Trivial L-functions for the rational function field

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1 Introduction

Let k be a global function field, over the finite field E with q elements. Let k^s be a separable closure of k, and let E^s be the separable closure of E in k^s.

The L-function $L(V, s)$ of a semi-simple l-adic Galois representation

$$\mathrm{Gal}(k^s/k) \longrightarrow \mathrm{GL}(V)$$

contains very little of the local information involved in its definition. Indeed, the cancellation of the local terms in the infinite Euler product defining $L(V, s)$ ultimately results in a function which a is quotient of two polynomials in q^{-s}. This led Weil to define [13, p.10] the notion of a (formal) Dirichlet series belonging to k, which keeps track of the local terms.

In this paper, we will study an extreme case of cancellation over the rational function field $k = E(T)$. If we assume that the geometric Galois group $\mathrm{Gal}(k^s/kE^s)$ has no non-trivial invariants on V and that the degree $f(V)$ of the Artin conductor of V is twice the dimension of V, then $L(V, s)$ is a polynomial of degree 0 in q^{-s} with constant coefficient 1. Hence $L(V, s) = 1$ is a constant function! We call these trivial L-functions for the rational function field (although we will see that they arise in many non-trivial situations).

After a brief introduction to the cohomological theory of Weil, Grothendieck, and Deligne, we will present several examples of Galois representations of the rational function field $k = E(T)$ with trivial L-functions. In all of our irreducible examples, the representation V remains geometrically irreducible and the set S of ramified places is contained in $\{\infty, 0, 1\} \subset \mathbb{P}^1(E)$. More precisely, V corresponds either to an irreducible representation of $\pi_1(\mathbb{G}_m) = \pi_1(\mathbb{P}^1 - \{\infty, 0\})$ which is tamely ramified at $T = 0$, or to an irreducible representation of $\pi_1(\mathbb{P}^1 - \{\infty, 0, 1\})$ which is tamely ramified at all three places.

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2 The degree and denominator of $L(V, s)$

Let V be a semi-simple l-adic representation of $\mathrm{Gal}(k^s/k)$, which is pure of weight w and defined over the finite extension M_λ of $\mathbb{Q}_l$. Fix a complex embedding $\iota : M_\lambda \to \mathbb{C}$. Then the complex L-function of V is defined by the Euler product (cf. [3, p. 173])

$$L(V, s) = \prod_v \det(1 - F_v q_v^{-s} | V^{I_v})^{-1}.$$

Here I_v is the inertia subgroup of a decomposition group D_v at the place v, and F_v is a geometric Frobenius element generating the quotient D_v/I_v. Each characteristic polynomial has coefficients in the field M_λ which we view as a subfield of $\mathbb{C}$ via the complex embedding ι.

Let S be the finite set of ramified places (those where I_v acts non-trivially on V). For $v \notin S$ the eigenvalues of F_v on V have complex absolute value $q^{w/2}$. Hence the Euler product defining $L(V, s)$ converges and defines an analytic function in the right half plane $Re(s) > 1 + w/2$.

Let X_0 be the complete, non-singular curve of genus g over E with function field k, let $U = X_0 - S$, and let $j : U \to X_0$ be the inclusion morphism. The representation V corresponds to a lisse, l-adic sheaf $\mathscr{F}$ on U, and $j_*\mathscr{F}$ is a constructible l-adic sheaf on X_0. (For a readable introduction to l-adic sheaves, see [3, Ch I,%12].) We use X and $j_*\mathscr{F}$ to denote the corresponding objects over the separable closure E^s of E. We then have Grothendieck's cohomological formula for the L-function [11], [3, p. 174].

$$L(V, s) = \frac{\det(1 - Fq^{-s}|H^1(X, j_*\mathscr{F}))}{\det(1 - Fq^{-s}|H^0(X, j_*\mathscr{F})).\det(1 - Fq^{-s}|H^2(X, j_*\mathscr{F}))}$$

Hence $L(V, s)$ is a rational function of q^{-s} and has a meromorphic continuation to the entire complex plane.

Deligne proved that all of the eigenvalues of F on $H^i(X, j_*\mathscr{F})$ have complex absolute value $q^{(w+i)/2}$ [1, Thm 2]. Hence there is no cancellation in the alternating product for $L(V, s)$, and the L-function is a polynomial in q^{-s} if and only if $H^0(X, j_*\mathscr{F}) = H^2(X, j_*\mathscr{F}) = 0$. This vanishing of even cohomology occurs precisely when the geometric Galois group $\mathrm{Gal}(k^s/kE^s)$ has no non-trivial invariants on (the semi-simple representation) V(cf. [3, Ch I, %5]). We will henceforth assume that this is the case, and will say that V has no geometric invariants. Then $L(V, s)$ is a polynomial in q^{-s}. From the definition of $L(V, s)$ as an Euler product it follows that the constant coefficient of this polynomial is equal to 1.

For $v \in S$ we define the Artin conductor $\mathfrak{f}_v(V) = \dim \mathrm{Hom}_{I_v}(V, A_v)$, where A_v is the local Artin representation over $\mathbb{Q}_l$ (cf. [12, Ch VI, pp. 97-106]). We define the global Artin conductor as the effective divisor $\sum_{v\in S} \mathfrak{f}_v(V)(v)$ on X_0, and let $f(V) = \sum_{v\in S} \mathfrak{f}_v(V) \deg(v) \geq 0$ be its degree. The degree of $L(V, s)$ as a polynomial in q^{-s} is given by Raynaud's Euler characteristic formula [10]

$$\deg L(V, s) = (2g - 2) \dim(V) + f(V).$$

Summarizing all of the above results, we have the following.

Proposition 1 *Assume that the representation V has no geometric invariants and Artin conductor of degree $f(V)$. Then the L-function $L(V,s)$ is a polynomial in q^{-s} of degree $(2g-2)\dim(V)+f(V)$ with constant coefficient equal to* 1.

3 The rational function field

We henceforth assume that k is the rational function field $E(T)$, so the curve X_0 has genus $g=0$ and is isomorphic to $\mathbb{P}^1$ over E with the three E-rational points $T=\infty, T=0, T=1$ marked. Let V be an l-adic representation of $\mathrm{Gal}(k^s/k)$ with no geometric invariants. Then by the previous proposition, we have the inequality

$$f(V) \geq 2\dim(V).$$

as the degree of a polynomial is non-negative. Moreover, when equality holds, $L(V,s)$ is a polynomial of degree 0 in q^{-s} with constant coefficient 1, so $L(V,s)=1$ is a constant function. In this case, we say that the L-function of V is trivial. (By Proposition 1 the only other representations V with trivial L-functions are everywhere unramified representations with no geometric invariants of the Galois group of an elliptic function field, where the curve X_0 has genus $g=1$.) All our examples of representations with trivial L-function for the rational function field have ramification set $S\subset\{\infty,0,1\}$ and fall into two distinct types.

The first type is where V is ramified at $S=\{\infty,0\}$ and is tamely ramified at $T=0$. Hence $\mathscr{F}_0$ is a representation of $\pi_1(\mathbb{G}_m)$, with $j_*\mathscr{F}$ is tamely ramified at $T=0$. In the case of tame ramification, we have a simple formula for the local Artin conductor: $\mathfrak{f}_0(V)=\dim(V/V^{I_0})$. Let $c_0(V)$ be the dimension of the space of I_0-invariants on V, so $0\leq c_0(V)\leq\dim(V)$. When the L-function of V is trivial, we must have

$$\begin{aligned}\mathfrak{f}_0(V)&=\dim(V)-c_0(V)\\ \mathfrak{f}_\infty(V)&=\dim(V)+c_0(V).\end{aligned}$$

In particular, whenever $c_0(V)>0$, the representation V must be wildly ramified at $T=\infty$.

The second type is where V is ramified at $S=\{\infty,0,1\}$ and tamely ramified at all three places. Hence $\mathscr{F}$ corresponds to a representation of $\pi_1^{tame}(\mathbb{P}^1-\{\infty,0,1\})$. Here the geometric tame fundamental group is known to be is isomorphic to the quotient of the free profinite group on three elements, corresponding to generators g_v of the tame inertia groups I_v, by the single relation $g_\infty g_0 g_1=1$. In this case, we have conductors

$$\mathfrak{f}_\infty(V)=\dim(V)-c_\infty(V)$$

$$\mathfrak{f}_0(V) = \dim(V) - c_0(V)$$
$$\mathfrak{f}_1(V) = \dim(V) - c_1(V)$$

where $c_v(V)$ is the dimension of the space of $\langle g_v \rangle$-invariants on V. When the L-function of V is trivial, we must have the equality $c_\infty(V) + c_0(V) + c_1(V) = \dim(V)$

Besides the relation on local conductors when the L-function of V is trivial, we also get a non-trivial relation on the local root numbers. Fix a non-trivial character $\psi = \prod \psi_v$ of the adèles $\mathbb{A}_k$ of k which is trivial on k and unramified outside of S and a Haar measure $dx = \prod dx_v$ of volume 1 on $\mathbb{A}_k/k$. Assume further that dx_v has volume 1 on the ring of integers of k_v, for all $v \notin S$. Then for a representation V with trivial L-function we find

$$\prod_{v \in S} \epsilon(V_v, \psi_v, dx_v, s) = 1.$$

Indeed, this product gives the global epsilon factor $\epsilon(V, s) = 1$ which appears in the functional equation of $L(V, s) = 1$, as the local epsilon factors at places $w \notin S$ are all equal to 1. In particular, the global root number $\epsilon(V, \frac{w+1}{2}) = +1$, and this gives a non-trivial relation on the local root numbers $\epsilon(V_v, \psi_v, dx_v, \frac{w+1}{2})$ at the places $v \in S$.

4 Artin L-functions

We first consider the case studied by Artin, where the representation V factors through a finite quotient $\mathrm{Gal}(K/k)$ of the Galois group. We will describe several such representations of the Galois group of the rational function field $k = E(T)$ with trivial L-functions.

A simple example when q is odd is the one-dimensional representation V given by the non-trivial character χ of the Galois group of the quadratic extension $K = k(\sqrt{T})$. This character is unramified outside of $S = \{\infty, 0\}$ and tamely ramified at these two places. Hence it has has global Artin conductor $(\infty) + (0)$ of degree $2 = 2\dim(V)$. Since χ is ramified, it is non-trivial when restricted to the geometric Galois group, so $L(V, s) = L(\chi, s) = 1$. When q is even, one can replace this example by the non-trivial character χ of the separable quadratic extension $K = k(x)$ where $x^2 + x = T$. This is unramified outside of $S = \{\infty\}$, and has Artin conductor $2(\infty)$.

More generally, let χ be a non-trivial character of the abelian Galois group $\mathrm{Gal}(K/k) = E^*$ of the Kummer extension $K = k(x)$ with $x^{q-1} = T$, or the abelian Galois group $\mathrm{Gal}(K/k) = E^+$ of the Artin-Schreier extension $K = k(x)$ with $x^q - x = T$. Then $\mathfrak{f}(\chi) = (\infty) + (0)$ in the first case and $\mathfrak{f}(\chi) = 2(\infty)$ in the second. Hence the conductor of χ has degree $f(\chi) = 2\dim(V)$ in both cases. Since χ is ramified, it is non-trivial when restricted to the geometric Galois group. Hence $L(\chi, s) = 1$. Since this holds for all non-trivial characters of $\mathrm{Gal}(K/k)$, the ratio of zeta functions is also trivial: $\zeta_K(s)/\zeta_k(s) = 1$.

The ratio of zeta functions will be trivial precisely when the Galois extension K of k also has genus 0 and field of constants E. This also occurs for some non-abelian Galois groups which act on the projective line. Let E_n be the unique extension of degree n of E contained in E^s. The finite group $\mathrm{PGL}_2(E) = \mathrm{PGL}_2(q)$ acts on the projective line $\mathbb{P}^1$ over E by fractional linear transformations. It acts transitively on $\mathbb{P}^1(E)$, with stabilizer a Borel subgroup B, and transitively on $\mathbb{P}^1(E_2) - \mathbb{P}^1(E)$ with stabilizer a non-split torus T. The remaining orbits on $\mathbb{P}^1(E^s) - \mathbb{P}^1(E_2)$ are all free. Since one of them is $\mathbb{P}^1(E_3) - \mathbb{P}^1(E)$, there is a unique $\mathrm{PGL}_2(E)$ covering $\mathbb{P}^1 \to \mathbb{P}^1 = X_0$ over E which maps the E-orbit to ∞, the E_2-orbit to 0, and the E_3-orbit to 1. This covering gives a Galois extension of rational function fields with Galois group $\mathrm{Gal}(K/k) = \mathrm{PGL}_2(E)$. By construction, it is ramified at the set $S = \{\infty, 0\}$ with inertia subgroups B and T respectively. Since K has genus 0 and the same field E of constants, the ratio $\zeta_K(s)/\zeta_k(s) = 1$. It follows that if V is any irreducible non-trivial representation of the Galois group $\mathrm{Gal}(K/k)$, then $L(V, s) = 1$. These are representations of the first type – tamely ramified at $T = 0$ and wildly ramified at $T = \infty$ (once $\dim(V) > 1$).

Associated to the finite subgroups G of $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$, we can make coverings $\mathbb{P}^1 \to X_0$ over F which are tamely ramified at the three places $\{\infty, 0, 1\}$. These give representations V of the second type. The tame inertia subgroups are cyclic groups of order

$(2, 2, n)$ for the dihedral group $G = D_n$ of order $2n$,

$(2, 3, 3)$ for the tetrahedral group $G = A_4$ of order 12,

$(2, 3, 4)$ for the octahedral group $G = S_4$ of order 24,

$(2, 3, 5)$ for the icosahedral group $G = A_5$ of order 60.

These coverings come from a reduction of the corresponding Galois coverings in characteristic zero. To obtain good reduction and to be sure that the action of G is defined over E, we assume that $q \equiv 1(\mathrm{mod\ m})$, where m is the order of the finite subgroup G. Then for any irreducible non-trivial representation V of G we have $L(V, s) = 1$.

An interesting case where *almost* all irreducible representations V of $\mathrm{Gal}(K/k)$ satisfy $L(V, s) = 1$ is for the Deligne-Lusztig curve Y over E with equation $x^{q+1} + y^{q+1} + z^{q+1} = 0$ in $\mathbb{P}^2$. Here the group $\mathrm{PU}_3(E) = \mathrm{PU}_3(q)$ acts on Y over E_2 with quotient isomorphic to $\mathbb{P}^1 = X_0$, so the function field $K = E_2(Y)$ gives a Galois extension of the rational function field $k = E_2(T)$. This covering is ramified at the set $S = \{\infty, 0\}$ and tamely ramified at $T = 0$. For all irreducible complex representations V of $\mathrm{Gal}(K/k) = \mathrm{PU}_3(E)$, except for the trivial representation and the unipotent cuspidal representation of dimension $q^2 - q$, we have $L(V, s) = 1$. There is a similar result [4] for the other two families of Deligne-Lusztig curves, associated to the Suzuki groups and the Ree groups in characteristics 2 and 3. These also give examples which are tamely ramified at $T = 0$ and wildly ramified at $T = \infty$.

5 Elliptic curves and their symmetric power representations

We next consider the case when the Galois representation $V = V_A$ is given by the l-adic Tate module of an elliptic curve A over $k = E(T)$. Then $\dim(V) = 2$. If we assume that the j-invariant of A is not an element of the finite field E, then the geometric Galois group has no invariants on V, and our inequality for the degree of the conductor is

$$f(V_A) \geq 4.$$

The L-function of A will be trivial precisely when $f(V_A) = 4$. A simple example (for odd q) where this equality holds is for the Legendre curve with equation (cf. [5])

$$y^2 = x(x-1)(x-T)$$

This has conductor $\mathfrak{f}(A) = (1) + (0) + 2(\infty)$. Note that all three places in S are tamely ramified in V_A: the tame inertia groups at $T = 1$ and $T = 0$ map to principal unipotent elements and the tame inertia group at $T = \infty$ maps to the product of a principal unipotent element with a central involution. A wildly ramified example in characteristic 2 is the curve [6]

$$y^2 + T^{-1}xy = x^3 + T^{-2}x$$

where the conductor is $\mathfrak{f}(A) = (0) + 3(\infty)$. Here tame inertia at $T = 0$ maps to a principal unipotent element, and the inertia group at $T = \infty$ maps to a subgroup of $\mathrm{PGL}(V_A)$ isomorphic to A_4.

At some point we should warn the reader that the triviality of the L-function of a Galois representation V does not imply the triviality of the L-functions of those Galois representations made from tensor operations on V. We can illustrate this with the symmetric powers $\mathrm{Sym}^n(V_A)$ of the representation coming from the Legendre elliptic curve. If $n \geq 2$ is even, this has Artin conductor

$$\mathfrak{f}(\mathrm{Sym}^n(V_A)) = n(1) + n(0) + n(\infty).$$

If $n \geq 1$ is odd, the conductor is

$$\mathfrak{f}(\mathrm{Sym}^n(V_A)) = n(1) + n(0) + (n+1)(\infty).$$

On the other hand, the dimension of $\mathrm{Sym}^n(V_A)$ is equal to $(n+1)$. Hence $L(\mathrm{Sym}^n(V_A), s)$ is a polynomial of degree $(n-2)$ in q^{-s} when n is even, and of degree $(n-1)$ in q^{-s} when n is odd. In particular, we get a trivial L-function only when $n = 1$ or $n = 2$.

6 Rigid local systems

Let A be the Legendre elliptic curve over $k = E(T)$, where q is odd. The Galois representation $W = \mathrm{Sym}^2(V_A) \otimes \det(V_A)^{-1} = \mathrm{Sym}^2(V_A)(1)$ is orthogonal, of determinant 1, so gives a homomorphism

$$\mathrm{Gal}(k^s/k) \longrightarrow \mathrm{SO}_3(\mathbb{Q}_l) = \mathrm{SO}(W)$$

with $L(W, s) = 1$. This is an example of a rigid local system for the group $G = \mathrm{SO}_3 \cong \mathrm{PGL}_2$.

More generally, let G be a split simple group over $\mathbb{Z}$. We say that a homomorphism

$$\mathrm{Gal}(k^s/k) \longrightarrow \mathrm{Aut}(G)(E_\lambda)$$

is a rigid local system for G if the composite adjoint representation on the Lie algebra

$$\mathrm{Gal}(k^s/k) \longrightarrow \mathrm{Aut}(G)(E_\lambda) \longrightarrow \mathfrak{g}(E_\lambda)$$

has a trivial L-function. The conditions for rigidity are: there are no geometric invariants on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and the conductor of the adjoint representation has degree $f(\mathfrak{g}) = 2\dim(G)$. The equivalent cohomological conditions for rigidity, that $H^i(X, j_*\mathscr{F}(\mathfrak{g})) = 0$ for $i = 0, 1, 2$, imply that the original homomorphism to $\mathrm{Aut}(G)$ has no deformations which preserve the local monodromy at the ramified places. In all of the examples below, the projection

$$\mathrm{Gal}(k^s/k) \longrightarrow \mathrm{Out}(G)(E_\lambda) = \mathrm{Out}(G)(\mathbb{Z})$$

factors through a tame Galois extension K/k of genus 0. Note that $\mathrm{Out}(G)(\mathbb{Z})$ is a finite group, isomorphic to S_1,S_2, or S_3.

Katz [8] has made an extensive study of rigid local systems in the case where $G = \mathrm{PGL}_n = \mathrm{PGL}(V)$ and where the projection to $\mathrm{Out}(G)$ is trivial. In fact, Katz studies homomorphisms $\mathrm{Gal}(k^s/k) \to \mathrm{PGL}(V)$ which lift to $\mathrm{GL}(V)$, so $\mathscr{F}_0$ is a lisse l-adic sheaf of rank n on $\mathbb{P}^1 - S$. In this case, the adjoint representation $\mathfrak{pgl}(V)$ occurs on the space $\mathrm{End}(V)_0$ of endomorphisms of V with trace zero. The adjoint representation $\mathfrak{gl}(V)$ of $\mathrm{GL}(V)$ is on the full space $\mathrm{End}(V)$ of endomorphisms of V. Since this has the invariant subspace of scalar endomorphisms, Katz only demands the vanishing of $H^1(X, j_*\mathscr{F}(\mathfrak{gl}(V)))$.

If a rigid local system is tamely ramified at the place v, the local conductor $\mathfrak{f}_v(\mathfrak{g}) = \dim \mathfrak{g}/\mathfrak{g}^{I_v}$ is strictly less than $\dim \mathfrak{g} = \dim G$. Indeed, a single automorphism of the simple Lie algebra $\mathfrak{g}$ has a non-trivial invariant subalgebra. In fact, let τ_v be a generator of the image of the tame inertia group at v in $\mathrm{Out}(G)$, viewed as a pinned automorphism of G, and let $G(\tau_v)$ be the subgroup it fixes. Then for a tamely ramified place v , we have the inequality

$$\mathfrak{f}_v(\mathfrak{g}) \leq \dim(G) - \mathrm{rank}(G(\tau_v)).$$

Some interesting examples of rigid local systems for simple adjoint groups G were recently constructed by Heinloth, Ngo, and Yun [7] using techniques of geometric Langlands theory. Their work was extended by Yun. We describe these examples briefly here.

The local systems of the first type come from regular elliptic classes σ in the extended Weyl group $W.\mathrm{Out}(G)$ of G. (In [7] this class is either the Coxeter class or the twisted Coxeter class.) Let m be the order of σ and let f be the order of the image τ of σ in $\mathrm{Out}(G)$. We assume that $q \equiv 1(\mathrm{mod\ f})$. Let R be the root system of G. Then there are rigid local systems

$$\pi_1(\mathbb{P}^1 - \{\infty, 0\}) = \pi_1(\mathbb{G}_m) \longrightarrow \mathrm{Aut}(G)(\mathbb{Q}_l(\mu_p))$$

which are tamely ramified at $T = 0$ and wildly ramified at $T = \infty$, such that the conductor of the adjoint representation is given by $\mathfrak{f}_0(\mathfrak{g}) = \dim(G) - \#R/m$ and $\mathfrak{f}_\infty(\mathfrak{g}) = \dim(G) + \#R/m$. We note that $\#R/m$ is an integer, as the cyclic group generated by σ acts freely on the set of roots R. A generator of tame inertia at $T = 0$ maps to $u \times \tau$, with u unipotent in $G(\tau)$. When p does not divide m, the inertia group at ∞ maps to the finite subgroup $H[p].\langle\sigma\rangle$, where H is a maximal split torus in G. (When p divides m, the image of wild inertia is probably not contained in a maximal torus.) In the case where $G = \mathrm{PGL}(n)$ and σ is the Coxeter class, the corresponding rigid local system of rank n on $\mathbb{G}_m$ was first constructed by Deligne [2] using Kloosterman sums. Katz [9] determined the Zariski closure of the image of the Galois group in this case.

The rigid local systems of the second type (constructed by Yun) give homomorphisms

$$\pi_1(\mathbb{P}^1 - \{\infty, 0, 1\}) \longrightarrow \mathrm{Aut}(G)(\mathbb{Q}_l)$$

which are tamely ramified at all three places. These examples exist when $q \equiv 1(\mathrm{mod}\ 4)$. A generator of tame inertia at $T = 1$ maps to a regular unipotent element in G, so $c_1(\mathfrak{g}) = \mathrm{rank}(G)$. Furthermore, $c_\infty(\mathfrak{g}) = c_0(\mathfrak{g}) = \#R/2$. These examples lift to characteristic zero, over the field of Gaussian numbers.

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