreducible complex representation of the group $GL_2(k)$, and let $\omega: k^*\to \mathbf{C}^*$ be the central character of π. We will assume later that π is generic, or equivalently, that π is infinite-dimensional.

Let S be the two-dimensional torus $\mathrm{Res}_{E/k}\mathbf{G}_m$, and let χ be an irreducible complex representation of the group $S(k)=E^*$. Since E has rank 2 over k, we have an embedding of groups:

$$S(k)\simeq \mathrm{Aut}_E(E)\to GL_2(k)\simeq \mathrm{Aut}_k(E)$$

We will consider the tensor product $\pi\otimes\chi$ as an irreducible representation of the group $GL_2(k)\times S(k)$, and wish to restrict this representation to the diagonally embedded subgroup $S(k)$.

The central local problem is to compute the space of coinvariants $\mathrm{Hom}_{S(k)}(\pi\otimes\chi,\mathbf{C})$. If this is non-zero, we must have

$$\omega\cdot\mathrm{Res}(\chi)=1 \qquad (*)$$

as a character of k^*. Indeed, $\omega\cdot\mathrm{Res}(\chi)$ gives the action of $k^*\subset E^*$ on all vectors in $\pi\otimes\chi$.

We will henceforth assume that (*) holds. Then $\pi \otimes \chi$ is an irreducible representation of $G(k)$, with

$$G = (GL_2 \times S)/\Delta\mathbf{G}_m,$$

and we wish to restrict it to the subgroup $T(k)$, where T is the diagonally embedded, one-dimensional torus $S/\mathbf{G}_m$.

5. Unitary similitudes

The group G defined above is a group of unitary similitudes. Indeed, let k be a field, and let $E \subset B$ be an étale, quadratic algebra E over k, contained in a quaternion algebra B over k. The k-algebra B is then graded: $B = B_+ + B_-$ with

$$B_+ = E$$

$$B_- = \{b \in B : be = \bar{e}b \quad \text{all}\, e \in E\}$$

Both B_+ and B_- are free E-modules of rank 1. The pairing $\phi : B \times B \to E$, defined by

$$\begin{aligned}\phi(b_1, b_2) &= (b_1\bar{b}_2)_+ \\ &= \text{first component of } b_1\bar{b}_2,\end{aligned}$$

is a non-degenerate Hermitian form on the free E module B of rank 2.

The group $GU(B, \phi)$ of unitary similitudes has k-valued points isomorphic to $B^* \times E^*/\Delta k^*$. To give a specific isomorphism, we let the pair (b, e) in $B^* \times E^*$ act on $x \in B$ by

$$(b, e)(x) = exb^{-1}.$$

Then Δk^* acts trivially on B, and the similitude factor for ϕ is $\mathbf{N}e/\mathbf{N}b$ in k^*.

When E is split, the algebra B is a matrix algebra. If we take $B = \mathrm{End}_k(E) = M_2(k)$, using the basis $\langle e_1, e_2 \rangle$ of orthogonal idempotents, then the k-valued points of $GU(B, \phi)$ are isomorphic to $GL_2(k) \times k^*$. The pair $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda\right)$ acts on $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda\right)(x) = \begin{pmatrix} \lambda & \\ & 1 \end{pmatrix} \cdot x \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

The unit element $1 \in B$ satisfies $\phi(1,1) = 1$, and the subgroup fixing this vector is the diagonally embedded torus $T = S/\mathbf{G}_m$, which acts as the unitary group of the line B_-.

Conversely, if (V, ϕ) is a non-degenerate unitary space of dimension 2 over E, and v is a vector in V satisfying $\phi(v, v) = 1$, we may give V the structure of a quaternion algebra over k, containing the quadratic algebra E. Indeed

$$V = E \cdot v + W, \quad \text{with} \quad W = (Ev)^\perp$$

and we define multiplication by

$$(\alpha v + w)(\alpha' v + w') = (\alpha\alpha' - \phi(w, w'))v + (\alpha w' + \overline{\alpha}' w).$$

The group $GU(V, \phi)(k)$ is then isomorphic to $B^* \times E^*/\Delta k^*$, with B the quaternion algebra so defined.

6. The L-group and its symplectic representation

The L-group of $G = GU(V, \phi)$ depends only on E, not on the quaternion algebra B. If E is split, $G \simeq GL_2 \times \mathbf{G}_m$ and

$${}^L G \simeq GL_2 \times \mathbf{G}_m.$$

If E is a field, ${}^L G$ is a semi-direct product

$${}^L G \simeq (GL_2 \times \mathbf{G}_m) \bowtie \mathrm{Gal}(E/k).$$

The action of the generator τ of $\mathrm{Gal}(E/k)$ is by

$$\tau(\lambda, g) = (\lambda \cdot \det g, g \cdot (\det\, g)^{-1}).$$

In all cases, the L-group ${}^L G$ has a four-dimensional symplectic representation

$$\rho : {}^L G \to Sp_4$$

with kernel isomorphic to $\mathbf{G}_m$ and image contained in the normalizer of a Levi factor in a Siegel parabolic. We will encounter this representation in §8.

7. Inner forms

We return to the case when k is a local field. If E is split, there is only one possible similitude group, corresponding to the quaternion algebra $M_2(k)$ of 2×2 matrices:

$$G(k) = (GL_2(k) \times E^*)/\Delta k^* \simeq GL_2(k) \times k^*.$$

When E is a field, it embeds into $M_2(k)$ as well as into the unique quaternion division algebra B over k (since E is a field, $k \neq \mathbf{C}$). This gives two similitude groups

$$G(k) = (GL_2(k) \times E^*)/\Delta k^*$$

$$G'(k) = (B^* \times E^*)/\Delta k^*.$$

Both contain the diagonally embedded torus

$$T(k) = E^*/k^*,$$

and the latter is compact modulo its center.

If π is an irreducible, complex representation of $GL_2(k)$ with central character ω, and χ is a character of E^*, which satisfies $\omega \cdot \mathrm{Res}(\chi) = 1$, then $\pi \otimes \chi$ is an irreducible representation of $G(k)$. If π is square-integrable, it corresponds to a unique finite-dimensional, irreducible, complex representation π' of B^*. The representation π' is described in [J-L]; it is characterized by

$$\mathrm{Tr}(g|\pi) + \mathrm{Tr}(g|\pi') = 0$$

for all elliptic, semi-simple classes g, and has central character ω. If E is a field, then $\pi' \otimes \chi$ gives an irreducible representation of $G'(k)$, which can be restricted to the diagonally embedded torus $T(k)$.

In the next section, we will compute the complex vector spaces:

$$\mathrm{Hom}_{T(k)}(\pi \otimes \chi, \mathbf{C}),$$

$$\mathrm{Hom}_{T(k)}(\pi' \otimes \chi, \mathbf{C}).$$

8. Langlands parameters

Let W_k denote the Weil group of k, and normalize the isomorphism $k^* \simeq W_k^{ab}$ of local class field theory to map a uniformizing parameter to a geometric Frobenius element, in the non-Archimedean case.

The representation π of $GL_2(k)$ has a local Langlands parameter [G-P]

$$\sigma_\pi : W_k' \to GL_2(\mathbf{C}).$$

Here W_k' is the Weil-Deligne group, and we normalize the parameter so that $\det(\sigma_\pi) = \omega$.

The character χ of E^* has conjugate χ^τ defined by $\chi^\tau(\alpha) = \chi(\overline{\alpha})$. Since $\chi \cdot \chi^\tau(\alpha) = \chi(\alpha\overline{\alpha}) = \omega^{-1}(\alpha\overline{\alpha})$, we have

$$\chi^\tau = \chi^{-1} \cdot (\omega \circ \mathbf{N})^{-1}.$$

When E is split, so $E^* = k^* \times k^*$, we have $\chi = (\eta_1, \eta_2)$ and $\chi^\tau = (\eta_2, \eta_1)$. In general, the pair (χ, χ^τ) gives a homomorphism, up to conjugacy

$$\sigma_\chi : W_k \to (\mathbf{C}^* \times \mathbf{C}^*) \rtimes \mathrm{Aut}(E/k).$$

The kernel of $\alpha : W_k \to \langle \pm 1 \rangle$ maps to the subgroup $\mathbf{C}^* \times \mathbf{C}^*$ via (χ, χ^τ), and $\mathrm{Aut}(E/k)$ permutes the two factors. The complex group on the right is $GO_2(\mathbf{C})$; in this optic, σ_χ is given by the induced representation $\mathrm{Ind}(\chi) = \mathrm{Ind}(\chi^\tau)$.

We view σ_π as a homomorphism from W_k' to $GSp(V)$, with $\dim V = 2$, having similitude factor ω. We view σ_χ as a homomorphism from W_k to $GO(V')$, with $\dim V' = 2$, having similitude factor $\mathrm{Res}(\chi) = \mathrm{Res}(\chi^\tau)$. Since $\omega \cdot \mathrm{Res}(\chi) = 1$, the tensor product $V \otimes V'$ is a four-dimensional, symplectic representation of the Weil-Deligne group:

$$\sigma_\pi \otimes \sigma_\chi : W_k' \to Sp(V \otimes V') = Sp_4(\mathbf{C}).$$

This is the composition of the Langlands parameter of the representation $\pi \otimes \chi$ of $G(k)$

$$\sigma_{\pi\otimes\chi} : W_k' \to {}^L G(\mathbf{C})$$

with the symplectic representation

$$\rho : {}^L G(\mathbf{C}) \to Sp_4(\mathbf{C}),$$

mentioned in §6. The key point is that the image of this symplectic representation lands in the normalizer of a Levi factor.

9. Local ϵ-factors

In particular, we have an equality of Langlands L and ϵ-factors with the Artin-Weil L and ϵ-factors:

$$L(\pi \otimes \chi, \rho, s) = L(\sigma_\pi \otimes \sigma_\chi, s)$$

$$\epsilon(\pi \otimes \chi, \rho, \psi, dx, s) = \epsilon(\sigma_\pi \otimes \sigma_\chi, \psi, dx, s).$$

We normalize the L-function so that the functional equation relates s to $1-s$, and the central point is $s = 1/2$. Since this is the only representation ρ of ${}^L G$ we will consider, we suppress it in the notation and write $L(\pi \otimes \chi, s)$ for $L(\pi \otimes \chi, \rho, s)$.

Since $\sigma_\pi \otimes \sigma_\chi$ is symplectic, we can also normalize the local constant following [G4]. If dx is the unique Haar measure on k which is self-dual for Fourier transform with respect to ψ, then

$$\epsilon(\pi \otimes \chi) = \epsilon(\sigma_\pi \otimes \sigma_\chi, \psi, dx, 1/2)$$

depends only on $\sigma_\pi \otimes \sigma_\chi$, and satisfies

$$\epsilon(\pi \otimes \chi)^2 = 1.$$

Since the representation ρ of ${}^L G$ is not faithful, certain non-isomorphic representations of $G(k)$ have the same local L- and ϵ-factors. Since $\sigma_\chi = \sigma_{\chi^\tau}$ as representations to $GO_2(\mathbf{C})$, we have

$$L(\pi \otimes \chi, s) = L(\pi \otimes \chi^\tau, s)$$

$$\epsilon(\pi \otimes \chi) = \epsilon(\pi \otimes \chi^\tau).$$

Also, if η is any character of k^*, and we define

$$\pi^* = \pi \otimes (\eta \circ \det)$$

$$\chi^* = \chi \otimes (\eta \circ \mathbf{N})^{-1},$$

then $\sigma_{\pi^*} \otimes \sigma_{\chi^*} = \sigma_\pi \otimes \sigma_\chi$, so

$$L(\pi^* \otimes \chi^*, s) = L(\pi \otimes \chi, s)$$

$$\epsilon(\pi^* \otimes \chi^*) = \epsilon(\pi \otimes \chi).$$

Note, however, that when $\eta \neq 1$ the representation $\pi \otimes \chi$ and $\pi^* \otimes \chi^*$ of $G(k)$ are *not* isomorphic, as they have distinct central characters. Their restrictions to the subgroup $U_2 \subset GU_2$ *are* isomorphic, and U_2 contains the torus T.

Finally, since the dual of the representation $\pi \otimes \chi$ of $G(k)$ is isomorphic to

$$\begin{aligned}(\pi \otimes \chi)^\vee &\simeq \pi^\vee \otimes \chi^\vee \\ &\simeq (\pi \otimes \omega(\det)^{-1}) \otimes \chi^{-1} \\ &\simeq \pi^* \otimes (\chi^\tau)^*\end{aligned}$$

with $\eta = \omega^{-1}$, we have

$$L((\pi \otimes \chi)^\vee, s) = L(\pi \otimes \chi, s)$$

$$\epsilon((\pi \otimes \chi)^\vee) = \epsilon(\pi \otimes \chi).$$

10. Local linear forms

We can now state the main local result, which is due to Tunnell and Saito (cf. [T], [S]). Recall that the representation π is generic if it has a non-zero linear functional on which the unipotent radical of a Borel subgroup acts via a non-trivial character. A fundamental result states that this occurs precisely when π is infinite dimensional. Also, recall that π has square-integrable matrix coefficients if and only if it lies in the discrete series for GL_2.

Theorem. *Assume that π is generic. Then* $\dim \operatorname{Hom}_{T(k)}(\pi \otimes \chi, \mathbf{C}) \leq 1$, *with equality holding precisely when*

$$\epsilon(\pi \otimes \chi) = \alpha \cdot \omega(-1).$$

If $\dim \operatorname{Hom}_{T(k)}(\pi \otimes \chi, \mathbf{C}) = 0$, *then π is square-integrable and E is a field. In this case, we have* $\dim \operatorname{Hom}_{T(k)}(\pi' \otimes \chi, \mathbf{C}) = 1$.

Informally speaking, this says that the $T(k)$ co-invariants in the representation

$$(\pi \otimes \chi) + (\pi' \otimes \chi)$$

have dimension 1, and the location of the co-invariants is given by the sign:

$$\epsilon(\pi \otimes \chi)/\alpha \cdot \omega(-1).$$

Note that this result is compatible with the identities

$$\epsilon(\pi \otimes \chi) = \epsilon(\pi \otimes \chi^{\tau}) = \epsilon(\pi^* \otimes \chi^*),$$

where $\pi^* = \pi \otimes (\eta \circ \det)$ and $\chi^* = \chi \otimes (\eta \circ \mathbf{N})^{-1}$.

11. Local test vectors

We can refine the result on $T(k)$-invariant linear forms ℓ on $\pi \otimes \chi$ or $\pi' \otimes \chi$ in favorable cases, by giving *test vectors* on which the non-zero invariant linear forms are non-zero. These vectors will lie on a line $\langle v \rangle$ fixed by a compact subgroup M in $G(k)$ or $G'(k)$, and M will be well-defined up to $T(k)$-conjugacy.

For k non-Archimedean, the favorable cases are when either the representation π, or the character χ, is unramified. In this case, $\langle v \rangle$ will be the fixed space of an open, compact subgroup M (cf. [G-P]). We will give a construction of M from the point of view of Hermitian lattices, first in the case when π is unramified, and then in the case when χ is unramified.

Let A be the ring of integers of k, and let $\mathcal{O}_E$ be the integral closure of A in E. Since either ω or $\mathrm{Res}(\chi)$ is unramified, and $\omega \cdot \mathrm{Res}(\chi) = 1$, both are trivial on the subgroup A^*. Writing $\omega = \eta^{-2}$, where η is an unramified character of k^*, and twisting π by $\eta(\det)$ and χ by $\eta(\mathbf{N})^{-1}$, we may assume that $\omega = \mathrm{Res}(\chi) = 1$.

Let c be the conductor ideal of χ, so χ is trivial when restricted to $(1 + c\mathcal{O}_E)$. Since χ is a character of E^*/k^*, c is the extension of an ideal of A, and the order

$$\mathcal{O} = A + c\mathcal{O}_E$$

is stable under conjugation. Since

$$\mathcal{O}^* = A^*(1 + c\mathcal{O}_E),$$

the character χ of E^* is trivial, when restricted to the subgroup $\mathcal{O}^*$.

Now assume that π is unramified. Then, by the theorem in the previous section, we have a $T(k)$-invariant linear form on $\pi \otimes \chi$. The quaternion algebra $B = \mathrm{End}_k(E)$ contains the quadratic algebra $B_+ = \mathrm{End}_E(E) = E$, and $B_- = E\tau$ is the E-submodule of antilinear maps. The associated Hermitian space has

$$\phi(\alpha + \beta\tau, \alpha + \beta\tau) = \mathbf{N}\alpha - \mathbf{N}\beta,$$

and contains the $\mathcal{O}$-lattice of rank 2:

$$L = \mathrm{End}_A(\mathcal{O}) \supset \mathcal{O} + \mathcal{O}\tau.$$

We define the open compact subgroup M of $G(k) = GU(B, \phi)$ as the stabilizer of L:

$$\begin{aligned} M = GU(L, \phi) &\simeq \mathrm{Aut}_A(\mathcal{O}) \times \mathcal{O}^*/\Delta A^* \\ &\simeq GL_2(A) \times \mathcal{O}^*/\Delta A^*. \end{aligned}$$

Note that ϕ, when restricted to L, takes values in $\mathcal{O}^\vee = \mathrm{Hom}(\mathcal{O}, A) \subset E$. Since $L \cap E = \mathcal{O}$, the intersection of M with $T(k) = E^*/k^*$ is $\mathcal{O}^*/A^*$. The proof that M fixes a unique line in $\pi \otimes \chi$, and that non-zero vectors on this line are test vectors for the $T(k)$-invariant linear form, is given in [G-P, §3].

The construction of $M \subset G(k)$ which we have just given, when π is unramified, appears to be a natural one. But we could also have taken the $\mathcal{O}$-lattice $L' = \mathrm{End}_A(P)$, where $P = \mathcal{O}\alpha$ is any proper $\mathcal{O}$-submodule of E, with $\alpha \in E^*$. The resulting stabilizer $M' = \mathrm{Aut}_A(P) \times \mathcal{O}^*/\Delta A^*$ in $G(k)$ is then the conjugate of M by the image of α in $T(k) = E^*/k^*$. This gives an action of the quotient group $E^*/k^* \cdot \mathcal{O}^*$ on the open compact subgroups M we have defined, and hence on their fixed lines $(\pi \otimes \chi)^M = \langle v \rangle$.

We now turn to the construction of M in the case when the character χ is unramified. The Hermitian lattice

$$L = \mathcal{O}_E + \mathcal{O}_E \cdot w$$

is determined by $\phi(w, w)$ in $A - \{0\}$, up to multiplication by $\mathbf{N}\mathcal{O}_E^*$. If E/k is unramified, $\mathbf{N}\mathcal{O}_E^* = A^*$, and $L = L_n$ is completely determined by $n = \mathrm{ord}(\phi(v, v))$. We let n be the conductor of π, and let $M = M_n$ be the subgroup of $GU(V, \phi)$ stabilizing L_n.

When E is split, M_n is open and compact in $G(k) \simeq GL_2(k) \times k^*$, of the form $R_n^* \times A^*$, where R_n is an Eichler order of conductor n. Hence M_n fixes a line in $\pi \otimes \chi$, giving test vectors for the T-invariant form, by [G-P,§4]. When E is the unramified quadratic field extension, $GU(V, \phi)$ is isomorphic to $G(k)$ when n is even, and to $G'(k)$ when n is odd.

Since $\epsilon(\pi \otimes \chi) = (-1)^n$ in this case, we find that M_n contains $T(k)$, so that the line fixed by M_n in $\pi \otimes \chi$ or $\pi' \otimes \chi$ provides test vectors [G-P, prop. 2.6].

Now assume E/k is tamely ramified and that π has conductor $n+1 \geq 1$. Then $\mathbf{N}\mathcal{O}_E^*$ has index 2 in A^*, and there are two Hermitian lattices L_n and L_n' of the form $\mathcal{O}_E + \mathcal{O}_E w$, with $\mathrm{ord}(\phi(w,w)) = n$. One has $M_n = GU(L_n, \phi)$ open and compact in $G(k)$, and the other has $M_n' = GU(L_n', \phi)$ open and compact in $G'(k)$. Both contain the compact subgroup $T(k)$, and fix a line in $\pi \otimes \chi$ and $\pi' \otimes \chi$, respectively—depending on the sign of $\epsilon(\pi \otimes \chi)$. This line provides test vectors for the $T(k)$-invariant linear form.

For k real and E complex, the torus $M = T(k)$ is compact. Its fixed space in either $\pi \otimes \chi$ or $\pi' \otimes \chi$ provides test vectors for the invariant form.

The remaining case is when k is Archimedean, and the algebra E is split. Let K be the maximal compact subgroup of $GL_2(k) = \mathrm{Aut}_k(E)$, which fixes the positive definite form $\mathrm{Tr}(e^2)$, when $k = \mathbf{R}$, and $\mathrm{Tr}(ee')$, when $k = \mathbf{C}$ and $(z,w)' = (\overline{z}, \overline{w})$. Let L be the maximal compact subgroup of k^*, so $K \times L$ is a maximal compact subgroup in $G(k) \simeq GL_2(k) \times k^*$.

Let $W \otimes \chi$ be the minimal $(K \times L)$-type in the representation $\pi \otimes \chi$. By construction, the intersection

$$M = T(k) \cap (K \times L)$$

is isomorphic to L. One shows that the M-invariants in $W \otimes \chi$ have dimension ≤ 1. The favorable situation is when the M-invariants have dimension 1: this line provides test vectors for the $T(k)$-invariant linear form [P].

12. An explicit local formula

We end the local section with a formula for the central critical value of the L-function, which provides a model for the global case.

Assume that k is non-Archimedean, with ring of integers A, and that E is split over

k. Let $B = \mathrm{End}_k(E)$, and fix an isomorphism

$$B \simeq M_2(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in k \right\}$$

such that

$$E = B_+ = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\}$$

$$B_- = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\}.$$

We assume $\pi \otimes \chi$ is a generic representation of $G(k) = B^* \times E^* / \Delta k^*$, and that the character χ of E^* is unramified.

Let U be the unipotent subgroup of G with

$$U(k) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \times 1 \right\}.$$

We fix an isomorphism $G(k) \simeq GL_2(k) \times k^*$, so that T maps to the torus with points

$$T(k) = \left\{ g_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \times \lambda \right\}.$$

The element g_λ acts by conjugation on $U(k)$, and the isomorphism has been chosen so that this action is given by multiplication by λ. Via the chosen isomorphism, the representation $\pi \otimes \chi$ of $G(k)$ corresponds to a representation $\pi \otimes \eta$ of $GL_2(k) \times k^*$. The representation $\pi \otimes \chi^\tau$ corresponds to the representation $\pi \otimes \eta'$, with $\eta \cdot \eta' \cdot \omega = 1$, and the contragradient $(\pi \otimes \chi)^\vee$ corresponds to the representation $\pi \cdot \omega^{-1} \otimes \eta^{-1} = \pi \cdot \omega^{-1} \otimes \eta' \omega$. Here we use the notation $\pi \cdot \alpha$, for a character α of k^*, to denote the representation $\pi \otimes \alpha(\det)$ of $GL_2(k)$. Note that

$$L(\pi \otimes \chi, s) = L(\pi \cdot \eta, s) L(\pi \cdot \eta', s)$$

as the representation σ_χ is the direct sum of the characters η and η'.

Let $\psi : U(k) \to S^1$ be a non-trivial character, with kernel $U(A)$, and let

$$m : \pi \otimes \chi \to \mathbf{C}$$

be a non-zero linear form on which $U(k)$ acts by ψ. This exists, and is unique up to scaling, by the genericity of $\pi \otimes \chi$.

Let $n \geq 0$ be the conductor of π, and

$$K_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \lambda \right\}$$

be the subgroup of $GL_2(A) \times A^*$ with $c \equiv 0 \pmod{\mathcal{P}_A^n}$. Then K_n fixes a unique line $\langle v \rangle$ in $\pi \otimes \chi$, by results of Casselman [C]. Moreover, one has $m(v) \neq 0$, by [G-P, Lemma 4.1]. Indeed, in the Kirillov model, the linear form m is given by $f \to f(1)$, and the spherical line $\langle v \rangle$ is explicitly determined in the function space. If we take the unique spherical vector on this line with $m(v) = 1$, then

$$\int_{k^*} m(g_\lambda v) \cdot |\lambda|^{s-1/2} \; d^*\lambda =$$
$$\int_{k^*} m\left(\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix} v_0 \right) \eta(\lambda) \cdot |\lambda|^{s-1/2} \; d^*\lambda,$$

where $v = v_0 \otimes 1$, and $d^*\lambda = \frac{d\lambda}{|\lambda|}$ with $\int_A d\lambda = 1$. But

$$m(gv_0) = W_0(g)$$

is the classical Whittaker function on $GL_2(k)$, normalized so that $W_0(e) = 1$. Hence, by [J-L],

$$\int_{k^*} m(g_\lambda v_0) \cdot |\lambda|^{s-1/2} d^*\lambda = L(\pi \cdot \eta, s)$$

is the Hecke L-series of the representation $\pi \cdot \eta$.

On the other hand, the map

$$\ell : \pi \otimes \chi \to \mathbf{C}$$

defined on vectors w by the integral

$$\ell(w) = \int_{k^*} m(g_\lambda w) \; d^*\lambda$$

gives (when the indefinite integral is convergent) a $T(k)$-invariant linear form on $\pi \otimes \chi$. Hence the line $\langle v \rangle$ provides test vectors for $\pi \otimes \chi$, whenever $L(\pi \cdot \eta_1, s)$ does *not* have a pole at $s = 1/2$, and we have the formula

$$\ell(v) = m(v) \cdot L(\pi \cdot \eta, 1/2)$$

for *any* vector v on that line.

The same considerations apply to the contragradient representation $(\pi \otimes \chi)^\vee$. Denoting the Whittaker linear functional by $m^\vee$, its T-invariant integral by $\ell^\vee$, and the K_n-line of test vectors by $\langle v^\vee \rangle$, we obtain the formula

$$\ell^\vee(v^\vee) = m^\vee(v^\vee) \cdot L(\pi \cdot \eta', 1/2).$$

Indeed, π is replaced by $\pi \cdot \omega^{-1}$ and η by $\eta^{-1} = \eta' \omega$. Multiplying the two formulas, we obtain

$$\ell(v) \cdot \ell^\vee(v^\vee) = m(v) \cdot m^\vee(v^\vee) \cdot L(\pi \otimes \chi, 1/2).$$

Since $\langle v \rangle$ is the line of K_n-invariants in $\pi \otimes \chi$, and $\langle v^\vee \rangle$ is the line of K_n-invariants in $(\pi \otimes \chi)^\vee$, we have

$$\langle v, v^\vee \rangle \neq 0$$

under the canonical pairing of $\pi \otimes \chi$ with $(\pi \otimes \chi)^\vee$. Hence, our final formula may be rewritten as

$$\frac{\ell(v)\ell^\vee(v^\vee)}{\langle v, v^\vee \rangle} \cdot \frac{\langle v, v^\vee \rangle}{m(v)m^\vee(v^\vee)} = L(\pi \otimes \chi, 1/2).$$

It is this form which we will generalize to the global case. Note that $m, m^\vee, v$, and $v^\vee$ are only determined up to scaling, but that the left-hand side of the formula is well defined.

13. Adèlic groups

We now turn to the global theory. Let k be a global field, with ring of adèles $\mathbf{A}$, and let E be an étale quadratic extension of k. Let

$$\pi \otimes \chi = \hat{\bigotimes_v}(\pi_v \otimes \chi_v)$$

be an admissable, irreducible representation of the adèlic group

$$G(\mathbf{A}) = GL_2(\mathbf{A}) \times \mathbf{A}_E^*/\Delta\mathbf{A}^*.$$

Then the local components π_v and χ_v are unramified, for almost all finite places v. Consequently, the set

$$S = \{v : \epsilon(\pi_v \otimes \chi_v) \neq \alpha_v \omega_v(-1)\} \tag{13.1}$$

is finite.

Assume that each local component π_v is infinite-dimensional. Then, by our local results, we may define an admissable, irreducible representation

$$\pi' \otimes \chi = \hat{\bigotimes_v}(\pi'_v \otimes \chi_v)$$

of the locally compact group

$$G_{S,\mathbf{A}} = \prod_v G'(k_v),$$

where $G' = G$ at the places not in S, and G' is the non-trivial local inner form of G at the places in S. Thus $G'(k_v)$ is defined by the split quaternion algebra over k_v, for the places not in S, and by the quaternion division algebra over k_v at the places in S. The representation $\pi' \otimes \chi$ has the property that

$$\mathrm{Hom}_{T(\mathbf{A})}(\pi' \otimes \chi, \mathbf{C})$$

is of dimension 1.

We emphasize that $G_{S,\mathbf{A}}$ need *not* be the adèlic points of a group G' defined over k. This will only be the case when the cardinality of the set S is even. In that case, let B be the quaternion algebra over k ramified at S, which exists by the results of global class field theory. Then, if we define

$$G'(k) = B^* \times E^*/\Delta k^*,$$

we have

$$G'(\mathbf{A}) = G_{S,\mathbf{A}}.$$

In this case, $G_{S,\mathbf{A}}$ not only contains the diagonally embedded adèlic group $T(\mathbf{A})$, but also contains the discrete subgroup $G'(k)$. Their intersection is the subgroup $T(k)$.

14. A special case

We now discuss a special case of the above, which is important in arithmetic applications. For this, we assume that k is a number field, and that the set S defined in (13.1) *contains all Archimedean places v of k.* This hypothesis has a number of surprising consequences, and seems essential if we wish to obtain an algebraic theory.

First it implies that the number field k is totally real, and the quadratic extension E of k is totally complex, so is a CM field. Indeed, at each place v of S the algebra $E \otimes k_v$ is a field.

Next, since π_v is square-integrable at each real place of k, and π'_v is finite-dimensional, we find that the representation $(\pi' \otimes \chi)_\infty = \otimes_{v|\infty} \pi'_v \otimes \chi_v$ is a finite-dimensional representation of the group $G_{S,\infty}$, which is compact modulo its split center.

We will sometimes *further* assume that the representation $(\pi' \otimes \chi)_\infty$ of $G_{S,\infty}$ is the *trivial* representation. This means that, for each real place v of k, the representation π_v is the discrete series of weight 2 for $PGL_2(k_v)$, and χ_v is the trivial character of E_v^*/k_v^*. If we also assume that the adèlic representation $\pi \otimes \chi$ of $G(\mathbf{A})$ is automorphic and cuspidal, then the last hypothesis means that π corresponds to a Hilbert modular form, of weight $(2, 2, \ldots, 2)$, with central character ω of finite order, split at infinity, and that χ is a Hecke character of $\mathbf{A}_E^*$, of finite order, with $\chi|\mathbf{A}^* = \omega^{-1}$.

15. Automorphic representations

We henceforth assume that the adèlic representation $\pi\otimes\chi$ is automorphic and cuspidal, so appears as a submodule in the space of cusp forms

$$\mathcal{F}(G) = \mathcal{F}(G(k)\backslash G(\mathbf{A})).$$

We will also assume that the Hecke character χ of $\mathbf{A}_E^*$ is unitary.

Fixing an embedding:

$$i : \pi \otimes \chi \to \mathcal{F}(G)$$

of $G(\mathbf{A})$-modules, gives a $G(k)$-invariant linear form

$$\ell : \pi \otimes \chi \to \mathbf{C}$$

defined by evaluating the function $i(w)$ on the identity in G:

$$\ell(w) = i(w)(1).$$

From the linear form ℓ, we can recover the embedding i by: $i(w)(g) = \ell(gw)$. Both i and ℓ are well-defined, up to scaling, as the multiplicity of $\pi\otimes\chi$ in $\mathcal{F}(G)$ is equal to 1 (cf. [J-L]).

Since π is automorphic, its central character ω is an idèle class character. The same is true for the quadratic character α corresponding to E. Hence

$$\prod_v (\alpha\omega)_v(-1) = 1.$$

We therefore find that

$$\epsilon(\pi \otimes \chi) = \prod_v \epsilon(\pi_v \otimes \chi_v) = (-1)^{\#S}$$

with S the finite set of places where

$$\epsilon(\pi_v \otimes \chi_v) \neq \alpha_v\omega_v(-1).$$

This is the global sign in the functional equation of the Rankin L-function $L(\pi\otimes\chi, s)$. In particular when $\#S$ is even, we will study the central critical value $L(\pi\otimes\chi, 1/2)$. When $\#S$ is odd, we will study the central critical derivative $L'(\pi\otimes\chi, 1/2)$, as $L(\pi\otimes\chi, 1/2) = 0$.

16. When $\#S$ is even

First, assume $\#S$ is even. We then have defined a group G' over k with $G'(\mathbf{A}) = G_{S,\mathbf{A}}$, as well as an irreducible representation $\pi' \otimes \chi$ of $G'(\mathbf{A})$. By a fundamental theorem of Jacquet-Langlands [J-L], the representation $\pi' \otimes \chi$ is also automorphic and cuspidal, and appears with multiplicity 1 in the space $\mathcal{F}(G')$ of cusp forms on G'.

We define a $T(\mathbf{A})$-invariant linear form m on $\mathcal{F}(G')$ by the formula

$$m(f) = \int_{T(k)\backslash T(\mathbf{A})} f(t)\, dt.$$

Here f is a function (with rapid decay) on $G'(\mathbf{A})$ (which is left $G'(k)$-invariant), $f(t)$ denotes its restriction to $T(\mathbf{A})$, and dt is a non-zero invariant measure on the locally compact abelian group $T(\mathbf{A})$ (which is unique up to scaling).

When E is a field, $T(k)\backslash T(\mathbf{A}) = E^* \cdot \mathbf{A}^* \backslash \mathbf{A}_E^*$ is compact. One example of an invariant measure is Tamagawa measure, which gives this quotient volume 2. When E is the split quadratic algebra, $G'(\mathbf{A}) \simeq G(\mathbf{A}) \simeq GL_2(\mathbf{A}) \times \mathbf{A}^*$, and $T(\mathbf{A})$ is embedded as the subgroup $\begin{pmatrix} t & \\ & 1 \end{pmatrix} \times t$. The integral defining m converges, owing to the rapid decay of f.

If we choose a $G'(\mathbf{A})$-equivariant embedding

$$i : \pi' \otimes \chi \hookrightarrow \mathcal{F}(G')$$

we may restrict m to the image, to obtain an element $m \circ i$ in the one-dimensional vector space $\mathrm{Hom}_{T(\mathbf{A})}(\pi' \otimes \chi, \mathbf{C})$. If ℓ is the $G'(k)$-invariant linear form corresponding to the embedding i, then

$$m \circ i(w) = \int_{T(k)\backslash T(\mathbf{A})} \ell(tw)\, dt = Av_T(\ell)(w).$$

The main global result in this case is due to Waldspurger [W].

Theorem. *The $T(\mathbf{A})$-invariant linear form $m \circ i = Av_T(\ell)$ is non-zero on $\pi' \otimes \chi$ if and only if $L(\pi \otimes \chi, 1/2) \neq 0$.*

17. Global test vectors

We can refine this result, if we are in the favorable situation where test vectors exist in $\pi'_v \otimes \chi_v$, for *all* places v of k. In this case, we let $\langle w_v \rangle$ be the line of local test vectors, and let

$$w = \bigotimes_v w_v$$

be a basis of the tensor product line in $\pi' \otimes \chi$. Then, by our local results, the linear form $Av_T(\ell)$ is non-zero if and only if

$$Av_T(\ell)(w) \neq 0.$$

Of course, this value depends on the choice of ℓ and w, both of which are only defined up to scalars, as well as the choice of invariant measure dt on $T(\mathbf{A})$ used to define the average. To obtain a number which depends only on $\pi' \otimes \chi$, we choose a $G'(k)$-invariant form $\ell^\vee : (\pi' \otimes \chi)^\vee \to \mathbf{C}$ on the contragradient representation, and a test vector $w^\vee$ for $\ell^\vee$ in $(\pi' \otimes \chi)^\vee$. The space

$$\mathrm{Hom}_{\Delta G'(\mathbf{A})}((\pi' \otimes \chi) \otimes (\pi' \otimes \chi)^\vee, \mathbf{C})$$

has dimension equal to 1, by Schur's lemma. The linear form $Av_{\Delta G'}(\ell \otimes \ell^\vee)$, defined by the integral

$$Av_{\Delta G'}(\ell \otimes \ell^\vee)(u \otimes u^\vee) = \int_{G'_{ad}(k)\backslash G'_{ad}(\mathbf{A})} \ell(gu)\ell^\vee(gu^\vee)\, dg$$

is a non-zero basis element, where dg is an invariant (positive) measure on the adèlic points of the adjoint group $Z\backslash G' = G'_{ad}$.

To verify that $Av_{\Delta G'}(\ell \otimes \ell^\vee)$ is non-zero, we observe that if we use ℓ to embed $\pi' \otimes \chi$ as a sub-module of $\mathcal{F}(G')$, so u corresponds to the function f'_u on $G'(k)\backslash G'(\mathbf{A})$ with $\ell(gu) = f'_u(g)$, then the contragradient $(\pi' \otimes \chi)^\vee$ embeds as the functions $g \mapsto \overline{f'_u(g)}$ and the form $\ell^\vee$ is given by evaluation at the identity. Consequently, if $u^\vee$ in $(\pi' \otimes \chi)^\vee$ is the conjugate of f'_u, we find

$$Av_{\Delta G'}(\ell \otimes \ell^\vee)(u \otimes u^\vee) = \int_{Z(\mathbf{A})G'(k)\backslash G'(\mathbf{A})} f'_u(g)\overline{f'_u(g)}\, dg$$

$$= \langle f'_u, f'_u \rangle.$$

This Petersson product is positive, so is non-zero.

Since the test vector w in $(\pi' \otimes \chi)$ is determined by its M-invariance (and K-type), we find that $w^\vee$ can be taken as the conjugate function of f'_w in $(\pi' \otimes \chi)^\vee$. If we do so, we find that:

$$Av_T(\ell)(w) = \int_{T(k)\backslash T(\mathbf{A})} f'_w(t)\, dt$$

$$Av_T(\ell^\vee)(w^\vee) = \int_{T(k)\backslash T(\mathbf{A})} \overline{f'_w(t)}\, dt = \overline{Av_T(\ell)(w)}$$

$$Av_{\Delta G'}(\ell \otimes \ell^\vee)(w \otimes w^\vee) = \int_{G'_{ad}(k)\backslash G'_{ad}(\mathbf{A})} f'_w(g)\overline{f'_w(g)}\, dg > 0$$

These results hold for any positive, invariant measures dt and dg on the adèlic groups $T(\mathbf{A})$ and $G'_{ad}(\mathbf{A})$. We now use the product measures:

$$dt = \bigotimes_v dt_v$$

$$dg = \bigotimes_v dg_v$$

which come from our test vector. Namely, at each finite place v of k, we let M_v be the open compact subgroup of $G'(k_v)$ which fixes the test vector w_v, and define dt_v and dg_v by:

$$\int_{M_v \cap T(k_v)} dt_v = 1$$

$$\int_{M_v / M_v \cap Z(k_v)} dg_v = 1.$$

At each Archimedean place,we let dt_v and dg_v be the canonical Haar measure $|\omega_v|$ defined in [G-G]. For example, if $k_v = \mathbf{R}$ and $E_v = \mathbf{C}$, so $T(k_v) = \mathbf{C}^*/\mathbf{R}^*$, we have

$$\int_{T(k_v)} dt_v = 2\pi.$$

If E_v is split, so $T(k_v) \simeq k_v^*$, dt_v is the usual measure on the multiplicative group.

Now the quantities $Av_T(\ell)(w)$, $Av_T(\ell^\vee)(w^\vee)$, and $Av_{\Delta G'}(\ell \otimes \ell^\vee)(w \otimes w^\vee)$ are all defined. The ratio:

$$A(\pi' \otimes \chi) = \frac{Av_T(\ell)(w) \cdot Av_T(\ell^\vee)(w^\vee)}{Av_{\Delta G'}(\ell \otimes \ell^\vee)(w \otimes w^\vee)}$$

is a real number ≥ 0, which is zero *precisely* when $L(\pi \otimes \chi, 1/2) = 0$. This is an invariant of $\pi' \otimes \chi$ and $(\pi' \otimes \chi)^\vee$, which is *independent* of the choices of ℓ, $\ell^\vee$, w and $w^\vee$. There should be a simple formula, expressing $L(\pi \otimes \chi, 1/2)$ as a product $R(\pi \otimes \chi) \cdot A(\pi' \otimes \chi)$, where $R(\pi \otimes \chi)$ is a positive real number, given by periods of $\pi \otimes \chi$. We will make this more precise in the next section.

18. An explicit global formula

To clarify the invariant $A(\pi' \otimes \chi)$, and to prepare for the discussion when $\#S$ is odd, we will obtain an explicit formula for $A(\pi' \otimes \chi)$ under the hypotheses that S has even cardinality and contains all Archimedean places, and that

$$(\pi' \otimes \chi)_\infty \quad \text{is the trivial representation of } G_{S,\infty}.$$

Recall that this implies that k is totally real, that E is a CM field, that π corresponds to a Hilbert modular form of weight $(2, 2, \ldots, 2)$ with central character ω of finite order, split at infinity, and that χ has finite order, with restriction ω^{-1} to $\mathbf{A}^*$.

We also assume that the conductors of π and χ are relatively prime, so a global test vector $w = w_\infty \otimes \omega_f$ exists in $\pi' \otimes \chi$. We let $M \subset G_{S,f} = G'(\mathbf{A}_f)$ be the open compact subgroup fixing w_f in $(\pi' \otimes \chi)_f$. By hypothesis, $G'(k \otimes \mathbf{R}) = G_{S,\infty}$ fixes w_∞.

Choose a $G'(k)$-invariant linear form (unique up to scaling, by the multiplicity 1 theorem):

$$\ell : \pi' \otimes \chi \to \mathbf{C}.$$

Then, using our test vector w, we get a function

$$f_w(g) = \ell(gw)$$

on the double coset space $G'(k)\backslash G'(\mathbf{A})/G'(k\otimes\mathbf{R})\times M$, which is identified with a function on

$$G'(k)\backslash G'(\mathbf{A}_f)/M,$$

since f_w is constant on $G'(k\otimes\mathbf{R})$. We wish to compute:

$$A_T = \int_{T(k)\backslash T(\mathbf{A})} f_w(t)\; dt$$
$$A_{G'} = \int_{G'_{ad}(k)\backslash G'_{ad}(\mathbf{A})} |f_w(g)|^2\; dg$$

for the Haar measures dt, dg defined in the previous section. Then

$$A(\pi'\otimes\chi) = \frac{A_T\cdot \overline{A}_T}{A_{G'}}.$$

Let $J = M\cap T(\mathbf{A}_f)$, so the restriction of f_w to $T(\mathbf{A}_f)\hookrightarrow G'(\mathbf{A}_f)$ is a function on the finite set $T(k)\backslash T(\mathbf{A}_f)/J$. Recall that A is the ring of integers in k, $\mathcal{O} = A + c\mathcal{O}_E$ the order of conductor c in $\mathcal{O}_E$, and $J = \hat{\mathcal{O}}^*/\hat{A}^*$. By our choice of measures

$$\int_{T(k\otimes\mathbf{R})\times J} dt = (2\pi)^d,$$

where d is the degree of k. Let

$$u = \#(\mathcal{O}^*/A^*) = \#(J\cap T(k)).$$

Then

$$A_T = \int_{T(k)\backslash T(\mathbf{A})} f_w(t)\; dt = (2\pi)^d\cdot\frac{1}{u}\sum_{T(k)\backslash T(\mathbf{A}_f)/J} f_w(t)$$

Similarly, if $M_{ad} = M/M\cap Z(\mathbf{A}_f)$ is the image of M in $G_{ad}(\mathbf{A}_f)$, then $|f_w(g)|^2$ is a function on the finite set $G'_{ad}(k)\backslash G'_{ad}(\mathbf{A}_f)/M_{ad}$. For each double coset, we define the integer

$$e_g = \#(M_{ad}\cap g^{-1}G'_{ad}(k)g).$$

We then have the formulas

$$\int_{G'_{ad}(k\otimes \mathbf{R})\times M_{ad}} dg = (2\pi)^{2d}$$

$$A_{G'} = (2\pi)^{2d} \cdot \sum_{G'_{ad}(k)\backslash G'_{ad}(\mathbf{A}_f)/M_{ad}} \frac{1}{e_g} \cdot |f_w(g)|^2.$$

The ratio $A(\pi' \otimes \chi) = |A_T|^2/A_{G'}$ has a nice description in the language of "algebraic modular forms" (cf. [G2]). Consider the finite-dimensional vector space of functions

$$V = \{F : G'(k)\backslash G'(\mathbf{A}_f)/M \to \mathbf{C}\}.$$

The finite abelian group

$$Z(k)\backslash Z(\mathbf{A}_f)/M \cap Z(\mathbf{A}_f) = E^*\backslash \mathbf{A}^*_{E,f}/\hat{\mathcal{O}}^* = \mathrm{Pic}(\mathcal{O})$$

acts on V by the formula

$$zF(g) = F(gz) = F(zg),$$

and V decomposes as a direct sum of eigenspaces $V(\chi)$ for the characters of $\mathrm{Pic}(\mathcal{O})$. Since $\chi \cdot \overline{\chi} = 1$, each eigenspace has a Hermitian inner product

$$\langle F, G\rangle = \sum_{G'_{ad}(k)\backslash G'_{ad}(\mathbf{A}_f)/M_{ad}} \frac{1}{e_g} F(g)\overline{G(g)}.$$

The representation $\pi' \otimes \chi$, and our choice of linear form ℓ and test vector w, give an element $f_w(g)$ in $V(\chi)$. The homomorphism

$$T \to G'$$

of groups over k gives a map of finite sets

$$\phi : T(k)\backslash T(\mathbf{A}_f)/J \to G'(k)\backslash G'(\mathbf{A}_f)/M.$$

This, in turn, gives an element F_T in V, which is defined by

$$F_T(g) = \frac{1}{u} \cdot \#\{t : \phi(t) = g\}.$$

We define the projection

$$F_T(\chi)(g) = \frac{1}{\#\mathrm{Pic}\,\mathcal{O}} \sum_{\mathrm{Pic}(\mathcal{O})} \chi^{-1}(z) F_T(zg),$$

which is an element of $V(\chi)$. We then have the inner product formula

$$\begin{aligned}\langle f_w, F_T(\chi)\rangle &= \sum_g \frac{1}{e_g} f_w(g)\overline{F_T(\chi)(g)} \\ &= \frac{1}{u}\sum_t f_w(t).\end{aligned}$$

Indeed, $F_T(\chi)$ is supported on $T \cdot Z$, and $T \cap Z = 1$.

Now let $F_T(\pi' \otimes \chi)$ be the projection of the function F_T to the $(\pi' \otimes \chi)$-eigenspace of V, or equivalently the projection of $F_T(\chi)$ to the π'-eigenspace of $V(\chi)$. Since this eigenspace is spanned by f_w, we have the inner product formula:

$$F_T(\pi' \otimes \chi) = \frac{\langle F_T(\chi), f_w\rangle}{\langle f_w, f_w\rangle} \cdot f_w.$$

Consequently, we obtain the formula

$$\begin{aligned}\langle F_T(\pi' \otimes \chi), F_T(\pi' \otimes \chi)\rangle &= \frac{|\langle F_T(\chi), f_w\rangle|^2}{\langle f_w, f_w\rangle} \\ &= \frac{|A_T|^2}{A_{G'}} \\ &= A(\pi' \otimes \chi).\end{aligned}$$

This shows, among other things, that $A(\pi' \otimes \chi)$ is algebraic, and lies in the field of definition of $\pi' \otimes \chi$.

It suggests, by our previous work on special values [G3], that the factor $R(\pi \otimes \chi)$ in the formula

$$L(\pi \otimes \chi, 1/2) = R(\pi \otimes \chi) \cdot A(\pi' \otimes \chi)$$

has the form

$$R(\pi \otimes \chi) = \frac{1}{\sqrt{\mathbf{N}D}}(f_0, f_0),$$

where f_0 is a new form in π with $a_1(f_0) = 1$, and $(,)$ is a normalized Petersson inner product on G_{ad}. There is work of Zhang and Xue in this direction, but the precise result is not yet proved. One may want to renormalize the measures dt and dg so that

$$R(\pi \otimes \chi) = \mathrm{Res}_{s=1} L(\pi \otimes \chi, ad, s),$$

as this formulation, using the adjoint L-function at $s = 1$, would make sense for more general π and χ.

19. When $\#S$ is odd

We now consider the case when $\#S$ is odd. Then $\epsilon(\pi \otimes \chi) = -1$ and $L(\pi \otimes \chi, 1/2) = 0$. In this case, the adèlic group $G_{S,\mathbf{A}}$ does *not* contain a natural discrete subgroup (like $G'(k)$ in the case when $\#S$ is even), so it is unclear what it means for the representation $\pi' \otimes \chi$ to be automorphic.

To generalize Waldspurger's theorem to a result on $L'(\pi \otimes \chi, 1/2)$, we need to construct a representation $\mathcal{F}$ of $G_{S,\mathbf{A}}$, analogous to the space of cusp forms on G'. This representation should contain $\pi' \otimes \chi$, and have a naturally defined $T(\mathbf{A})$-invariant linear form. We will construct the representation $\mathcal{F}$ in the special case when k is a number field, the group $G_{S,\infty} = \prod_{v|\infty} G'_v(k_v)$ is compact modulo its center, and the representation $(\pi' \otimes \chi)_\infty$ of $G_{S,\infty}$ is the *trivial* representation.

More generally, one should be able to construct $\mathcal{F}$ whenever the set S contains all Archimedean places of k. In the number field case, this implies that the representation $(\pi' \otimes \chi)_\infty$ is finite-dimensional. When this representation is non-trivial, however, the approach to the first derivative sketched in the next few sections involves the theory of heights on local systems over curves, which is not yet complete. However, there is much preliminary work in the area (cf. [Br], [Z2]).

20. Shimura varieties

We now assume that k is a number field and that the set S has odd cardinality, and contains all Archimedean places. Then k is totally real and E is a CM field.

Let v be a real place of k, and let $B(v)$ be the quaternion algebra over k which is ramified at $S - \{v\}$. Since E_w is a field, for all $w \in S$, E embeds as a subfield of $B(v)$. Let $G' = G(v)$ be the corresponding group of unitary similitudes, with k-points $B(v)^* \times E^*/\Delta k^*$. The torus T with $T(k) = E^*/k^*$ embeds diagonally, as the subgroup of G' fixing a vector u with $\phi(u,u) = 1$. The adèlic pair $T(\mathbf{A}_f) \to G'(\mathbf{A}_f)$ is isomorphic to $T(\mathbf{A}_f) \to G_{S,\mathbf{A}_f}$, independent of the choice of real place v.

Consider the homomorphism

$$h : \mathbf{C}^* \to T(k \otimes \mathbf{R}) = \prod_{w|\infty} (\mathbf{C}^*/\mathbf{R}^*)_w$$

given by

$$h(z) = (z(\text{mod } \mathbf{R}^*), 1, 1, \ldots, 1),$$

where the first coordinate corresponds to the real place v. The inclusion $T \to G'$ then gives rise to a homomorphism

$$h : \mathbf{C}^* \to G'(k \otimes \mathbf{R}).$$

The data (T,h) defines a Shimura variety $M(T,h)$ over $\mathbf{C}$, with an action of $T(\mathbf{A}_f)$. If $J \subset T(\mathbf{A}_f)$ is compact and open, then

$$M(T,h)^J(\mathbf{C}) = T(k)\backslash T(\mathbf{A}_f)/J$$

is a finite set of points.

The data (G',h) define a Shimura variety $M(G',h)$ over $\mathbf{C}$, with an action of $G'(\mathbf{A}_f) = G_{S,\mathbf{A}_f}$. If $K \subset G_{S,\mathbf{A}_f}$ is compact and open, then

$$M(G',h)^K(\mathbf{C}) = G'(k)\backslash(X \times G_{S,\mathbf{A}_f}/K),$$

where X is the Riemann surface of the $G'(k \otimes \mathbf{R})$ conjugacy class of h. We have

$$\begin{aligned} X &\simeq G'(k \otimes \mathbf{R})/Z(h) \\ &= G'(k_v)/Z \cdot T(k_v) \\ &\simeq GL_2(\mathbf{R})/\mathbf{C}^* \simeq \mathcal{H}^{\pm}. \end{aligned}$$

Hence $M(G',h)^K(\mathbf{C})$ is a disjoint union of a finite number of connected hyperbolic Riemann surfaces with finite volume. They are compact, unless $k = \mathbf{Q}$ and $S = \{\infty\}$, in which case G' is quasi-split and there are finitely many cusps.

The theory of canonical models provides models for $M(T,h)$ and $M(G',h)$ over their reflex fields. In this case, the reflex field is E in both cases, embedded in $(E \otimes k_v) = \mathbf{C}$ by the place v. The actions of $T(\mathbf{A}_f)$ and $G_{S,\mathbf{A}_f}$ are both defined over E, and the morphism $M(T,k) \to M(G',h)$ is defined over E and is $T(\mathbf{A}_f)$-equivariant. The connected components of $M(T,h)$ and $M(G',h)$ are all defined over the maximal abelian extension E^{ab} of E in $\mathbf{C}$, and the action of $\mathrm{Gal}(E^{ab}/E)$ on these components is given by Shimura's reciprocity law. For proofs of these assertions, see [D] and [Ca].

21. Nearby quaternion algebras

In fact, the varieties $M(T,h) \to M(G',h)$ over E do not depend on the choice of a real place v of k, which was used to define G' and the Shimura varieties over $(k_v \otimes E) = \mathbf{C}$. They depend only on S, and we will denote them $M(T) \to M(G_S)$. More precisely, if w is any real place of k, and $G' = G(w)$ is the form of G coming from the quaternion algebra ramified at $S - \{w\}$, we have an isomorphism of Riemann surfaces

$$M(G_S)^K(E_w) = G'(k)\backslash(G'(k_w)/Z \cdot T(k_w) \times G_{S,\mathbf{A}_f}/K).$$

If $J = K \cap T(\mathbf{A}_f)$, then

$$M(T)^J(E_w) = T(k)\backslash(1 \times T(\mathbf{A}_f)/J),$$

and the morphism $M(T) \to M(G_S)$ over E_w is given by the map $T \to G'$.

More generally, if w is a non-Archimedean place in S, and $G' = G(w)$ is the inner form of G made from the quaternion algebra over k which is ramified at $S - \{w\}$, we also have a rigid analytic uniformization of the points of the curve $M(G_S)$ over $E_w = E \otimes k_w$. For simplicity, we describe this in the case where $K \subset G_{S,\mathbf{A}_f}$ has the form $K = K_w \times K^w$, and K_w is the unique maximal compact subgroup (for the general case, see [Dr]). In this case $J_w = T(k_w)$, and all the points of $M(T)^J$ are rational over E_w. We have an isomorphism of rigid analytic spaces:

$$M(G_S)^K(E_w) \simeq G'(k)\backslash(G'(k_w)/Z \cdot T(k_w) \times G_{S,\mathbf{A}_f^w}/K^w).$$

Here

$$\begin{aligned} G'(k_w)/Z \cdot T(k_w) &\simeq GL_2(k_w)/E_w^* \\ &\simeq \mathbf{P}^1(E_w) - \mathbf{P}^1(k_w) \end{aligned}$$

are the E_w-points of Drinfeld's upper half plane. The inclusion of

$$M(T)(E_w) \simeq T(k)\backslash(1 \times T(\mathbf{A}_f^w)/J^w)$$

is again described by group theory. The components of $M(G_S)^K$ containing these "special points" are rational over E_w; the general components are rational over the maximal unramified extension of E_w.

There is also a slightly weaker result for non-Archimedean places w which are *not* in S. Here we get a rigid analytic description of the "supersingular locus" on $M(G_S)$ over E_w, when E_w is the unramified quadratic field extension of k_w. Let $G' = G(w)$ be the inner form of G, corresponding to the quaternion algebra ramified at $S \cup \{w\}$, and assume $K \subset G_{S,\mathbf{A}^f}$ has the form $K = K_w \times K^w$, with K_w the unique hyperspecial maximal compact subgroup of $G(k_v)$ which contains $T(k_v)$. Then $M(G_S)^K$ has a model over the ring of integers $\mathcal{O}_w$ of E_w, with good reduction (mod w). The points reducing to supersingular points (mod w) give a rigid analytic space

$$M(G_S)^K(E_w)_{\text{s-sing}} \simeq G'(k)\backslash(G'(k_w)/Z \cdot T(k_w) \times G_{S,\mathbf{A}_f^w}/K^w).$$

The supersingular points in the residue field $\mathbf{F}_w$ are given by

$$M(G_S)^K(\mathbf{F}_w)_{\mathrm{s-sing}} \simeq G'(k)\backslash G_{S,\mathbf{A}_f^w}/K^w,$$

and the reduction map is given by group theory. Note that

$$G'(k_w)/Z \cdot T(k_w) \simeq D_w^*/E_w^*$$

is analytically isomorphic to an open disc over $\mathcal{O}_w$. The special points in $M(T)^J$ are all rational over E_w, and their map to the supersingular locus in $M(G_S)^K$ is given by group theory. Again, not all of the components of $M(G_S)^K$ are rational over E_w, but those containing the special points are.

To recapitulate, the morphism of Shimura varieties $M(T) \to M(G_S)$ over E depends only on the pair of groups $T \to G$ and the odd set S of places, which contains all real places of k. The local study of this morphism over the completion $E_w = E \otimes k_w$ involves the quaternion algebra over k with ramification locus

$$S - \{w\}, \quad \text{when} \quad w \in S$$

$$S \cup \{w\}, \quad \text{when} \quad w \notin S \quad \text{and} \quad E_w \quad \text{is a field.}$$

These are all quaternion algebras at distance one from the set S.

22. The global representation

We are now prepared to construct the representation $\mathcal{F}$ of $G_{S,\mathbf{A}} = G_{S,\infty} \times G_{S,\mathbf{A}_f}$, with a $T(\mathbf{A})$-invariant linear form, under the hypothesis that

$$(\pi' \otimes \chi)_\infty = \mathbf{C} \tag{22.1}$$

is the trivial representation. We will define a natural representation $\mathcal{F}_f$ of $G_{S,\mathbf{A}_f}$, using the Shimura curve $M(G_S)$, as well as a $T(\mathbf{A}_f)$-invariant linear form on it, using the morphism $M(T) \to M(G_S)$. We will then put

$$\mathcal{F} = \mathbf{C} \otimes \mathcal{F}_f$$

as a representation of $G_{S,\infty} \times G_{S,\mathbf{A}_f}$.

Let $\mathrm{Pic}^0(M(G_S)^K)$ be the group of line bundles on the curve $M(G_S)^K$ which have degree zero on each geometric component, and $\mathrm{Pic}^0(M(G_S)^K)(E)$ those line bundles which are rational over E. If $K' \subset K$, we have a finite covering of curves over E

$$\beta : M(G_S)^{K'} \to M(G_S)^K.$$

This induces a homomorphism, by pull-back:

$$\beta^* : \mathrm{Pic}^0(M(G_S)^K)(E) \to \mathrm{Pic}^0(M(G_S)^{K'})(E).$$

The map β^* is an injection, modulo torsion. We define

$$\mathcal{F}_f = \lim_{\rightarrow} \mathrm{Pic}^0(M(G_S)^K)(E) \otimes \mathbf{C},$$

where the direct limit maps are now injective.

The direct limit is a representation of $G_{S,\mathbf{A}_f}$, and the fixed space of any open compact subgroup K

$$(\mathcal{F}_f)^K = \mathrm{Pic}^0(M(G_S)^K)(E) \otimes \mathbf{C}$$

is a finite-dimensional complex vector space, by the Mordell-Weil theorem.

The tangent space to $\mathrm{Pic}^0(M(G_S)^K)$ is the cohomology group $H^0(M(G_S)^K, \Omega^1)$. On the direct limit

$$H_f = \lim_{\rightarrow} H^0(M(G_S)^K, \Omega^1) \otimes \mathbf{C}$$

the group $G_{S,\mathbf{A}_f}$ acts via a direct sum of the representations $(\pi' \otimes \chi)_f$, where $\pi \otimes \chi$ appears in the space of cusp forms for G, is square integrable at all local places in S, and satisfies (22.1).

Since the action of the endomorphisms of abelian varieties in characteristic zero is faithfully represented by their action on the tangent space, these are the *only* representations which *can* appear in the action of $G_{S,\mathbf{A}_f}$ on $\mathcal{F}_f$. Whereas their multiplicity in H_f

is equal to 1, their multiplicity in $\mathcal{F}_f$ is predicted by a generalization of the conjecture of Birch and Swinnerton-Dyer.

Conjecture. *The multiplicity of* $(\pi' \otimes \chi)_f$ *in* $\mathcal{F}_f$ *is equal to* $\operatorname{ord}_{s=1/2} L(\pi \otimes \chi, s)$.

In particular, since $\#S$ is odd, we have $L(\pi \otimes \chi, 1/2) = 0$, and the multiplicity of $(\pi' \otimes \chi)_f$ in $\mathcal{F}_f$ should be positive (and odd). The same holds for the multiplicity of $\pi' \otimes \chi$ in $\mathcal{F} = \mathbf{C} \otimes \mathcal{F}_f$, where $\mathbf{C}$ is the trivial representation of $G_{S,\infty}$.

23. The global linear form

We now use the torus T, and the zero cycle $j : M(T) \to M(G_S)$ over E, to define a $T(\mathbf{A}_f)$-invariant linear form $\ell : \mathcal{F}_f \to \mathbf{C}$.

For each open compact $K \subset G_{S,\mathbf{A}_f}$, we put $J = K \cap T(\mathbf{A}_f)$, which is open, compact in $T(\mathbf{A}_f)$. We define the 0-cycle

$$m(T)^K = j_*(M(T)^J)$$

on the curve $M(G_S)^K$ over E.

Since $M(G_S)^K$ is hyperbolic, for each component c, there is a class δ_c in $\operatorname{Pic}(M(G_S)^K)(\overline{E}) \otimes \mathbf{Q}$ which has degree 1 on c, and degree 0 on all other components. Indeed, if the component c is $X = \mathcal{H}/\Gamma$ over $\mathbf{C}$, then the divisor class

$$d_c = K_X + \sum_{\substack{x \text{ elliptic} \\ \text{or cuspidal}}} \left(1 - \frac{1}{e_x}\right)(x)$$

has positive degree, where K_X is the canonical class, and e_x is the order of the cyclic stabilizer Γ_x of x. This pulls back correctly for coverings given by the subgroups of finite index in Γ. We put

$$\delta_c = d_c / \deg(d_c),$$

and define

$$m_0(T)^K = m(T)^K - \sum_c \deg_c(m(T)^K) \cdot \delta_c.$$

This class lies in $\mathrm{Pic}^0(M(G_S)^K)(E) \otimes \mathbf{Q}$, as $m(T)^K$ has equal degree on conjugate components, and the δ_c are similarly conjugate.

The canonical height pairing of Néron and Tate, on the Jacobian of $M(G_S)^K$ over E, gives a linear form $\ell_K : (\mathcal{F}_f)^K \to \mathbf{C}$, defined by

$$\ell_K(d) = \langle d, m_0(T)^K \rangle.$$

These forms come from a single linear form ℓ on $\mathcal{F}_f$, for when $K' \subset K$ we have a commutative diagram

$$\begin{array}{ccc} M(T)^{J'} & \xrightarrow[j']{} & M(G_S)^{K'} \\ \downarrow & & \downarrow \pi \\ M(T)^J & \xrightarrow[j]{} & M(G_S)^K \end{array}$$

Hence $m_0(T)^K = \pi_*(m_0(T)^{K'})$, and if d is in $(\mathcal{F}_f^0)^K$:

$$\begin{aligned} \ell_{K'}(\pi^* d) &= \langle \pi^* d, m_0(T)^{K'} \rangle \\ &= \langle d, \pi_* m_0(T)^{K'} \rangle \\ &= \langle d, m_0(T)^K \rangle \\ &= \ell_K(d) \end{aligned}$$

The form ℓ is clearly $T(\mathbf{A}_f)$-invariant, so gives a $T(\mathbf{A})$-invariant form (also denoted ℓ) on $\mathcal{F} = \mathbf{C} \otimes \mathcal{F}_f$. This induces a map

$$\ell_* : \mathrm{Hom}_{G_{S,\mathbf{A}}}(\pi' \otimes \chi, \mathcal{F}) \to \mathrm{Hom}_{T(\mathbf{A})}(\pi' \otimes \chi, \mathbf{C}).$$

We know the space $\mathrm{Hom}_{T(\mathbf{A})}(\pi' \otimes \chi, \mathbf{C})$ is one-dimensional, by the local theory and the construction of $G_{S,\mathbf{A}}$. The space $\mathrm{Hom}_{G_{S,\mathbf{A}}}(\pi' \otimes \chi, \mathcal{F})$ was conjectured to have odd dimension, equal to the order of $L(\pi \otimes \chi, s)$ at $s = 1/2$.

The analog of Waldspurger's theorem in this context is the following.

Conjecture. *The map ℓ_* is non-zero if and only if $L'(\pi\otimes\chi,1/2)\neq 0$.*

24. Global test vectors

We can refine this conjecture, and obtain a statement generalizing [G-Z], in the situation where a global test vector exists (i.e., when the conductors of π and χ are relatively prime). Let $K\subset G_{S,\mathbf{A}_f}$ be the open compact subgroup fixing a line of test vectors in $(\pi'\otimes\chi)_f$. Then one wants a formula relating $L'(\pi\otimes\chi,1/2)$ to the height pairing of the $(\pi'\otimes\chi)_f$-eigencomponent of $m_0(T)^K$ with *itself*. This formulation does not require a determination of the entire space $\mathrm{Hom}_{G_{S,\mathbf{A}}}(\pi'\otimes\chi,\mathcal{F})$.

In the spirit of the explicit formula in §18, I would guess that

$$L'(\pi\otimes\chi,1/2)=\frac{(f_0,f_0)}{\sqrt{\mathbf{N}D}}\langle m_0(T)^K(\pi'\otimes\chi)_f, m_0(T)^K(\pi'\otimes\chi)_f\rangle.$$

Zhang [Z1] has done fundamental work in this direction. In the case when $k=\mathbf{Q}$, E is imaginary quadratic, all primes p dividing N (the conductor of π) are split in E, and $c=\mathrm{conductor}(\chi)=1$, we have $S=\{\infty\}$. Furthermore,

$$K\subset GL_2(\hat{\mathbf{Z}})\times\hat{\mathcal{O}}_E^*/\Delta\hat{\mathbf{Z}}^*$$

is the subgroup of those $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ in $GL_2(\hat{\mathbf{Z}})$ with $c\equiv 0 \pmod N$. The desired formula is the one I proved with Zagier, twenty years ago.

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