A RIGID IRREGULAR CONNECTION ON THE PROJECTIVE LINE

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To Victor Kac and Nick Katz on their 65th birthdays

Abstract. In this paper we construct a connection ∇ on the trivial G-bundle on $\mathbb{P}^1$ for any simple complex algebraic group G, which is regular outside of the points 0 and ∞, has a regular singularity at the point 0, with principal unipotent monodromy, and has an irregular singularity at the point ∞, with slope $1/h$, the reciprocal of the Coxeter number of G. The connection ∇, which admits the structure of an oper in the sense of Beilinson and Drinfeld, appears to be the characteristic 0 counterpart of a hypothetical family of ℓ-adic representations, which should parametrize a specific automorphic representation under the global Langlands correspondence. These ℓ-adic representations, and their characteristic 0 counterparts, have been constructed in some cases by Deligne and Katz. Our connection is constructed uniformly for any simple algebraic group, and characterized using the formalism of opers. It provides an example of the geometric Langlands correspondence with wild ramification. We compute the de Rham cohomology of our connection with values in a representation V of G, and describe the differential Galois group of ∇ as a subgroup of G.

1. Introduction

The Langlands correspondence relates automorphic representations of a split reductive group G over the ring of adèles of a global field F and ℓ-adic representations of the Galois group of F with values in a (slightly modified) dual group of G (see Section 2). On the other hand, the trace formula gives us an effective tool to find the multiplicities of automorphic representations satisfying certain local conditions. In some cases one finds that there is a unique irreducible automorphic representation with prescribed local behavior at finitely many places. A special case of this, analyzed in [G3], occurs when F is the function field of the projective line $\mathbb{P}^1$ over a finite field k, and G is a simple group over k. We specify that the local factor at one rational point of $\mathbb{P}^1$ is the Steinberg representation, the local factor at another rational point is a simple supercuspidal representation constructed in [GR], and that the local representations are unramified at all other places of F. In this case the trace formula shows that there is an essentially unique automorphic representation with these properties. Hence the corresponding family of ℓ-adic representations of the Galois group of F to the dual group $\check{G}$ should also be unique. An interesting open problem is to find it.

Due to the compatibility of the local and global Langlands conjectures, these ℓ-adic representations should be unramified at all points of $\mathbb{P}^1$ except for two rational points 0 and ∞. At 0 it should be tamely ramified, and the tame inertia group should map

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to a subgroup of $\check{G}$ topologically generated by a principal unipotent element. At ∞ it should be wildly ramified, but in the mildest possible way.

Given a representation V of the dual group $\check{G}$, we would obtain an ℓ-adic sheaf on $\mathbb{P}^1$ (of rank $\dim V$) satisfying the same properties. The desired lisse ℓ-adic sheaves on $\mathbb{G}_m$ have been constructed by P. Deligne [D1] and N. Katz [Katz2] in the cases when $\check{G}$ is SL_n, Sp_{2n}, SO_{2n+1} or G_2 and V is the irreducible representation of dimension $n, 2n, 2n+1$, and 7, respectively. However, there are no candidates for these ℓ-adic representations known for other groups $\check{G}$.

In order to gain a better understanding of the general case, we consider an analogous problem in the framework of the geometric Langlands correspondence. Here we switch from the function field F of a curve defined over a finite field to an algebraic curve X over the complex field. In the geometric correspondence (see, e.g., [F2]) the role of an ℓ-adic representation of the Galois group of F is played by a flat $\check{G}$-bundle on X (that is, a pair consisting of a principal $\check{G}$-bundle on X and a connection ∇, which is automatically flat since $\dim X = 1$). Hence we look for a flat $\check{G}$-bundle on $\mathbb{P}^1$ having regular singularity at a point $0 \in \mathbb{P}^1$ with regular unipotent monodromy and an irregular singularity at another point $\infty \in \mathbb{P}^1$ with the smallest possible slope $1/h$, where h is the Coxeter number of $\check{G}$ (see [D2] and Section 5 for the definition of slope). By analogy with the characteristic p case discussed above, we expect that a flat bundle satisfying these properties is unique (up to the action of the group $\mathbb{G}_m$ of automorphisms of $\mathbb{P}^1$ preserving the points $0, \infty$).

In this paper we construct this flat $\check{G}$-bundle for any simple algebraic group $\check{G}$. A key point of our construction is that this flat bundle is equipped with an oper structure. The notion of oper was introduced by A. Beilinson and V. Drinfeld [BD2] (following the earlier work [DS]), and it plays an important role in the geometric Langlands correspondence. An oper is a flat bundle with an additional structure; namely, a reduction of the principal $\check{G}$-bundle to a Borel subgroup $\check{B}$ which is in some sense transverse to the connection ∇. In our case, the principal $\check{G}$-bundle on $\mathbb{P}^1$ is actually trivial, and the oper $\check{B}$-reduction is trivial as well. If N is a principal nilpotent element in the Lie algebra of a Borel subgroup opposite to $\check{B}$ and E is a basis vector of the highest root space for $\check{B}$ on $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$, then our connection takes the form

$$\nabla = d + N\frac{dt}{t} + Edt, \tag{1.1}$$

where t is a parameter on $\mathbb{P}^1$ with a simple zero at 0 and a simple pole at ∞. We also give in Section 5.1 a twisted analogue of this formula, associated to an automorphism of $\check{G}$ of finite order preserving $\check{B}$ (answering a question raised by P. Deligne).

For any representation V of $\check{G}$ our connection gives rise to a flat connection on the trivial vector bundle of rank $\dim V$ on $\mathbb{P}^1$. We examine this connection more closely in the special cases analyzed by Katz in [Katz2]. In these cases Katz constructs not only the ℓ-adic sheaves, but also their counterparts in characteristic 0, so we can compare with his results. These special cases share the remarkable property that a regular unipotent element of $\check{G}$ has a single Jordan block in the representation V. For this reason our oper connection can be converted into a scalar differential operator of order equal to $\dim V$ (this differential operator has the same differential Galois group as the original connection). We compute this operator in all of the above cases and find perfect agreement with the differential operators constructed by Katz [Katz2]. This strongly suggests that our connections are indeed the characteristic 0 analogues of the special ℓ-adic representations whose existence is predicted by the Langlands correspondence and the trace formula.

Another piece of evidence is the vanishing of the de Rham cohomology of the intermediate extension to $\mathbb{P}^1$ of the $\mathcal{D}$-module on $\mathbb{G}_m$ defined by our connection with values in the adjoint representation of $\check{G}$. This matches the expectation that the corresponding cohomology of the ℓ-adic representations also vanish, or equivalently, that the their global L-function with respect to the adjoint representation of $\check{G}$ is equal to 1. We give two proofs of the vanishing of this de Rham cohomology. The first uses non-trivial results about the principal Heisenberg subalgebras of the affine Kac–Moody algebras due to V. Kac [Kac1, Kac2]. The second uses an explicit description of the differential Galois group of our connection and its inertia subgroups [Katz1].

Since the first de Rham cohomology is the space of infinitesimal deformations of our local system (preserving its formal types at the singular points 0 and ∞) [Katz3, BE, A], its vanishing means that our local system on $\mathbb{P}^1$ is rigid. We also prove the vanishing of the de Rham cohomology for small representations considered in [Katz2]. This is again in agreement with the vanishing of the cohomology of the corresponding ℓ-adic representations shown by Katz. Using our description of the differential Galois group of our connection and a formula of Deligne [D2] for the Euler characteristic, we give a formula for the dimensions of the de Rham cohomology groups for an arbitrary representation V of $\check{G}$.

Finally, we describe some connections which are closely related to ∇, and others which are analogous to ∇ coming from subregular nilpotent elements. We also use ∇ to give an example of the geometric Langlands correspondence with wild ramification.

The paper is organized as follows. In Section 2 we introduce the concepts and notation relevant to our discussion of automorphic representations. In Section 3 we give the formula for the multiplicity of automorphic representations from [G3]. This formula implies the existence of a particular automorphic representation. In Section 4 we summarize what is known about the corresponding family of ℓ-adic representations. We then switch to characteristic 0. In Section 5 we give an explicit formula for our connection for an arbitrary complex simple algebraic group, as well as its twisted version. In Section 6 we consider the special cases of representations on which a regular unipotent element has a single Jordan block. In these cases our connection can be represented by a scalar differential operator. These operators agree with those found earlier by Katz [Katz2].

We then take up the question of computation of the de Rham cohomology of our connection. After some preparatory material presented in Sections 7–9 we prove vanishing of the de Rham cohomology on the adjoint and small representations in Sections 10 and 11, respectively. We also show that the de Rham cohomology can be non-trivial for other representations using the case of SL_2 as an example in Section 12. In Section 13 we determine the differential Galois group of our connection. We then use it in Section 14 to give a formula for the dimensions of the de Rham cohomology for an arbitrary finite-dimensional representation of $\check{G}$. In particular, we give an alternative proof of the vanishing of de Rham cohomology for the adjoint and small representations. In Section 15 we discuss some closely related connections.

Finally, in Section 16 we describe what the geometric Langlands correspondence should look like for our connection.

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2. Simple algebraic groups over global function fields

Let k be a finite field, of order q. Let G be an absolutely almost simple algebraic group over k (which we will refer to as a simple group for brevity). The group G is quasi-split over k, and we fix a maximal torus $A \subset B \subset G$ contained in a Borel subgroup of G over k. Let k' be the splitting field of G, which is the splitting field of the torus A, and put $\Gamma = \mathrm{Gal}(k'/k)$. Then Γ is a finite cyclic group, of order 1, 2, or 3. Let Z denote the center of G, which is a finite, commutative group scheme over k.

Let $\check{G}$ denote the complex dual group of G. This comes with a pinning $\check{T} \subset \check{B} \subset \check{G}$, as well as an action of Γ which permutes basis vectors $X_{-\alpha}$ of the simple negative root spaces. The principal element $N = \sum X_{-\alpha}$ in $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$ is invariant under Γ ([G1]). Let $\check{Z}$ denote the finite center of $\check{G}$, which also has an action of Γ.

There is an element ϵ in $Z(\check{G})^\Gamma$ which satisfies $\epsilon^2 = 1$ and is defined as follows. Let 2ρ be the co-character of $\check{T}$ which is the sum of positive co-roots, and define

$$\epsilon = (2\rho)(-1).$$

Since the value of ρ on any root is integral, ϵ lies in $Z(\check{G})$. It is also fixed by Γ. We have

$$\epsilon = 1 \quad \longleftrightarrow \quad \rho \text{ is a co-character of } \check{T}.$$

In order to avoid choosing a square root of q in the construction of Galois representations, we will use the following modification $\check{G}_1$ of $\check{G}$, which was suggested by Deligne. Let $\check{G}_1 = \check{G} \times \mathbb{G}_m/(\epsilon \times -1)$. We have homomorphisms $\mathbb{G}_m \to \check{G}_1 \to \mathbb{G}_m$ with composite $z \mapsto z^2$, The group Γ acts on $\check{G}_1$, trivially on $\mathbb{G}_m$. The co-character group $X_\bullet(\check{T}_1)$ contains the direct sum $X_\bullet(\check{T}) \times X_\bullet(\mathbb{G}_m)$ with index 2. The advantage of passing to $\check{G}_1$ is that we can choose a co-character

$$\eta : \mathbb{G}_m \to \check{T}_1$$

fixed by Γ which satisfies

$$\langle \eta, \alpha \rangle = 1$$

for all simple roots α of $\check{G}$. (This is impossible to do for $\check{G}$ when $\epsilon \neq 1$.) Having chosen η, we let $w(\eta) \in \mathbb{Z}$ be defined by composite map $\mathbb{G}_m \xrightarrow{\eta} \check{G}_1 \to \mathbb{G}_m$, $z \mapsto z^{w(\eta)}$. Then $w(\eta)$ is odd precisely when $\epsilon \neq 1$.

Let X be a smooth, geometrically connected, complete algebraic curve over k, of genus g. Let $F = k(X)$ be the global function field of X. We fix two disjoint, non-empty sets S, T of places v of F, and define the degrees

$$\deg(S) = \sum_{v \in S} \deg v \ \geq \ 1,$$
$$\deg(T) = \sum_{v \in T} \deg v \ \geq \ 1.$$

A place v of F corresponds to a $\mathrm{Gal}(\overline{k}/k)$-orbit on the set of points $X(\overline{k})$. The degree $\deg v$ of v is the cardinality of the orbit. Let

$$M_G = \bigoplus_{d \geq 2} V_d(1-d)$$

be the motive of the simple group G over $F = k(X)$ ([G2]). The spaces V_d, of invariant polynomials of degree d, are all rational representations of the finite, unramified quotient Γ of $\mathrm{Gal}(F^s/F)$. The Artin L-function of $V = V_d$, relative to the sets S and T, is defined by

$$L_{S,T}(V,s) = \prod_{v \notin S} \det(1 - \mathrm{Fr}_v\, q_v^{-s}|V)^{-1} \prod_{v \in T} \det(1 - \mathrm{Fr}_v\, q_v^{1-s}|V).$$

Here $\mathrm{Fr}_v = \mathrm{Fr}^{\deg v}$, where Fr is the Frobenius generator of Γ, $x \mapsto x^q$, and $q_v = q^{\deg v}$. This is known to be a polynomial of degree $\dim V(2g - 2 + \deg S + \deg T)$ in q^{-s} with integral coefficients and constant coefficient 1 ([We]). We define

$$L_{S,T}(M_G) = \prod_{d \geq 2} L_{S,T}(V_d, 1-d),$$

which is a non-zero integer. In the next section, we will use the integer $L_{S,T}(M_G)$ to study spaces of automorphic forms on G over F. We end this section with some examples.

Let $2 = d_1, d_2, \ldots, d_{\mathrm{rk}(G)} = h$ be the degrees of generators of the algebra of invariant polynomials of the Weyl group, where $\mathrm{rk}(G) = \dim A$ is the rank of G over the splitting field k' and h is the Coxeter number. If G is split,

$$L_{S,T}(M_G) = \prod_{i=1}^{\mathrm{rk}(G)} \zeta_{S,T}(1 - d_i),$$

where $\zeta_{S,T}$ is the zeta-function of the curve $X - S$ relative to T. Now assume that G is not split, but that G is not of type D_{2n}. Then each V_d has dimension 1, Γ has order 2, and Γ acts non-trivially on V_d if and only if d is odd. Hence

$$L_{S,T}(M_G) = \prod_{d_i \text{ even}} \zeta_{S,T}(1 - d_i) \prod_{d_i \text{ odd}} L_{S,T}(\epsilon, 1 - d_i),$$

where ϵ is the non-trivial quadratic character of Γ.

3. Automorphic representations

Let $\mathbb{A}$ be the ring of adèles of the function field $F = k(X)$. Then $G(F)$ is a discrete subgroup, with finite co-volume, in $G(\mathbb{A})$. Let L denote the discrete spectrum, which is a $G(\mathbb{A})$-submodule of $L^2(G(F)\backslash G(\mathbb{A}))$. Any irreducible representation π of $G(\mathbb{A})$ has finite multiplicity $m(\pi)$ in L.

We will count the sum of multiplicities over irreducible representations $\pi = \widehat{\bigotimes}\pi_v$ with specified local behavior. Specifically, for $v \notin S \cup T$, we insist that π_v be an unramified irreducible representation of $G(F_v)$, in the sense that the open compact subgroup $G(\mathcal{O}_v)$ fixes a non-zero vector in π_v. At places $v \in S$, we insist that π_v is the Steinberg representation of $G(F_v)$. Finally, at places $v \in T$, we insist that π_v is a simple supercuspidal representation of $G(F_v)$, of the following type (cf. [GR]). We let $\chi_v : P_v \to \mu_p$ be a given affine generic character of a pro-p-Sylow subgroup $P_v \subset G(\mathcal{O}_v)$. We recall that χ_v is non-trivial on the simple affine root spaces of the Frattini quotient of P_v. (This is the affine analogue of a generic character of the unipotent radical of a Borel subgroup.) Extend χ_v to a character of $Z(q_v) \times P_v$ which is trivial on $Z(q_v)$; then the compactly induced representation $\mathrm{Ind}_{P_v \times Z(q_v)}^{G(F_v)}(\chi_v)$ of $G(F_v)$ is multiplicity-free, with $\#Z(\check{G})^\Gamma$ irreducible summands. We insist that π_v be a summand of this induced representation.

The condition imposed on π_v only depends on the I_v-orbit of the generic character χ_v, where I_v is the Iwahori subgroup of $G(F_v)$ which normalizes P_v. The quotient group I_v/P_v is isomorphic to $A(q_v)$, where A is the torus in the Borel subgroup, and there are $(q_v - 1) \cdot \#Z(q_v)$ different orbits. We note that affine generic characters form a principal homogeneous space for the group $A^{\mathrm{ad}}(q_v) \times \mathbb{F}_{q_v}^\times$, where the latter group maps a local parameter t to λt.

Using the simple trace formula, and assuming that some results of Kottwitz [Kot] on the vanishing of the local orbital integrals of the Euler–Poincaré function on non-elliptic classes extend from characteristic zero to characteristic p, we obtain the following formula for multiplicities [G3]:

Assume that p does not divide $\#Z(\check{G})$. Then

$$\sum_{\pi \text{ as above}} m(\pi) = L_{S,T}(M_G) \cdot \frac{\#Z(\check{G})^\Gamma}{\prod_{v \in S} \#Z(\check{G})^{\Gamma_v}(-1)^{\mathrm{rk}(G)_v}} \cdot \frac{\#Z(q)}{\prod_{v \in T} \#Z(q_v)(-1)^{\mathrm{rk}(G)_v}},$$

where $\mathrm{rk}(G)_v$ is the rank of G over F_v.

In the special case where $X = \mathbb{P}^1$ has genus 0, $S = \{0\}$ and $T = \{\infty\}$, we have

$$L_{S,T}(V_d, s) = 1, \qquad \text{for all } d, s.$$

Hence

$$L_{S,T}(M_G) = 1.$$

Since $\Gamma_v = \Gamma$ and $q_v = q$ in this case, we find that

$$\sum m(\pi) = 1.$$

Hence there is a unique automorphic representation in the discrete spectrum which is Steinberg at 0, simple supercuspidal for a fixed orbit of generic characters at ∞, and unramified elsewhere. This global representation is defined over the field of definition of π_∞, which is a subfield of $\mathbb{Q}(\mu_p)$. We will consider its Langlands parameter in the next section.

4. ℓ-adic sheaves on $\mathbb{G}_m$

Consider the automorphic representation π of the simple group G over $F = k(t)$, described at the end of Section 3. This representation is defined over the field $\mathbb{Q}(\mu_p)$. Its local components π_v are unramified irreducible representations of $G(F_v)$, for all $v \neq 0, \infty$. The representation π_0 is the Steinberg representation, and π_∞ is a simple supercuspidal representation.

Associated to π and the choice of a co-character $\eta : \mathbb{G}_m \to \check{T}_1 \subset \check{G}_1$, as described in Section 2, we should (conjecturally) have a global Langlands parameter. This will be a continuous homomorphism

$$\varphi_\lambda : \mathrm{Gal}(F^s/F) \to \check{G}_1(\mathbb{Q}(\mu_p)_\lambda) \rtimes \Gamma$$

for every finite place λ of $\mathbb{Q}_\ell(\mu_p)$ dividing a rational prime $\ell \neq p$. Here we view $\check{G}_1$ as a pinned, split group over $\mathbb{Q}$. The homomorphism φ_λ should be unramified outside of 0 and ∞, and map Fr_v to the η-normalized Satake parameter for π_v [G4], which is a semi-simple conjugacy class in $\check{G}_1(\mathbb{Q}(\mu_p)) \rtimes \Gamma_v$. If this is true, the projection of φ_λ to $\mathbb{G}_m(\mathbb{Q}(\mu_p)_\lambda)$ will be the $w(\eta)$-power of the ℓth cyclotomic character.

At 0, φ_ℓ should be tamely ramified. The tame inertia group should map to a $\mathbb{Z}_\ell$-subgroup of $\check{G}(\mathbb{Q}_\ell(\mu_p))$, topologically generated by a principal unipotent element. At ∞, φ_ℓ should be wildly ramified, but trivial on the subgroup $I_\infty^{1/h+}$ in the upper numbering filtration, where h is the Coxeter number of G. If p does not divide h, the image of inertia should lie in the normalizer $N(\check{T})$ of a maximal torus, and should have the form $E \cdot \langle n \rangle$. Here $E \subset \check{T}[p]$ is a regular subgroup isomorphic to the additive group of the finite field $\mathbb{F}_p(\mu_h)$ and n maps to a Coxeter element w of order h in the quotient group $N(\check{T})/\check{T}$. The element n is both regular and semi-simple in $\check{G}$, and satisfies $n^h = \epsilon \in Z(\check{G})$, with $\epsilon^2 = 1$.

If

$$\rho : \check{G}_1 \rtimes \Gamma \to GL(V)$$

is a representation over $\mathbb{Q}$, we would obtain from $\rho \circ \varphi_\lambda$ a lisse λ-adic sheaf $\mathcal{F}$ on $\mathbb{G}_m$ over k, with $\mathrm{rank}(\mathcal{F}) = \dim(V)$. Katz has constructed and studied these lisse sheaves in the cases when the Coxeter element in the Weyl group has a single orbit on the set of non-zero weights for $\check{T}$ on V (cf. Theorem 11.1 in [Katz2]). In all of these cases, the principal nilpotent element $N = \sum X_{-\alpha}$ has a single Jordan block on V. We list them in the table below.

$\check{G}$	$\dim V$
SL_n	n
Sp_{2n}	$2n$
SO_{2n+1}	$2n+1$
G_2	7

In all of these cases, there are no I_∞-invariants on $\mathcal{F}$ and $sw_\infty(\mathcal{F}) = 1$. If $j : \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ is the inclusion, Katz has shown that $H^i(\mathbb{P}^1, j_*\mathcal{F}) = 0$ for all i. Hence $L(\pi, V, s) = 1$.

More generally, consider the adjoint representation $\check{\mathfrak{g}}$. Then the sheaf $\mathcal{F}$ has rank equal to the dimension of $\check{G}$. In this case, we predict that

1) I_0 has $\mathrm{rk}(\check{G})$ Jordan blocks on $\mathcal{F}$;
2) I_∞ has no invariants on $\mathcal{F}$, and $sw_\infty(\mathcal{F}) = \mathrm{rk}(\check{G})$;
3) $H^i(\mathbb{P}^1, j_*\mathcal{F}) = 0$ for all i;
4) $L(\pi, \check{\mathfrak{g}}, s) = 1$.

Katz has verified this in the cases tabulated above, using the fact that the adjoint representation $\check{\mathfrak{g}}$ of $\check{G}$ appears in the tensor product of the representation V and its dual.

It is an open problem to construct the λ-adic sheaf $\mathcal{F}$ on $\mathbb{G}_m$ over the finite field k for general groups $\check{G}$. We consider an analogous problem, for local systems on $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$, in the next section.

5. The connection

Let $\check{G}$ be a simple algebraic group over $\mathbb{C}$ and $\check{\mathfrak{g}}$ its Lie algebra. Fix a Borel subgroup $\check{B} \subset \check{G}$ and a torus $\check{T} \subset \check{B}$. For each simple root α_i, we denote by $X_{-\alpha_i}$ a basis vector in the root subspace of $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$ corresponding to $-\alpha_i$. Let E be a basis vector in the root subspace of $\check{\mathfrak{g}}$ corresponding to the maximal root θ. Set

$$N = \sum_{i=1}^{\mathrm{rk}(\check{\mathfrak{g}})} X_{-\alpha_i}.$$

We define a connection ∇ on the trivial $\check{G}$-bundle on $\mathbb{P}^1$ by the formula

$$\nabla = d + N\frac{dt}{t} + E dt, \tag{5.1}$$

where t is a parameter on $\mathbb{P}^1$ with a simple zero at 0 and a simple pole at ∞.

The connection ∇ is clearly regular outside of the points $t = 0$ and ∞, where the differential forms $\frac{dt}{t}$ and dt have no poles. We now discuss the behavior of (5.1) near the points $t = 0$ and ∞. It has regular singularity at $t = 0$, with the monodromy being a regular unipotent element of $\check{G}$. It also has a double pole at $t = \infty$, so is irregular, and its slope there is $1/h$, where h is the Coxeter number. Here we adapt the definition of slope from [D2], Theorem 1.12: a connection on a principal $\check{G}$-bundle with irregular singularity at a point x on a curve X has slope $a/b > 0$ at this point if the following holds. Let s be a uniformizing parameter at x, and pass to the extension given by adjoining the b^{th} root of s: $u^b = s$. Then the connection, written using the parameter

u in the extension and a particular trivialization of the bundle on the punctured disc at x should have a pole of order $a+1$ at x, and its polar part at x should not be nilpotent.

To see that our connection has slope $1/h$ at the point ∞, suppose first that $\check{G}$ has adjoint type. In terms of the uniformizing parameter $s = t^{-1}$ at ∞, our connection has the form

$$d - N\frac{ds}{s} - E\frac{ds}{s^2}.$$

Now take the covering given by $u^h = s$. Then the connection becomes

$$d - hN\frac{du}{u} - hE\frac{du}{u^{h+1}}.$$

If we now make a gauge transformation with $g = \rho(u)$ in the torus $\check{T}$, where ρ is the co-character of $\check{T}$ which is given by half the sum of the positive co-roots for $\check{B}$, this becomes

$$d - h(N+E)\frac{du}{u^2} - \rho\frac{du}{u}. \tag{5.2}$$

The element $N+E$ is regular and semi-simple, by Kostant's theorem [Kos]. Since the pole has order $(a+1) = 2$ with $a = 1$, the slope is indeed $1/h$. If $\check{G}$ is not of adjoint type, then $g = \rho(u)$ might not be in $\check{T}$, but it will be after we pass to the covering obtained by extracting a square root of u. The resulting slope will be the same.

Note that $\exp(\rho/h)$ is a Coxeter element in $\check{G}$, which normalizes the maximal torus centralizing the regular element $N+E$. We therefore have a close analogy between the local behavior of our connection ∇ and the desired local parameters in the λ-adic representation φ_λ at both zero and infinity.

The connection we have defined looks deceptively simple. We now describe how we used the theory of opers to find it. We recall from [BD2] that a (regular) oper on a curve U is a $\check{G}$-bundle with a connection ∇ and a reduction to the Borel subgroup $\check{B}$ such that, with respect to any local trivialization of this $\check{B}$-reduction, the connection has the form

$$\nabla = d + \sum_{i=1}^{\mathrm{rk}(\check{\mathfrak{g}})} \psi_i X_{-\alpha_i} + \mathbf{v}, \tag{5.3}$$

where the ψ_i are nowhere vanishing one-forms on U and $\mathbf{v}$ is a regular one-form taking values in the Borel subalgebra $\check{\mathfrak{b}} = \mathrm{Lie}(\check{B})$. Here $X_{-\alpha_i}$ are non-zero vectors in the root subspaces corresponding to the negative simple roots $-\alpha_i$ with respect to any maximal torus $\check{T} \subset \check{B}$. Since the group $\check{B}$ acts transitively on the set of such tori, this definition is independent of the choice of $\check{T}$.

Any $\check{B}$-bundle on the curve $U = \mathbb{G}_m$ may be trivialized. Therefore the space of $\check{G}$-opers on $\mathbb{G}_m$ may be described very concretely as the quotient of the space of connections (5.3), where

$$\psi_i \in \mathbb{C}[t,t^{-1}]^\times dt = \mathbb{C}^\times \cdot t^{\mathbb{Z}} dt,$$

and $\mathbf{v} \in \check{\mathfrak{b}}[t,t^{-1}]dt$, modulo the gauge action of $\check{B}[t,t^{-1}]$.

We now impose the following conditions at the points 0 and ∞. First we insist that at $t = 0$ the connection has a pole of order 1, with principal unipotent monodromy.

The corresponding $\check{B}[t,t^{-1}]$-gauge equivalence class contains a representative (5.3) with $\psi_i \in \mathbb{C}^\times t^{-1}dt$ for all $i = 1,\dots,\mathrm{rk}(\check{\mathfrak{g}})$ and $\mathbf{v} \in \check{\mathfrak{b}}[t]dt$. By making a gauge transformation by a suitable element in $\check{T}$, we can make $\psi_i = \frac{dt}{t}$ for all i.

Second, we insist that the one-form $\mathbf{v}$ has a pole of order 2 at $t = \infty$, with leading term in the highest root space $\check{\mathfrak{g}}_\theta$ of $\check{\mathfrak{b}}$, which is the minimal non-zero orbit for $\check{B}$ on its Lie algebra. Then $\mathbf{v}(t) = Edt$, with E a non-zero vector of $\check{\mathfrak{g}}_\theta$. These basis vectors are permuted simply-transitively by the group $\mathbb{G}_m$ of automorphisms of $\mathbb{P}^1$ preserving 0 and ∞ (that is, rescalings of the coordinate t). Our oper connection therefore takes the form

$$\nabla = d + N\frac{dt}{t} + Edt. \tag{5.4}$$

This shows that the oper satisfying the above conditions is unique up to the automorphisms of $\mathbb{P}^1$ preserving 0 and ∞.

We note that opers of this form (and possible additional regular singularities at other points of $\mathbb{P}^1$) have been considered in [FF], where they were used to parametrize the spectra of quantum KdV Hamiltonians. Opers of the form (5.4), where E is a regular element of $\check{\mathfrak{g}}$ (rather than nilpotent element generating the maximal root subspace $\check{\mathfrak{g}}_\theta$, as discussed here), have been considered in [FFT, FFR]. Finally, irregular connections (not necessarily in oper form) with double poles and regular semi-simple leading terms have been studied in [B].

5.1. **Twisted version.** In this paper we focus on the case of a "constant" group scheme over $\mathbb{G}_m$ corresponding to $\check{G}$. More generally, we may consider group schemes over $\mathbb{G}_m$ twisted by automorphisms of $\check{G}$. The connection ∇ has analogues for these group schemes which we now describe. It is natural to view them as complex counterparts of the λ-adic representation φ_λ for non-split groups. This answers a question raised by Deligne.

Let σ be an automorphism of $\check{G}$ of order n. It defines an automorphism of the Lie algebra $\check{\mathfrak{g}}$, which we also denote by σ. Define a group scheme $\check{G}_\sigma$ on $X = \mathbb{G}_m$ as follows. Let $\widetilde{X} = \mathbb{G}_m$ be the n-sheeted cyclic cover of X with a coordinate z such that $z^n = t$. Then $\check{G}_\sigma$ is the quotient of $\widetilde{X} \times \check{G}$, viewed as a group scheme over X, by the automorphism $\widetilde{\sigma}$ which acts by the formula $(z,g) \mapsto (ze^{2\pi i/n}, \sigma(g))$. It is endowed with a (flat) connection. Hence we have a natural notion of a $\check{G}_\sigma$-bundle on X with a (flat) connection. We now give an example of such an object which is a twisted analogue of the flat bundles described above.

Take the trivial $\check{G}_\sigma$-bundle on X. Then a connection on it may be described concretely as an operator

$$\nabla = d + A(z)dz,$$

where $A(z)dz$ is a $\widetilde{\sigma}$-invariant $\check{\mathfrak{g}}$-valued one-form on $\widetilde{X}$.

Note that $\check{G}_\sigma$ and $\check{G}_{\sigma'}$ are isomorphic if σ and σ' differ by an inner automorphism of $\check{G}$. Hence, without loss of generality, we may, and will, assume that σ is an automorphism of $\check{G}$ preserving the pinning we have chosen (thus, σ has order $1, 2$ or 3). In particular, it permutes the elements $X_{-\alpha_i} \in \check{\mathfrak{g}}$. Then, for $A(z)$ as above, $zA(z)$ defines an element of the twisted affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\sigma$, see [Kac2]. The lower nilpotent subalgebra of this Lie algebra has generators $\widehat{X}_{-\alpha_i}, i = 0, \ldots, \ell_\sigma$, where ℓ_σ is the rank of the σ-invariant Lie subalgebra of $\check{\mathfrak{g}}$, and $\widehat{X}_{-\alpha_i}, i = 1, \ldots, \ell_\sigma$, are σ-invariant linear combinations of the elements $X_{-\alpha_j}$ of $\check{\mathfrak{g}}$ (viewed as elements of the constant Lie subalgebra of $\widehat{\check{\mathfrak{g}}}_\sigma$). Explicit expressions for $\widehat{X}_{-\alpha_i}$ in terms of $\check{\mathfrak{g}}$ may be found in [Kac2].

In the untwisted case the corresponding generators of the affine Kac–Moody algebra (which is a central extension of $\check{\mathfrak{g}}[t, t^{-1}]$, see Section 10) are $\widehat{X}_{-\alpha_i}, i = i, \ldots, \ell = \mathrm{rk}(\check{\mathfrak{g}})$, and $\widehat{X}_{-\alpha_0} = Et$. Hence our connection (5.4) may be written as

$$\nabla = d + \sum_{i=0}^{\ell} \widehat{X}_{-\alpha_i} \frac{dt}{t}.$$

We now adapt the same formula in the twisted case and define the following connection on the trivial $\check{G}_\sigma$-bundle on $\mathbb{G}_m$:

$$\nabla = d + \sum_{i=0}^{\ell_\sigma} \widehat{X}_{-\alpha_i} \frac{dz}{z}. \tag{5.5}$$

We propose (5.5) as a complex counterpart of the λ-adic representation φ_λ discussed above in the case when the group G is quasi-split and $\Gamma = \langle \sigma \rangle$.

The notion of oper may be generalized to the twisted case as well. Namely, for any smooth complex curve X equipped with an unramified n-sheeted covering $\widetilde{X}$ we define the group scheme $\check{G}_\sigma$ in the same way as above. Since σ preserves a Borel subgroup $\check{B}$ of $\check{G}$, the group scheme $\check{G}_\sigma$ contains a group subscheme $\check{B}_\sigma$. The $\check{G}_\sigma$-opers on X (relative to $\widetilde{X}$) are then $\check{G}_\sigma$-bundles on X with a connection ∇ and a reduction to $\check{B}_\sigma$ satisfying the condition that locally the connection has the form

$$\nabla = d + \sum_{i=1}^{\ell_\sigma} \psi_i \widehat{X}_{-\alpha_i} + \mathbf{v}, \tag{5.6}$$

where ψ_i are nowhere vanishing one-forms and $\mathbf{v}$ takes values in the Lie algebra of $\check{B}_\sigma$. (As above, if σ acts by permutation of elements $X_{-\alpha_j}$ of $\check{\mathfrak{g}}$, then the $\widehat{X}_{-\alpha_i}$ are σ-invariant linear combinations of the $X_{-\alpha_j}$.)

It is clear that (5.5) is a $\check{G}_\sigma$-oper connection on $\mathbb{G}_m$.

6. Special cases

If V is any finite-dimensional complex representation of the group $\check{G}$, a connection ∇ on the principal $\check{G}$-bundle gives a connection $\nabla(V)$ on the vector bundle $\mathcal{F}$ associated to V. In our case, this connection is

$$\nabla(V) = d + N(V)\frac{dt}{t} + E(V)dt, \tag{6.1}$$

where $N(V)$ and $E(V)$ are the corresponding nilpotent endomorphisms of V.

In this section, we will provide formulas for the first order matrix differential operator $\nabla(V)_{td/dt}$, for some simple representations V of $\check{G}$. We will be able to convert these matrix differential operator into scalar differential operators, because in these cases $N(V)$ will be represented by a principal nilpotent matrix in $\mathrm{End}(V)$. This will allow us to compare our connection with the scalar differential operators studied by Katz in [Katz2].

Case I. $\check{G} = SL_n$ and V is the standard n-dimensional representation with a basis of vectors $v_i, i = 1, \ldots, n$, on which the torus acts according to the weights e_i. Since $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, n-1$ and $\theta = e_1 - e_n$, we can normalize the $X_{-\alpha_i}$ and E in such a way that

$$N(V) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, \qquad E(V) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{6.2}$$

Therefore the operator $\nabla_{td/dt}$ of the connection (5.1) has the form

$$\nabla_{td/dt} = t\frac{d}{dt} + \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & t \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \tag{6.3}$$

and hence corresponds to the scalar differential operator

$$(td/dt)^n + (-1)^{n+1}t. \tag{6.4}$$

Now let V be the dual of the standard representation. Then, choosing as a basis the dual basis to the basis $\{v_{n+1-i}\}_{i=1,\ldots,n}$ (in the reverse order), we obtain the same matrix differential operator (6.3). Hence the flat vector bundles associated to the standard representation of SL_n and its dual are isomorphic.

Case II. $\check{G} = Sp_{2m}$ and V is the standard $2m$-dimensional representation. We choose the basis $v_i, i = 1, \ldots, 2m$, in which the symplectic form is given by the formula

$$\langle v_i, v_j \rangle = -\langle v_j, v_i \rangle = \delta_{i,2m+1-j}, \qquad i < j.$$

The weights of these vectors are $e_1, e_2, \ldots, e_m, -e_m, \ldots, -e_2, -e_1$. Since $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, m-1$, $\alpha_m = 2e_m$ and $\theta = 2e_1$, we can normalize $N(V)$ and $E(V)$ in such a way that they are given by formulas (6.2).

Hence $\nabla_{td/dt}$ is also given by formula (6.3) in this case. It corresponds to the operator (6.4) with $n = 2m$.

Case III. $\check{G} = SO_{2m+1}$ and V is the standard $(2m+1)$-dimensional representation of SO_{2m+1} with the basis $v_i, i = 1, \ldots, 2m+1$, in which the inner product has the form

$$\langle v_i, v_j \rangle = (-1)^i \delta_{i,2m+2-j}.$$

The weights of these vectors are $e_1, e_2, \ldots, e_m, 0, -e_m, \ldots, -e_2, -e_1$. Since $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, m-1$, $\alpha_m = e_m$ and $\theta = e_1 + e_2$, we can normalize the $X_{-\alpha_i}$ and E in such a way that

$$(6.5)\qquad N(V)=\begin{pmatrix}0&0&0&\dots&0&0\\1&0&0&\dots&0&0\\0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\dots&\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&1&0\end{pmatrix},\qquad E(V)=\begin{pmatrix}0&0&0&\dots&1&0\\0&0&0&\dots&0&1\\0&0&0&\dots&0&0\\0&0&0&\dots&0&0\\\dots&\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&0&0\end{pmatrix}.$$

Therefore

$$(6.6)\qquad \nabla_{td/dt}=t\frac{d}{dt}+\begin{pmatrix}0&0&0&\dots&t&0\\1&0&0&\dots&0&t\\0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\dots&\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&1&0\end{pmatrix}.$$

We can convert this first order matrix differential operator into a scalar differential operator. In order to do this, we need to find a gauge transformation by an upper-triangular matrix which brings it to a canonical form, in which we have 1's below the diagonal and other non-zero entries occur only in the first row. This matrix is uniquely determined by this property and is given by

$$\begin{pmatrix}1&0&0&\dots&t\\0&1&0&\dots&0\\0&0&1&\dots&0\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&1\end{pmatrix}.$$

The resulting matrix operator is

$$t\frac{d}{dt}+\begin{pmatrix}0&0&0&\dots&2t&-t\\1&0&0&\dots&0&0\\0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\dots&\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&1&0\end{pmatrix}$$

which corresponds to the scalar operator

$$(td/dt)^{2m+1}-2t^2d/dt-t.$$

Case IV. $\check{G}=G_2$ and V is the 7-dimensional representation.

The Lie algebra g_2 of G_2 is a subalgebra of so_7. Both the nilpotent elements N and E in so_7 may be simultaneously chosen to lie in this subalgebra, where they are equal to the corresponding elements for g_2. Hence $\nabla_{td/dt}$ is equal to the operator (6.6) with $m=3$, which corresponds to the scalar differential operator

$$(td/dt)^7-2t^2d/dt-t.$$

There is one more case when N is a regular nilpotent element in $\mathrm{End}(V)$; namely, when $\check{G} = SL_2$ and $V = \mathrm{Sym}^n(\mathbb{C}^2)$ is the irreducible representation of dimension $n+1$. We have already considered the cases $n = 1$ and $n = 2$ (the latter corresponds to the standard representation of $\check{G} = SO_3$). These representations, and the cases considered above, are the only cases when an oper may be written as a scalar differential operator.

The scalar differential operators we obtain agree with those constructed by Katz in [Katz2]. More precisely, to obtain his operators in the case of SO_{2m+1} and G_2 we need to rescale t by the formula $t \mapsto -\frac{1}{2}t$.

In the above examples, the connection matrix was the same for Sp_{2n} as it was for SL_{2n}, and was the same for G_2 as it was for SO_7. The same phenomenon will occur for the pairs $SO_{2n+1} < SO_{2n+2}, G_2 < D_4$, and $F_4 < E_6$, for the reason explained in Section 13.

7. De Rham cohomology

In the next sections of this paper, we will calculate the cohomology of the intermediate extension of our local system to $\mathbb{P}^1$, with values in a representation V of $\check{G}$. In particular, we will show that this cohomology vanishes for the adjoint representation $\check{\mathfrak{g}}$, as well as for the small representations tabulated in Section 6. This is further evidence that our connection is the characteristic 0 analogue of the ℓ-adic Langlands parameter. It also implies that our connection is rigid, in the sense that it has no infinitesimal deformations preserving the formal types at 0 and ∞, as such deformations form an affine space over the first de Rham cohomology group [Katz3, BE, A].

We begin with some general remarks on algebraic de Rham cohomology for a principal $\check{G}$-bundle with connection ∇ on the affine curve $U = \mathbb{G}_m$. Any complex representation V of $\check{G}$ then gives rise to a flat vector bundle $\mathcal{F}(V)$ on U, where the connection is $\nabla(V)$.

Since U is affine, with the ring of functions $\mathbb{C}[t, t^{-1}]$, the connection $\nabla(V)$ gives a $\mathbb{C}$-linear map

$$\nabla(V): \quad V[t, t^{-1}] \to V[t, t^{-1}]\frac{dt}{t}. \tag{7.1}$$

Any $\check{G}$-bundle on U may be trivialized. Once we pick a trivialization of our bundle $\mathcal{F}$, we represent the connection as $\nabla = d + A$, where A is a one-form on U with values in the Lie algebra $\check{\mathfrak{g}}$. Let $A(V)$ be the corresponding one-form on U with values in $\mathrm{End}(V)$. We may write

$$A(V) = \sum_n A_n(V)\frac{dt}{t}$$

with $A_n(V) \in \mathrm{End}(V)$. If

$$f(t) = \sum_n v_n t^n,$$

we find that

$$\nabla(V)(f) = \sum_n w_n t^n \frac{dt}{t}$$

has coefficients w_n given by the formula

$$w_n = nv_n + \sum_{a+b=n} A_a(V)(v_b).$$

For our specific connection

$$\nabla = d + N(V)\frac{dt}{t} + Edt$$

we find

$$w_n = nv_n + N(V)(v_n) + E(V)(v_{n-1}).$$

The ordinary de Rham cohomology groups $H^i(U, \mathcal{F}(V))$ are defined as the cohomology of the complex (7.1):

$$H^0(U, \mathcal{F}(V)) = \operatorname{Ker} \nabla(V),$$
$$H^1(U, \mathcal{F}(V)) = \operatorname{Coker} \nabla(V).$$

Thus elements of $H^0(U, \mathcal{F}(V))$ are solutions of the differential equation $\nabla(V)(f) = 0$. For our particular connection, a solution $f = \sum v_n t^n$ corresponds to a solution to the system of linear equations

$$nv_n + N(V)(v_n) + E(V)(v_{n-1}) = 0 \tag{7.2}$$

for all n. For example, if v is in the kernel of both $N(V)$ and $E(V)$, then $f = v$ is a constant solution, with $v_n = 0$ for all $n \neq 0$ and $v_0 = v$.

We can also study the complex (7.1) with functions and one-forms on various subschemes of U. For example, the kernel and cokernel of

$$\nabla(V) : \quad V(\!(t)\!) \to V(\!(t)\!)\frac{dt}{t} \tag{7.3}$$

define the cohomology groups $H^0(D_0^\times, \mathcal{F}(V))$ and $H^1(D_0^\times, \mathcal{F}(V))$, where $D_0^\times$ is the punctured disc at $t = 0$. Likewise, the kernel and cokernel of

$$\nabla(V) : \quad V(\!(t^{-1})\!) \to V(\!(t^{-1})\!)\frac{dt}{t} \tag{7.4}$$

define the cohomology groups $H^0(D_\infty^\times, \mathcal{F}(V))$ and $H^1(D_\infty^\times, \mathcal{F}(V))$, where $D_\infty^\times$ is the punctured disc at $t = \infty$. We will also identify the kernel and cokernel of

$$\nabla(V) : \quad V[[t, t^{-1}]] \to V[[t, t^{-1}]]\frac{dt}{t} \tag{7.5}$$

as cohomology groups with compact support in Section 9.

The flat bundle $\mathcal{F}(V)$ on U defines an algebraic, holonomic, left $\mathcal{D}$-module on U (which has the additional property of being coherent as an $\mathcal{O}$-module). In the next section we will recall the definition of the intermediate extension $j_{!*}\mathcal{F}(V)$ in the category of algebraic, holonomic, left $\mathcal{D}$-modules, where $j : U \hookrightarrow \mathbb{P}^1$ is the inclusion. The de Rham cohomology of $j_{!*}\mathcal{F}(V)$ may be calculated in terms of some Ext groups in this category. We will establish the following result, which is in agreement with the results of N. Katz on the ℓ-adic cohomology with coefficients in the adjoint representation and small representations for the analogous ℓ-adic representations (in those cases in which they have been constructed).

Theorem 1. *Assume that* $\nabla = d + N\frac{dt}{t} + Edt$ *and that* V *is either the adjoint representation* $\check{\mathfrak{g}}$ *of* $\check{G}$ *or one of the small representations tabulated in Section 6. Then*

$$H^i(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = 0$$

for all i.

We will provide two proofs of this result. The first, given in Sections 10 and 11, uses the theory of affine Kac–Moody algebras and the relation between the cohomology of the intermediate extension of $\mathcal{F}(V)$ and solutions of the equation $\nabla(V)(f) = 0$ in various spaces. The second, given in Section 14, uses Deligne's Euler characteristic formula and a calculation of the differential Galois group of ∇. The latter proof gives a formula for the dimensions of $H^i(\mathbb{P}^1, j_{!*}\mathcal{F}(V))$ for any representation V of $\check{G}$.

8. The intermediate extension and its cohomology

Here we follow [Katz2], Section 2.9 and [BE] (see also [A]). Let $j : U \hookrightarrow \mathbb{P}^1$ be the inclusion. We consider the two functors

$$\begin{aligned} j_* &= \text{direct image}, \\ j_! &= \Delta \circ j_* \circ \Delta \end{aligned} \tag{8.1}$$

from the category of left holonomic $\mathcal{D}$-modules on U to the category of left holonomic $\mathcal{D}$-modules on $\mathbb{P}^1$. The functor j_* is right adjoint to the inverse image functor, and $j_!$ is defined using the duality functors Δ on these categories (see, e.g. [GM], Section 5).

We have

$$H^i(\mathbb{P}^1, j_*\mathcal{F}) = H^i(U, \mathcal{F}), \qquad H^i(\mathbb{P}^1, j_!\mathcal{F}) = H^i_c(U, \mathcal{F}).$$

The cohomology groups on $\mathbb{P}^1$ are the Ext groups in the category of holonomic $\mathcal{D}$-modules. The first equality follows from the adjointness property of j_*, and the second can be taken as the definition of the cohomology with compact support. Poincaré duality gives a perfect pairing

$$H^i(U, \mathcal{F}(V)) \times H^{2-i}_c(U, \mathcal{F}(V^*)) \to \mathbb{C},$$

where V^* is the representation of $\check{G}$ that is dual to V. Thus, we have $H^0_c(U, \mathcal{F}(V)) = 0$ and

$$H^i_c(U, \mathcal{F}(V)) \simeq H^{2-i}(U, \mathcal{F}(V^*))^*, \qquad i = 1, 2. \tag{8.2}$$

From the adjointness property of j_*, we obtain a map of $\mathcal{D}$-modules on $\mathbb{P}^1$

$$j_!\mathcal{F} \to j_*\mathcal{F}$$

whose kernel and cokernels are $\mathcal{D}$-modules supported on $\{0, \infty\}$. Let $j_{!*}\mathcal{F}$ be the image of $j_!\mathcal{F}$ in $j_*\mathcal{F}$. We will now show that

$$H^0(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = H^0(U, \mathcal{F}(V)), \tag{8.3}$$

$$H^1(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = \mathrm{Im}\left(H^1_c(U, \mathcal{F}(V)) \to H^1(U, \mathcal{F}(V))\right), \tag{8.4}$$

$$H^2(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = H^2_c(U, \mathcal{F}(V)). \tag{8.5}$$

We will also describe an exact sequence involving the cohomology groups on $D_0^\times$ and $D_\infty^\times$ which allows us to compute $H^1(\mathbb{P}^1, j_{!*}\mathcal{F}(V))$.

Let t_α be a uniformizing parameter at $\alpha = 0, \infty$ (t and t^{-1}, respectively) and let

$$\delta_\alpha = \mathbb{C}((t_\alpha))/C[[t_\alpha]]$$

be the left delta $\mathcal{D}$-module supported at α. We then have an exact sequence of $\mathcal{D}$-modules on $\mathbb{P}^1$

$$0 \to \bigoplus_\alpha H^0(D_\alpha^\times, \mathcal{F}) \otimes \delta_\alpha \to j_! \mathcal{F} \to j_* \mathcal{F} \to \bigoplus_\alpha H^1(D_\alpha^\times, \mathcal{F}) \otimes \delta_\alpha \to 0. \tag{8.6}$$

By the definition of $j_{!*}\mathcal{F}$, this gives two short exact sequences

$$0 \to \bigoplus_\alpha H^0(D_\alpha^\times, \mathcal{F}) \otimes \delta_\alpha \to j_! \mathcal{F} \to j_{!*}\mathcal{F} \to 0$$

$$0 \to j_{!*}\mathcal{F} \to j_*\mathcal{F} \to \bigoplus_\alpha H^1(D_\alpha^\times, \mathcal{F}) \otimes \delta_\alpha \to 0$$

We now take long exact sequence in cohomology and use the fact that

$$H^0(\mathbb{P}^1, \delta_\alpha) = 0,$$
$$H^1(\mathbb{P}^1, \delta_\alpha) = \mathbb{C}.$$

This gives a proof of (8.4)–(8.5), and patching our two long exact sequences along $H^1(\mathbb{P}^1, j_{!*}\mathcal{F})$ gives a six-term exact sequence

$$\begin{aligned} 0 \to H^0(U, \mathcal{F}) \to \bigoplus_\alpha H^0(D_\alpha^\times, \mathcal{F}) \to H_c^1(U, \mathcal{F}) \to \\ \to H^1(U, \mathcal{F}) \to \bigoplus_\alpha H^1(D_\alpha^\times, \mathcal{F}) \to H_c^2(U, \mathcal{F}) \to 0. \end{aligned} \tag{8.7}$$

We will compare it later with an exact sequence obtained from the snake lemma.

From the exact sequence (8.7) we deduce the following condition for the vanishing of cohomology.

Proposition 2. *For a flat vector bundle $\mathcal{F}$ on U, we have $H^i(\mathbb{P}^1, j_{!*}\mathcal{F}) = 0$ for all i if and only if*

(1) $H^0(U, \mathcal{F}) = H^0(U, \mathcal{F}^*) = 0$;
(2) $\dim H^0(D_0^\times) + \dim H^0(D_\infty^\times) = \dim H_c^1(U, \mathcal{F})$.

9. The dual complex

We have seen that the de Rham cohomology of the flat vector bundle $\mathcal{F}(V)$ on U can be calculated from the de Rham complex (7.1). Since compactly supported cohomology of $\mathcal{F}(V)$ is dual to the (ordinary) de Rham cohomology of $\mathcal{F}(V^*)$, it can be calculated from the complex dual to

$$\nabla(V^*) : \quad V^*[t, t^{-1}] \to V^*[t, t^{-1}] \frac{dt}{t}. \tag{9.1}$$

In this section we will identify this dual complex with the complex

$$-\nabla(V) : \quad V[[t, t^{-1}]] \to V[[t, t^{-1}]] \frac{dt}{t} \tag{9.2}$$

by using the residue pairing at $t = 0$, which is described in detail below.

Hence $H^1_c(U, \mathcal{F}(V))$ is identified with the kernel of (9.2) and $H^2_c(U, \mathcal{F}(V))$ with its cokernel. Using this identification, we will compare the six-term exact sequence (8.7) to the one obtained from the snake lemma.

We can also rewrite Proposition 2 in a form that depends only on solutions to $\nabla(f) = 0$.

Corollary 3. *The cohomology of the intermediate extension $j_{!*}\mathcal{F}$ on $\mathbb{P}^1$ vanishes if and only if*

(1) $\operatorname{Ker}\nabla(V) = 0$ *on* $V[t, t^{-1}]$ *and* $\operatorname{Ker}\nabla(V^*) = 0$ *on* $V^*[t, t^{-1}]$*;*

(2) *Every solution $f(t)$ to $\nabla(V)(f) = 0$ in $V[[t, t^{-1}]]$ can be written (uniquely) as a sum $f_0(t) + f_\infty(t)$, with f_0 and f_∞ in* $\operatorname{Ker}\nabla(V)$ *on $V((t))$ and $V((t^{-1}))$, respectively.*

We now turn to the identification of (9.2) with the dual of (9.1). Define a bilinear pairing on $f \in V[[t, t^{-1}]]$ and $\omega \in V^*[t, t^{-1}]\frac{dt}{t}$ by

$$\langle f, \omega\rangle = \operatorname{Res}_{t=0} S(f \cdot \omega), \tag{9.3}$$

where $f \cdot \omega$ is the product in $V \otimes V^*[[t, t^{-1}]]\frac{dt}{t}$ and $S : V \otimes V^* \to \mathbb{C}$ is the natural contraction, so $S(f \cdot \omega)$ is an element of $\mathbb{C}[[t, t^{-1}]]\frac{dt}{t}$. Explicitly, if $f = \sum v_n t^n$ and $\omega = \sum \omega_m t^m \frac{dt}{t}$, then

$$\langle f, \omega\rangle = \sum_{n+m=0} S(v_n \otimes \omega_m).$$

This pairing identifies the direct product vector space $V[[t, t^{-1}]]$ with the dual of the direct sum vector space $V^*[t, t^{-1}]\frac{dt}{t}$. A similar pairing identifies $V^*[[t, t^{-1}]]$ with the dual of the direct sum vector space $V[t, t^{-1}]\frac{dt}{t}$.

To complete the proof that this pairing identifies the dual of (9.1) with (9.2), we must show that the adjoint of $\nabla(V^*)$ is $-\nabla(V)$. Write

$$\nabla = d + \sum_m A_m t^m \frac{dt}{t},$$

with $A_m \in \check{\mathfrak{g}}$. Then, for $g = \sum w_n t^n$ and f as before we have

$$\nabla(V^*)(g) = dg + \sum_{m,n} A_m(V^*)(w_n) t^{m+n}\frac{dt}{t},$$

$$\nabla(V)(f) = df + \sum_{m,n} A_m(V)(v_n) t^{m+n}\frac{dt}{t}.$$

The desired adjoint identity

$$\langle\nabla(V)(f), g\rangle + \langle f, \nabla(V^*)(g)\rangle = 0$$

then follows from the two identities

$$\operatorname{Res}_{t=0}(g \otimes df + f \otimes dg) = 0,$$

$$S(A(V)v, w) + S(v, A(V^*)w) = 0.$$

We end this section with a reconstruction of the six-term exact sequence (8.7). The maps $\alpha(f) = (f, f)$ and $\beta(f, g) = f - g$ give an exact sequence of vector spaces

$$0 \to V[t, t^{-1}] \xrightarrow{\alpha} V(\!(t)\!) \oplus V(\!(t^{-1})\!) \xrightarrow{\beta} V[[t, t^{-1}]] \to 0.$$

Using $\nabla(V)$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V[t,t^{-1}] & \xrightarrow{\alpha} & V(\!(t)\!) \oplus V(\!(t^{-1})\!) & \xrightarrow{\beta} & V[[t,t^{-1}]] & \longrightarrow & 0 \\ & & {\scriptstyle\nabla}\big\downarrow & & {\scriptstyle(\nabla,\nabla)}\big\downarrow & & {\scriptstyle\nabla}\big\downarrow & & \\ 0 & \longrightarrow & V[t,t^{-1}]\frac{dt}{t} & \xrightarrow{\alpha} & V(\!(t)\!)\frac{dt}{t} \oplus V(\!(t^{-1})\!)\frac{dt}{t} & \xrightarrow{\beta} & V[[t,t^{-1}]]\frac{dt}{t} & \longrightarrow & 0 \end{array} \tag{9.4}$$

The 3 kernels and the 3 cokernels have been identified with the cohomology groups in (8.7). Do the morphisms in (8.7) come from an application of the snake lemma to (9.4)? We note that we have made two sign choices: in the pairing (9.3) we took the residue at $t = 0$, not at $t = \infty$ (which would have changed the sign), and in the map β we took $f_0 - f_\infty$, not $f_\infty - f_0$. We expect that with consistent choice of signs the morphisms in (8.7) do indeed come from (9.4) via the snake lemma.

10. The vanishing of adjoint cohomology

We now turn to the proof of Theorem 1, using Corollary 3. Specifically, when V is the adjoint representation of $\check{G}$ or a small representation we will show that any solution $f(t) = \sum v_n t^n$ of $\nabla(V)(f) = 0$ in $V[[t, t^{-1}]]$ satisfies

$$v_n = 0 \qquad \text{for all} \quad n < 0. \tag{10.1}$$

Next, we will use the following lemma.

Lemma 4. *Suppose that any solution $f = \sum v_n t^n \in V[[t, t^{-1}]]$ to $\nabla(V)(f) = 0$ satisfies property* (10.1) *and the same property holds if we replace V by V^*. Then $H^i(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = 0$ for all i.*

Proof. The equation $\nabla(V)(f) = 0$ implies that the components v_n satisfy

$$n v_n + N(V)(v_n) + E(V)(v_{n-1}) = 0. \tag{10.2}$$

If (10.1) is satisfied, then it follows that v_0 lies in the kernel of $N(V)$ on V. This also shows that there is a unique solution

$$f = \sum_{n \geq 0} v_n t^n$$

for any v_0 in the kernel of $N(V)$, because $N(V)$ is nilpotent so that the operator $n\,\mathrm{Id} + N(V)$ is invertible on V for all $n \neq 0$. Clearly then, if $v_0 \neq 0$, this solution has

$$v_n \neq 0 \qquad \text{for all} \quad n \geq 0. \tag{10.3}$$

Hence there cannot be a non-zero solution to $\nabla(V)(f) = 0$ that has finitely many non-zero components for positive powers of t. We obtain that

$$H^0(U, \mathcal{F}(V)) = H^0(D_\infty^\times, \mathcal{F}(V)) = 0$$

and

$$H^0(D_0^\times, \mathcal{F}(V)) \simeq H^1_c(U, \mathcal{F}(V)) \simeq \operatorname{Ker} N(V).$$

Together we the same properties for V replaced by V^*, this implies that all of the criteria of Corollary 3 for the vanishing of cohomology are met. $\square$

We now turn to the proof of the property (10.1) in the case when $V = \check{\mathfrak{g}}$. We will drop V in our notation and write ∇ for $\nabla(V)$, etc. The vector space $\check{\mathfrak{g}}[t, t^{-1}]$ is a $\mathbb{Z}$-graded Lie algebra, with the Lie bracket

$$[xt^n, yt^m] = [x, y]t^{n+m}.$$

There is a $\check{G}$-invariant inner product

$$\kappa : \check{\mathfrak{g}} \otimes \check{\mathfrak{g}} \to \mathbb{C}$$

given by the Killing form (which is a unique such inner product up to scaling). We define an inner product on $\check{\mathfrak{g}}[t, t^{-1}]$ by

$$\left\langle \sum x_n t^n, \sum y_m t^m \right\rangle = \sum_{n+m=0} \kappa(x_n, y_m). \tag{10.4}$$

The $\mathbb{Z}$-grading on $\check{\mathfrak{g}}[t, t^{-1}]$ is given by the differential operator td/dt. If we write a solution $f(t)$ to $\nabla(f) = 0$ in terms of its graded pieces: $f = \sum v_n t^n$, then (10.2) becomes

$$nv_n + [N, v_n] + [E, v_{n-1}] = 0 \tag{10.5}$$

for all n.

To find the solutions to (10.5), it is convenient to switch to a different $\mathbb{Z}$-grading of $\check{\mathfrak{g}}[t, t^{-1}]$ called the principal grading. Let ρ be again half the sum of positive coroots for the Borel subgroup $\check{B}$. Then

$$[\rho, N] = -N, \qquad [\rho, E] = (h-1)E.$$

The operator

$$d = h\, t\frac{d}{dt} - \operatorname{ad}\rho \tag{10.6}$$

has integer eigenvalues on $\check{\mathfrak{g}}[t, t^{-1}]$, and defines the principal grading with respect to which the element

$$p_1 = N + Et \tag{10.7}$$

has degree 1. If we write a solution $f = \sum y_n$ of $\nabla(f) = 0$ in its components for the principal grading, then (10.5) gives rise to the identities

$$ny_n + [\rho, y_n] + h[p_1, y_{n-1}] = 0 \tag{10.8}$$

for all n.

Since the eigenvalues of $\operatorname{ad}\rho$ on $\check{\mathfrak{g}}$ are the integers in the interval $[1-h, h-1]$, we see that the eigenvalues of d on $\check{\mathfrak{g}}t^n$ are the integers of the form $nh+e$, with $1-h \le e \le h-1$. In particular, as eigenvector y_m with eigenvalue m has the form

$$y_m \in \check{\mathfrak{t}}\, t^n,$$

if $m = nh$, where $\check{\mathfrak{t}} \subset \check{\mathfrak{g}}$ is the unique Cartan subalgebra containing ρ (so $\check{\mathfrak{t}}$ is the kernel of $\operatorname{ad}\rho$ on $\check{\mathfrak{g}}$). If $n = nh + e$ with $0 < e < h$, then y_m has the form

$$y_m = at^n + bt^{n+1}$$

with a of degree e and b of degree $(e-h)$ for $-\operatorname{ad}\rho$. From this one deduces that the component of $\check{\mathfrak{g}}[t,t^{-1}]$ of degree m with respect to the principal grading has dimension equal to the rank of $\check{G}$, except when m is congruent to an exponent e of $\check{G}$ (mod h), when the dimension is the rank of $\check{G}$ plus the multiplicity of that exponent. Note that the original grading operator td/dt preserves the components with respect to the principal grading.

V. Kac has studied the decomposition of $\check{\mathfrak{g}}[t,t^{-1}]$ under the action of $\operatorname{ad} p_1$, where $p_1 = N + Et$. His results are summarized in the following proposition. Set

$$\mathfrak{a} = \operatorname{Ker}(\operatorname{ad} p_1), \qquad \mathfrak{c} = \operatorname{Im}(\operatorname{ad} p_1).$$

Proposition 5 ([Kac1], Prop. 3.8)**.**

(1) *The Lie algebra* $\check{\mathfrak{g}}[t,t^{-1}]$ *has an orthogonal decomposition with respect to the inner product* (10.4)*,*

$$\check{\mathfrak{g}}[t,t^{-1}] = \mathfrak{a} \oplus \mathfrak{c}.$$

(2) $\mathfrak{a}$ *is a commutative Lie subalgebra of* $\check{\mathfrak{g}}[t,t^{-1}]$*. With respect to the principal grading,* $\mathfrak{a} = \bigoplus_{i\in I} \mathfrak{a}_i$*, where* I *is the set of all integers equal to the exponents of* $\check{\mathfrak{g}}$ *modulo the Coxeter number* h*, and* $\dim \mathfrak{a}_i$ *is equal to the multiplicity of the exponent* $i \bmod h$*.*

(3) *With respect to the principal grading,* $\mathfrak{c} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{c}_j$*, where* $\dim \mathfrak{c}_j = \operatorname{rk}(\check{\mathfrak{g}})$*, and the map* $\operatorname{ad} p_1 : \mathfrak{c}_j \to \mathfrak{c}_{j+1}$ *is an isomorphism for all* $j \in \mathbb{Z}$*.*

Let f be a solution to $\nabla(f) = 0$ and $f = \sum y_n$ its decomposition with respect to the principal grading. Then the components satisfy (10.8). We now have the following crucial lemma.

Lemma 6. *Suppose that the* y_n *satisfy the equations* (10.8) *and* $y_n \in \mathfrak{a}_n$ *for some* n*. Then* $y_m = 0$ *for all* $m \leq n$*.*

Proof. Applying (10.8), we obtain

$$t\frac{d}{dt}y_n = \frac{m}{h}y_n + \frac{1}{h}[\rho, y_n] \in \mathfrak{c}_n, \tag{10.9}$$

where $\mathfrak{c}_n$ is the degree n homogeneous component of $\mathfrak{c} = \operatorname{Im}(\operatorname{ad} p_1)$. Let us show that this is impossible to satisfy if $y_n \in \mathfrak{a}_n$ and $y_n \neq 0$.

Consider the affine Kac–Moody algebra $\widehat{\check{\mathfrak{g}}}$, which is the universal central extension $\widehat{\check{\mathfrak{g}}}$ of $\check{\mathfrak{g}}[t,t^{-1}]$ by one-dimensional center spanned by an element $\mathbf{1}$,

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\check{\mathfrak{g}}} \to \check{\mathfrak{g}}[t,t^{-1}] \to 0$$

The commutation relations in $\widehat{\check{\mathfrak{g}}}$ read

$$[At^n, Bt^m] = [A,B]t^{n+m} + n\kappa(A,B)\delta_{n,-m}\mathbf{1}.$$

According to [Kac2], Lemma 14.4, the inverse image of $\mathfrak{a}$ in $\widehat{\check{\mathfrak{g}}}$ is a (non-degenerate) Heisenberg Lie subalgebra $\mathfrak{a} \oplus \mathbb{C}\mathbf{1}$. Hence there exists $z \in \mathfrak{a}_{-n}$ such that

$$[y_n, z] \neq 0 \quad \text{in } \mathbb{C}\mathbf{1} \subset \widehat{\check{\mathfrak{g}}}.$$

Write, for $n = kh + e$ (where e is an exponent of $\check{\mathfrak{g}}$),

$$y_n = at^k + bt^{k+1}, \qquad z = a't^{-k} + b't^{-k-1},$$

as above, where a, b, b, b' are homogeneous elements of $\check{\mathfrak{g}}$ with respect to the grading defined by $-\operatorname{ad}\rho$ of degrees $e, e-h, -e, h-e$, respectively. Then we find that

$$[y_n, z] = (k\, \kappa(a, a') + (k+1)\, \kappa(b, b'))\mathbf{1} \neq 0.$$

But

$$k\, \kappa(a, a') + (k+1)\, \kappa(b, b') = \langle t\frac{d}{dt} \cdot y_n, z\rangle,$$

where in the right hand side we use the inner product defined by formula (10.4). This contradicts the condition that $t\frac{d}{dt} \cdot y_n \in \mathfrak{c}_n$, which is orthogonal to $\mathfrak{a}_{-n}$. Hence $y_n = 0$.

Now, equation (10.8) shows that if $y_n = 0$, then $y_{n-1} \in \mathfrak{a}_{n-1}$. Hence we find by induction that $y_m = 0$ for all $m \leq n$. □

Setting $n = 0$ in (10.8), we obtain

$$[\rho, y_0] + [p_1, y_{-1}] = 0.$$

Since $\check{\mathfrak{g}}[t, t^{-1}]_0 = \check{\mathfrak{t}} = \operatorname{Ker}(\operatorname{ad}\rho)$, this shows that

$$[p_1, y_{-1}] = 0.$$

Hence $y_{-1} \in \mathfrak{a}_{-1}$. Lemma 7 then implies that $y_n = 0$ for all $n < 0$. Thus, any solution $f \in \check{\mathfrak{g}}[[t, t^{-1}]]$ to $\nabla(f) = 0$ has the form $f = \sum_{n\geq 0} y_n$, and in particular belongs to $\check{\mathfrak{g}}[[t]]$. Writing this solution as $f = \sum v_n t^n$, we obtain that it satisfies property (10.1). Theorem 1 for the adjoint representation now follows from Lemma 4 because $\check{\mathfrak{g}}^* = \check{\mathfrak{g}}$.

11. Vanishing for small representations

Let V be one of the small representations considered in Section 6; that is, n-dimensional representation of SL_n, $2n$-dimensional representation of Sp_{2n}, $(2n+1)$-dimensional representation of SO_{2n+1}, or 7-dimensional representation of G_2. We denote the corresponding flat bundle on U by $\mathcal{F}(V)$. Since our connection for Sp_{2n} and G_2 is the same as for SL_{2n} and SO_7, respectively, it is sufficient to consider only the cases of SL_n and SO_{2n+1}.

We will now prove Theorem 1 when V is one of these representations. We will follow the argument used in the proof of Theorem 1 for the adjoint representation. Define a $\mathbb{Z}$-grading on $V[t, t^{-1}]$ compatible with the principal $\mathbb{Z}$-grading on $\check{\mathfrak{g}}[t, t^{-1}]$ defined by the operator d. The representation V has a basis $v_1, \ldots, v_p$ in which N appears as a lower Jordan block. We set

$$\deg v_i t^k = i - 1 + kh.$$

All graded components are one-dimensional in the case of SL_n. For SO_{2n+1} the components of degrees $kh, k \in \mathbb{Z}$, are two-dimensional. Components of all other degrees are one-dimensional. It is easy to see that the operator td/dt preserves the graded components.

We will use the operators $N(V)$ and $E(V)$ from Section 6. Denote again $N(V) + E(V)t$ by p_1. Let $f \in V[[t, t^{-1}]]$ be a solution to $\nabla(V)(f) = 0$. Decomposing $f = \sum y_r$ with respect to the above grading, we obtain the following system:

$$t\frac{d}{dt} y_r = -p_1 \cdot y_{r-1}, \qquad r \in \mathbb{Z}. \tag{11.1}$$

The role of Lemma 6 is now played by the following result.

Lemma 7. *Suppose that the y_r satisfy the equations* (11.1) *and $p_1 \cdot y_r = 0$ for some r. Then $y_m = 0$ for all $m \leq r$.*

Proof. In the case of SL_n we have $\operatorname{Ker} p_1 = 0$, so if $p_1 \cdot y_r = 0$, then $y_r = 0$. For SO_{2n+1}, $\operatorname{Ker} p_1$ is spanned by the vectors $v_1 t^k - v_{2n+1} t^{k-1}, k \in \mathbb{Z}$. Hence y_r is one of these vectors. Equation (11.1) tells us that

$$t\frac{d}{dt}(v_1 t^k - v_{2n+1} t^{k-1}) = k v_1 t^k - (k-1) v_{2n+1} t^{k-1}$$

is in the image of p_1. But the image of p_1 in this graded component is spanned by the vector $v_1 t^k + v_{2n+1} t^{k-1}$. Since $k \in \mathbb{Z}$, it is impossible to satisfy this condition. Therefore $y_r = 0$.

Now, if $y_r = 0$, then $p_1 \cdot y_{r-1} = 0$, by (11.1). Hence we obtain by induction that $y_m = 0$ for all $m \leq r$. □

Now observe that td/dt annihilates the component of $V[t, t^{-1}]$ of degree 1 (in fact, all components of degrees $1, \ldots, h-1$). Hence we find from (11.1) with $r = 1$ that $p_1 \cdot y_0 = 0$. Therefore it follows from Lemma 6 that $y_m = 0$ for all $m \leq 0$. Hence any solution $f = \sum f_n t^n$ to $\nabla(V)(f) = 0$ is a formal power series in t, that is, $f_n = 0$ for all $n < 0$. Theorem 1 for small representations now follows from Lemma 4 and the fact that $V^* = V$ in the case of SO_{2n+1}, and in the case of SL_n the flat bundles associated to V and V^* are isomorphic as well (see Case I in Section 6).

12. The case of SL_2

In the previous two sections we have shown that the de Rham cohomology of our connection vanishes in the adjoint representation as well as the small representations. This is because solutions to the equation $\nabla(V)(f) = 0$ enjoy special properties which do not hold for a general representation V. The key property of $\check{\mathfrak{g}}$ and the small representations that we have used is the fact that the weights of the torus of a principal SL_2 on V are all small (that is, the eigenvalues of $\operatorname{ad} \rho$ on V have absolute value less than h).

But for most other representations this is not the case and the de Rham cohomology does not vanish. As an example, we consider in this section the faithful irreducible representations $V = \operatorname{Sym}^{2k-1}$ of SL_2 of even dimension $2k = n \geq 2$. We will see that there are $k = n/2$ solutions in the space of formal power series in t and t^{-1} with coefficients in V, but that only the zero solution lies in $V((t^{-1}))$, and only one line of solutions lies in $V((t))$. Hence $H^1(\mathbb{P}^1, j_{!*}\mathcal{F}(V))$ has dimension $k - 1 = (n/2) - 1$.

We will see how to compute the dimensions of the de Rham cohomology of $j_{!*}\mathcal{F}(V)$ on $\mathbb{P}^1$, for any representation V of $\check{G}$, in Section 13.

In the case of SL_2 our connection looks as follows:

$$\nabla = d + F\frac{dt}{t} + E dt,$$

where $F = X_{-\alpha_1}$ and E are the standard generators of $\mathfrak{sl}_2$. Let $J = \operatorname{diag}[1, 2, \ldots, n]$. Make a change of variables $t = z^2$ and apply gauge transformation by z^J. We then obtain

$$\nabla = d - J\frac{dz}{z} + 2(E+F)dz.$$

Let V_{even} (resp., V_{odd}) be the subspace of V on which the eigenvalues of J are even (resp., odd). The de Rham complex on $U = \mathbb{G}_m$,

$$V[t,t^{-1}] \xrightarrow{\nabla} V[t,t^{-1}]dt,$$

is identified with the subcomplex

$$V_{\text{even}}[z^2,z^{-2}] \oplus V_{\text{odd}}[z^2,z^{-2}]z \xrightarrow{\nabla} V_{\text{even}}[z^2,z^{-2}]\frac{dz}{z} \oplus V_{\text{odd}}[z^2,z^{-2}]dz \tag{12.1}$$

of the de Rham complex

$$V[z,z^{-1}] \xrightarrow{\nabla} V[z,z^{-1}]dz \tag{12.2}$$

on the double cover of U. Introduce a $\mathbb{Z}$-grading on this complex by setting $\deg vz^k = k$.

Note that the operator $E+F$ is conjugate to $2\rho = \text{diag}[n-1, n-3, \ldots, -n+1]$. From now on we assume that n is even. Then the operator $E+F$ is invertible. Suppose that $y(z) = \sum y_m z^m$ is in the kernel of (12.2). Then we obtain the following system of equations:

$$(m\,\text{Id} - J)\cdot y_m = -2(E+F)\cdot y_{m-1} \tag{12.3}$$

on its homogeneous components. Let m be the largest integer such that $y_m = 0$ but $y_{m-1} \neq 0$. Then we should have $(E+F)\cdot y_{m-1} = 0$, which is impossible. This shows that $H^0(\mathbb{P}^1, j_{!*}(\mathcal{F}(V_n))) = 0$. Using duality, we obtain that $H^2(\mathbb{P}^1, j_{!*}(\mathcal{F}(V_n))) = 0$.

In order to compute $H^1(\mathbb{P}^1, j_{!*}(\mathcal{F}(V_n))) = 0$, we first compute the kernel of

$$V[[z,z^{-1}]] \xrightarrow{\nabla} V[[z,z^{-1}]]dz. \tag{12.4}$$

Note that the operator $m\,\text{Id} - J$ is invertible for all $m \neq 1, \ldots, n$. Let us choose any $y_n \in V_n$. Then we can find $y_m, m > n$, by using equation (12.3) and inverting $m\,\text{Id} - J$, and we can find $y_m, m < n$, by using (12.3) and inverting $E+F$. Thus, the kernel of (12.4) is isomorphic to V_n. An element of this kernel belongs to

$$V_{\text{even}}[[z^2,z^{-2}]] \oplus V_{\text{odd}}[[z^2,z^{-2}]]z$$

if and only if $y_n \in V_{\text{even}}$. Thus, we obtain that $H^1(U, \mathcal{F}(V_n))^*$ is isomorphic to V_{even}. Using the exact sequence (8.7), we obtain that $H^1(\mathbb{P}^1, j_{!*}(\mathcal{F}(V_n)))^*$ is the quotient of the above space of solutions of (12.3) by the subspace of those solutions for which $y_m = 0$ for $m \gg 0$ or $m \ll 0$.

Those are precisely the solutions for which there exists $m = 1, \ldots, n$ such that $y_{m-1} = 0$, but $y_m \neq 0$. We claim that there are no such solutions for $m \neq n$. Indeed, denote by $v_i, i = 1, \ldots, n$, an eigenvector of J with eigenvalue i. If $y_{m-1} = 0$, but $y_m \neq 0$, then y_m is a multiple of v_m. But $-2(E+F)(v_m)$ contains v_{m+1} with non-zero coefficient if $m < n$. Hence $-2(E+F)(v_m)$ cannot be in the image of $(m+1)\,\text{Id} - J$, and so (12.3) cannot be satisfied. If, on the other hand, $y_n = v_n$, then $y_m = 0$ for all $m < n$.

Thus, we obtain that there is a unique solution (up to a scalar) for which $y_m = 0$ for $m \gg 0$ or $m \ll 0$. Hence $\dim H^1(\mathbb{P}^1, j_{!*}(\mathcal{F}(V_n))) = (n/2) - 1$, which is non-zero if $n \geq 4$.

13. The differential Galois group

In this section, we determine the differential Galois group of our rigid irregular connection ∇ on $\mathbb{G}_m$, as well as its inertia subgroups (up to conjugacy) at $t = 0$ and $t = \infty$.

We first review some of the general theory, which is due to N. Katz [Katz1]. Fix the point $t = 1$ on $\mathbb{G}_m$. Then the fiber at this point gives a fiber functor from the category of flat complex algebraic vector bundles $(\mathcal{F}, \nabla)$ on $\mathbb{G}_m$ to the category of finite-dimensional vector spaces. The automorphism group of this fiber functor is, by definition, the differential Galois group of $\mathbb{G}_m$. This is a pro-algebraic group over $\mathbb{C}$, which we denote by $DG(\mathbb{G}_m)$; Katz calls this group $\pi_1^{\mathrm{diff}}(\mathbb{G}_m, 1)$.

Since the fiber functor preserves tensor products, the fundamental theorem of [DM], Ch. 2, gives an equivalence between the category of flat bundles on $\mathbb{G}_m$ with the category of finite-dimensional representations of the differential Galois group $DG(\mathbb{G}_m)$. If

$$\varphi : DG(\mathbb{G}_m) \to \mathrm{GL}(V)$$

corresponds to the flat bundle $(\mathcal{F}, \nabla)$, then

$$H^0(\mathbb{G}_m, \mathcal{F}) = \mathrm{Ker}\, \nabla = V^{DG(\mathbb{G}_m)}.$$

Under this equivalence, a principal $\check{G}$-bundle with connection ∇ on $\mathbb{G}_m$ defines a homomorphism

$$\varphi_\nabla : DG(\mathbb{G}_m) \to \check{G}$$

up to conjugacy. The image $\check{G}_\nabla$ is an algebraic subgroup of $\check{G}$, which we call the differential Galois group of ∇.

At the two points $t = 0$ and $t = \infty$ on $\mathbb{P}^1 - \mathbb{G}_m$, we have local inertia groups I_0 and I_∞ in $DG(\mathbb{G}_m)$, well-defined up to conjugacy. Each inertia group $I = I_\alpha$ is filtered by normal subgroups

$$P^{x+} \subset P^x \subset P \subset I$$

for rational $x > 0$ (called the slopes). The wild inertia subgroup P is a pro-torus over $\mathbb{C}$. As a pro-algebraic group over $\mathbb{C}$, I/P is isomorphic to the product of $\mathbb{G}_a$ with a pro-group A of multiplicative type, with character group $\mathbb{C}/\mathbb{Z}$. The additive part comes from local systems with regular singularity at α with unipotent monodromy, and the rest from the one-dimensional local systems $d - a\, dt_\alpha/t_\alpha$ with solutions t_α^a near $t_\alpha = 0$, with $a \in \mathbb{C}/\mathbb{Z}$. The tame monodromy is then $e^{2\pi i a} \in \mathbb{C}^\times$. The torsion in the character group of A is $\mathbb{Q}/\mathbb{Z}$, so the component group of A is the dual group $\widehat{\mathbb{Z}}(1) = \varprojlim \mu_n$, which is the profinite Galois group of $\mathbb{C}(\!(t_\alpha)\!)$.

The quotient group I/P acts on the pro-torus P by conjugation. Its connected component centralizes P; only the component group $\widehat{\mathbb{Z}}(1)$ acts non-trivially. On the character group $\mathbb{C}$ of the quotient torus P^x/P^{x+}, the element $a = 2\pi i m$ acts by multiplication by the root of unity e^{-ax}. Hence the component group acts through its finite quotient $\widehat{\mathbb{Z}}(1)/n\widehat{\mathbb{Z}}(1) = \mu_n$, where $nx \equiv 0 \pmod{\mathbb{Z}}$.

Now let

$$\nabla = d + N\frac{dt}{t} + Edt$$

be the connection introduced in Section 5. We will determine the differential Galois group $\check{G}_\nabla$ of $\check{G}$ by studying the images $\varphi_\nabla(I_0)$ and $\varphi(I_\infty)$ in $\check{G}$, and we begin with a discussion of the two homomorphisms

$$\begin{aligned} \varphi_\nabla : & I_0 \to \check{G}, \\ \varphi_\nabla : & I_\infty \to \check{G} \end{aligned}$$

up to conjugacy.

Since ∇ has a regular singular point at $t = 0$, the restriction of φ_∇ to I_0 is trivial on the wild inertia subgroup P_0. The resulting homomorphism is given by the analytic monodromy, which maps $a = 2\pi i n$ in $Z(1)$ to $\exp(-aN)$ in $\check{G}$. In particular, the image $\varphi_\nabla(I_0)$ is an additive subgroup $\mathbb{G}_a = \exp(zN)$ of $\check{G}$, containing the principal unipotent element $u = \exp(-2\pi i N)$.

To determine the image of the inertia group at $t = \infty$, we first make the assumption that ρ is a co-character of $\check{G}$. Let $u^h = s = t^{-1}$. Then by our earlier results in Section 5, over the extension $\mathbb{C}(\!(u)\!)$ of $\mathbb{C}(\!(s)\!)$ our connection is equivalent to

$$d - h(N+E)\frac{du}{u^2} - \rho\frac{du}{u}.$$

By a fundamental result of Kostant [Kos], the element $(N+E)$ is regular and semi-simple. Hence the highest order polar term of our connection over $\mathbb{C}((u))$ is diagonalizable, in any representation V of $\check{G}$. It follows that the slopes of this connection over $\mathbb{C}(\!(u)\!)$, as defined by Deligne [D2], Theorem 1.12, are either 0 or 1, the former occurring at the zero eigenspaces for $N+E$ on V and the latter occurring at the non-zero eigenspaces. Since Katz has shown [Katz1], Sections 1 and 2.2.11.2, that the slopes over the extension $\mathbb{C}(\!(u)\!)$ are h times the slopes over the original completion $\mathbb{C}(\!(s)\!)$, we see that the original slopes are either 0 or $1/h$. In particular, I_∞ is trivial on the subgroup $P^{1/h+}$. A similar argument works when ρ is not a co-character, using the extension of degree $2h$.

The image $\check{S} := \varphi(P_\infty)$ of wild inertia subgroup P_∞ is the smallest torus in $\check{G}$, whose Lie algebra contains the regular, semi-simple element $N+E$. The irregularity $\mathrm{Irr}_\alpha(V)$ of a representation V of $\check{G}$ at an irregular singular point α was defined by Deligne [D2], p. 110, and shown by Katz [Katz1], Sections 1 and 2.3, to be the sum of the slopes. From the above analysis, we deduce that

$$h\,\mathrm{Irr}_\infty(V) = \dim V - \dim V^{\check{S}}.$$

The full image $\check{H} = \varphi(I_\infty)$ normalizes $\check{S}$, and the quotient is generated by the element $n = (2\rho)(e^{\pi i/h})$. The element n is regular and semi-simple in $\check{G}$. Further, $\epsilon := n^h$ is a central involution in $\check{G}$ which is equal to identity if and only if ρ is a co-character of $\check{G}$. The element n acts on $N+E$ by multiplication by $e^{-2\pi i/h}$. The irreducibility of the

h^{th} cyclotomic polynomial over $\mathbb{Z}$ shows that $\check{S}$ has dimension $\phi(h) = \#(\mathbb{Z}/h\mathbb{Z})^\times$ and that the eigenvalues of n on $\mathrm{Lie}(\check{S})$ are the primitive h^{th} roots of unity.

Since n normalizes $\check{S}$, it also normalizes the centralizer $\check{T}'$ of $\check{S}$ in $\check{G}$, which is a maximal torus (note that it is different from the maximal torus $\check{T}$ considered in Section 5). The image w of n in $N(\check{T}')/\check{T}'$ is a Coxeter class in the Weyl group, and has order h.

The character group $X^*(\check{S})$ is the free quotient of $X^*(\check{T}')$ where w acts by primitive h^{th} roots of unity. Thus the characters of $\check{T}'$ which restrict to the trivial character of $\check{S}$ are generated by those $\lambda : \check{T}' \to \mathbb{G}_m$ where the $\langle w \rangle$-orbit of λ has size less than h.

This completes the description of the local inertia groups, and we now turn to the global differential Galois group $\check{G}_\nabla$.

Proposition 8. *Let $\check{G}_0 \subset \check{G}$ be an algebraic subgroup which contains $\varphi(I_\infty) = \check{H}$ and $\varphi(I_0) = \exp(zN)$. Then $\check{G}_0$ is reductive and contains the image of a principal SL_2 in $\check{G}$.*

Proof. Let $R(\check{\mathfrak{g}}_0)$ be the unipotent radical of $\check{G}_0$, and let $Z = \mathrm{Lie}(Z)$ be its center. We will show that $Z = 0$.

The group $\check{H} = \varphi(I_\infty)$ acts on $R(\check{G}_0)$ and Z. Since $\check{S}$ contains regular elements, every root $\alpha : \check{T}' \to \mathbb{G}_m$ restricts to a non-trivial character of $\check{S}$. Hence the action of $\check{H} = \langle \check{S}, n \rangle$ on $\check{\mathfrak{g}}$ decomposes as

$$\mathrm{Lie}(\check{T}') \oplus \bigoplus_{i=1}^{\mathrm{rk}(\check{\mathfrak{g}})} W_i,$$

where the W_i are irreducible representations of dimension h whose restriction to $\check{S}$ contains an entire w-orbit of roots.

Since Z is nilpotent and $\mathrm{Lie}(\check{T}')$ is semi-simple, Z must be the sum of certain W_i. But each W_i contains a non-zero vector $v_0 = \sum_{i=1}^h n^i(v)$ fixed by $\langle n \rangle$ (as $n^h = \epsilon$ is central and acts trivially on $\check{\mathfrak{g}}$). Since n is regular and semi-simple, v_0 is a semi-simple element and hence cannot be contained in Z. Therefore $Z = 0$.

Since $\check{G}_0$ is reductive and contains the principal unipotent element u, it contains a principal SL_2 [C]. We note that a principal embedding $SL_2 \to \check{G}$ maps the central element $-1 \in SL_2$ to the element $\epsilon \in \check{H} \subset \check{G}$. $\square$

Proposition 8 is a serious constraint on the global image $\check{G}_\nabla$, as the reductive subgroups of $\check{G}$ containing a principal SL_2 are severely limited by a result of [SS] which goes back to the work of Dynkin [Dy]. They are all simple, and their Lie algebras appear in one of the following maximal chains:

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{sp}_{2n} \longrightarrow \mathfrak{sl}_{2n}$$

$$\begin{tikzcd} & & \mathfrak{sl}_{2n+1} \\ \mathfrak{sl}_2 \arrow[r] & \mathfrak{so}_{2n+1} \arrow[ur] \arrow[dr] & \\ & & \mathfrak{so}_{2n+2} \end{tikzcd}$$

$$\begin{tikzcd} & & & \mathfrak{sl}_7 \\ \mathfrak{sl}_2 \arrow[r] & \mathfrak{g}_2 \arrow[r] & \mathfrak{so}_7 \arrow[ur] \arrow[dr] & \\ & & & \mathfrak{so}_8 \end{tikzcd}$$

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{f}_4 \longrightarrow \mathfrak{e}_6$$

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{e}_7$$

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{e}_8$$

In some of these cases, the subgroup $\check{G}_0$ cannot contain $\check{H}$, as its Coxeter number is less than the Coxeter number of $\check{G}$. Looking at the minimal cases remaining, we obtain

Corollary 9. *If $\check{G}$ is simple of type $A_{2n}, n \geq 1, C_n, n \geq 1, B_n, n \geq 4$, G_2, F_4, E_7, E_8, then $\check{G}_\nabla = \check{G}$.*

Indeed, by the above list of embeddings and Proposition 8, any $\check{G}_0 \subset \check{G}$ containing $\check{H}$ and $\langle n \rangle$ must be equal to $\check{G}$.

For the remaining cases, observe that the automorphism group Σ of the pinning of $\check{G}$ is known to be isomorphic to the outer automorphism group of $\check{G}$. This finite group fixes N and acts on the highest root space. If $\check{G}$ is not of type A_{2n}, it also fixes E. Hence Σ fixes the connection ∇, and its differential Galois group $\check{G}_\nabla$ is contained in $\check{G}^\Sigma$. Thus Corollary 9 gives the differential Galois group in all cases.

Corollary 10. *If $\check{G}$ is of type A_{2n-1}, then $\check{G}_\nabla$ is of type C_n, with center the kernel of the center of $\check{G}$ on the second exterior power representation.*

If $\check{G}$ is of type D_{2n+1} with $n \geq 4$, then $\check{G}_\nabla$ is of type B_n, with the center the kernel of the center of $\check{G}$ on the standard representation.

If $\check{G}$ is of type D_4 or B_3, then $\check{G}_\nabla$ is of type G_2.

If $\check{G}$ is of type E_6, then $\check{G}_\nabla$ is of type F_4.

14. The dimension of cohomology

We now use the calculation of the differential Galois group $\check{G}_\nabla$ of our connection, with its inertia subgroups at 0 and ∞, to determine the dimensions of the cohomology groups of the $\mathcal{D}$-modules $j_*\mathcal{F}(V)$, $j_!\mathcal{F}(V)$ and $j_{!*}\mathcal{F}(V)$ on $\mathbb{P}^1$ associated to a representation V of $\check{G}$.

We will assume that we are in one of the cases described in Corollary 9. Otherwise, the computation of cohomology reduces to one in a smaller group given by Corollary 9. We will also assume that V is irreducible non-trivial representation of $\check{G}$, so

$$H^0(\mathbb{P}^1, j_*\mathcal{F}(V)) = H^0(U, \mathcal{F}(V)) = V^{\check{G}} = 0,$$
$$H^2(\mathbb{P}^1, j_!\mathcal{F}(V)) = H^2_c(U, \mathcal{F}(V)) = \mathrm{Hom}_{\check{G}}(V^*, \mathbb{C}) = 0.$$

We will use Deligne's formula [D2], Section 6.21.1, for the Euler characteristic

$$\chi(H^\bullet(U, \mathcal{F})) = \chi(H^\bullet_c(U, \mathcal{F})) = \chi(U)\,\mathrm{rank}(\mathcal{F}) - \sum_\alpha \mathrm{Irr}_\alpha(\mathcal{F}).$$

In our case, $\chi(U) = 0$ and ∇ is regular at $\alpha = 0$. Hence

$$\dim H^1(\mathbb{P}^1, j_*\mathcal{F}) = \dim H^1(\mathbb{P}^1, j_!\mathcal{F}) = \mathrm{Irr}_\infty(\mathcal{F}).$$

The kernel of the map $H^1_c(U, \mathcal{F}) \to H^1(U, \mathcal{F})$ is isomorphic to the direct sum

$$H^0(D_0^\times, \mathcal{F}) \oplus H^0(D_\infty^\times, \mathcal{F}) = V^{I_0} \oplus V^{I_\infty}$$

by (8.7). Hence we obtain

Proposition 11. *If $\check{G}_\nabla = \check{G}$ and V is an irreducible, non-trivial representation of $\check{G}$ with associated flat vector bundle $\mathcal{F}(V)$ on $\mathbb{G}_m$, then*

$$H^0(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = H^2(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = 0,$$

$$d(V) := \dim H^1(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = \mathrm{Irr}_\infty(V) - \dim V^{I_0} - \dim V^{I_0}. \tag{14.1}$$

If we don't assume that $\check{G}_\nabla = \check{G}$ or that V is an irreducible, non-trivial representation of $\check{G}$, we obtain the formulas

$$\dim H^0(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = \dim H^2(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = \dim V^{\check{G}_\nabla},$$
$$\dim H^1(\mathbb{P}^1, j_{!*}\mathcal{F}(V)) = \mathrm{Irr}_\infty(V) - \dim V^{I_0} - \dim V^{I_0} + 2\dim V^{\check{G}_\nabla}.$$

Since

$$h\,\mathrm{Irr}_\infty(V) = \dim V - \dim V^{\check{S}},$$

as we have seen above, this allows us to compute the dimensions of the cohomology of the middle extension for any representation V of $\check{G}$, provided that we know the restriction of V to the three subgroups $\check{S}$, $\check{H}$ and $\check{G}_\nabla$, and the restriction of V to a principal SL_2. We will now make this more explicit.

The irreducible representations of SL_2 all have the form $\mathrm{Sym}^k, k \geq 0$. Hence we may write the restriction of V to the principal SL_2 as

$$V = \bigoplus_{k\geq 0} (\mathrm{Sym}^k)^{\oplus m(k)}$$

with multiplicities $m(k) \geq 0$. We then have

$$\dim V^{I_0} = \sum_{k \geq 0} m(k) \tag{14.2}$$

as $\varphi(I_0) = \exp(tN)$ fixes a unique line on each irreducible factor. We note that the parity of these k is determined by V: if $m(k) > 0$ we have

$$(-1)^k = \epsilon|_V.$$

The irreducible complex representations of $\check{H}$ have dimensions either h or 1. The irreducible W of dimension h restrict to a sum of h distinct non-trivial characters λ of $\check{S}$, in a single $\langle n \rangle$-orbit. The irreducible χ of dimension 1 are the representations trivial on $\check{S}$, and determined by $\chi(n)$, which lies in μ_{2h}. We may therefore write

$$V = \bigoplus_i \chi_i^{\oplus m(\chi_i)} \oplus \bigoplus_j W_j^{\oplus m(W_j)}. \tag{14.3}$$

If $m_i > 0$, then

$$\chi_i(n)^h = \epsilon|_V.$$

In terms of the decomposition (14.3), we have

$$\dim V^{I_\infty} = m(\chi_0), \tag{14.4}$$

where χ_0 is the trivial character, $\chi_0(n) = 1$. Each one-dimensional representation of $\check{H}$ is tame, and each irreducible h-dimensional representation has irregularity $1 = h \cdot (1/h)$. Hence

$$\begin{aligned} \mathrm{Irr}_\infty(V) &= \sum_j m(W_j) \\ &= \frac{1}{h} \cdot \#\{\text{non-trivial weight spaces for } \check{S} \text{ on } V\}. \end{aligned} \tag{14.5}$$

If we know the restriction of V to a principal SL_2 and to $\check{H}$, then formulas (14.1), (14.2), (14.4), and (14.5) allow us to determine the dimension $d(V)$ of $H^1(\mathbb{P}^1, j_{!*}\mathcal{F}(V))$. For example, when $V = \check{\mathfrak{g}}$ is the adjoint representation, we have $\epsilon = +1$ on V and

$$\begin{aligned} V &= \bigoplus_{i=1}^{\mathrm{rk}(\check{\mathfrak{g}})} \mathrm{Sym}^{2d_i - 2}, \\ &= \bigoplus_{i=1}^{\mathrm{rk}(\check{\mathfrak{g}})} \chi_i \oplus \bigoplus_{j=1}^{\mathrm{rk}(\check{\mathfrak{g}})} W_j, \end{aligned}$$

where the d_i are the degrees of invariant polynomials and $\chi_i \neq 1$ for all i. Hence $\mathrm{Irr}_\infty = \dim V^{I_0} = \mathrm{rk}(\check{\mathfrak{g}})$, $V^{I_\infty} = 0$, and so $d(V) = 0$. This gives the second proof of the rigidity of our connection (Theorem 1), for the groups $\check{G}$ in Corollary 9. (When $\check{G}_\nabla$ is a proper subgroup of $\check{G}$, we find that $\check{\mathfrak{g}} = \check{\mathfrak{g}}_\nabla \oplus V'$, where the representation V' of $\check{G}_\nabla$ also has $d(V') = 0$.)

For example, the adjoint representation $\mathfrak{e}_6$ of $\check{G} = E_6$ decomposes as a sum of two irreducible representations $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus V'$ for the differential Galois group $\check{G}_\nabla = F_4$, with $\dim V' = 26$. The restriction of V' to the principal SL_2 is the sum $\mathrm{Sym}^{16} \oplus \mathrm{Sym}^8$. The restriction of V' to $\check{H}$ is the sum $\chi_1 \oplus \chi_2 \oplus W_1 \oplus W_2$, with the χ_i non-trivial of order 3. Hence $\mathrm{Irr}_\infty(V') = \dim V'^{I_0} = 2, \dim V'^{I_\infty} = 0$, and $d(V') = 0$.

Similarly, for the non-trivial irreducible representations $V = \mathrm{Sym}^n$ of $\check{G} = SL_2$, we find that $d(V) = (n-1)/2$ when n is odd, $d(V) = (n-2)/2$ when n is congruent to 2 (mod 4), and $d(V) = (n-4)/2$ when n is divisible by 4, in agreement with the results of Section 12. In particular, $d(V) > 0$ whenever $n > 4$. For the spin representation V of dimension 2^n of $\check{G} = Spin_{2n+1}$, we find that $d(V) > 0$ for all $n > 7$.

We now investigate the difference

$$\begin{aligned} d(V) &= \mathrm{Irr}_\infty - \dim V^{I_0} - \dim V^{I_\infty} \\ &= \sum_j m(W_j) - \sum_{k\geq 0} m(k) - m(\chi_0) \end{aligned}$$

further, assuming that V is irreducible and non-trivial. This argument, which was shown to us by Mark Reeder, breaks into two cases, depending on whether $\epsilon = +1$ or -1 on V.

If $\epsilon = +1$ on V, we may assume (by passing to a quotient that acts faithfully on V) that $\epsilon = 1$ in $\check{G}$. Then $n^h = 1$ and $\check{H}$ is a semi-direct product $\langle n\rangle \ltimes \check{S}$. In this case, the map $SL_2 \to \check{G}$ factors through the quotient PGL_2.

We now count the dimension of the span of invariants for $\langle n\rangle$ on V. Since $\langle n\rangle$ is a subgroup of $\check{H}$ which fixes a line in each W_j, we find that

$$\begin{aligned} \dim V^{\langle n\rangle} &= \sum m(W_j) + m(\chi_0) \\ &= \mathrm{Irr}_\infty + m(\chi_0). \end{aligned}$$

Since n is conjugate to the element $n' = \rho(e^{2\pi i/h})$, which lies in the maximal torus A of the principal PGL_2, we have

$$\dim V^{\langle n\rangle} = \sum_{k\geq 0} m(k) \cdot \#\{\text{weights of } A \text{ on } \mathrm{Sym}^k \text{ with } a \equiv 0 \pmod{h}\}.$$

if $m(k) \neq 0$, then k is even and the weight $a = 0$ occurs once in the irreducible representation Sym^k. Since the weights a and $-a$ occur with the same multiplicity in V, we find that

$$\begin{aligned} \dim V^{\langle n\rangle} &= \sum_{k\geq 0} m(k) + 2\#\{\text{weights } a > 0 \text{ of } A \text{ on } V \text{ with } a \equiv 0 \pmod{h}\} \\ &= \dim V^{I_0} + 2\#\{\text{weights } a > 0 \text{ of } A \text{ on } V \text{ with } a \equiv 0 \pmod{h}\}. \end{aligned}$$

Hence, when $\epsilon = +1$ we find that

$$\begin{aligned} d(V) &= \mathrm{Irr}_\infty - \dim V^{I_0} - m(\chi_0) \\ &= 2(\#\{\text{weights } a > 0 \text{ of } A \text{ on } V \text{ with } a \equiv 0 \pmod{h}\} - m(\chi_0)). \end{aligned}$$

The fact that $d(V)$ is even is consistent with the fact that when V is self-dual and $\epsilon|_V = +1$, then V is orthogonal. Hence the first cohomology group $H^1(\mathbb{P}^1, j_{!*}\mathcal{F}(V))$ is symplectic.

If the highest weight $\check{\lambda}$ of V satisfies

$$\langle \check{\lambda}, \rho \rangle \leq h - 1,$$

then there are no weights $a > 0$ with $a \equiv 0 \pmod{h}$. Since $d(V) \geq 0$, this forces $m(\chi_0) = 0$ and $d(V) = 0$. This is what happens for the adjoint representation.

A similar analysis when $\epsilon|_V = -1$ gives the formula

$$\begin{aligned} d(V) = \#\{&\text{weights } 2k+1 \text{ for the torus of } SL_2 \text{ on } V \text{ with } k \equiv 0 \pmod{h} \text{ and } k \neq 0\} \\ &- m(\chi_1), \end{aligned}$$

where χ_1 is the character of $\check{H}$ which is trivial on $\check{S}$ and maps n to $e^{\pi i/h}$. if $\epsilon = n^h$ lies in $\check{S}$, then $m(\chi - 1) = 0$, as all weights of $\check{S}$ on V are non-trivial.

15. Nearby connections

There are several connections closely related to the rigid irregular connection ∇ that we have studied in this paper. First, there is the connection

$$\nabla_1 = d + N\frac{dt}{t}$$

which has regular singularities at both $t = 0$ and $t = \infty$. The monodromy of ∇_1 is generated by the principal unipotent element $u = \exp(2\pi i N)$ and the differential Galois group is the additive group $\exp(zN)$ of dimension 1 in $\check{G}$.

A second related connection is the canonical extension of our local differential equation at infinity, defined by Katz [K1, 2.4]. This is the connection

$$\nabla_2 = d + \left(N - \frac{1}{h}\rho\right)\frac{dt}{t} + Edt.$$

If we pass to the ramified extension given by $u^h = s = t^{-1}$, and make a gauge transformation by $g = \rho(u)$ (assuming that ρ is a co-character), the connection ∇_2 becomes equivalent to the connection

$$d - h(N+E)\frac{du}{u^2}.$$

This is regular at the unique point lying above $t = 0$. The differential Galois group of ∇_2 and its inertia group at infinity are both isomorphic to $\check{H}$. The inertia group at zero is cyclic, of order h or $2h$.

Finally, we have the following generalization of ∇ for the exceptional groups $\check{G}$ of types G_2, F_4, E_6, E_7, and E_8, which was suggested by the treatment of nilpotent elements and regular classes in the Weyl group in [Sp], Section 9. We thank Mark Reeder for bringing this argument to our attention. Let

$$\varphi' : \mathfrak{sl}_2 \to \check{\mathfrak{g}}$$

be a subregular $\mathfrak{sl}_2$ in the Lie algebra. The Dynkin labels of the semi-simple element $\varphi'(h)$ are equal to 2 on all of the vertices of the diagram for $\check{G}$, except the vertex corresponding to the (unique) root with the highest multiplicity m in the highest root, where the label is 0.

Let $d = h - m$. Then the highest eigenspace $\check{\mathfrak{g}}[2d-2]$ for $\varphi'(h)$ on $\check{\mathfrak{g}}$ has dimension equal to 1. Let E' be a basis, and let $N' = \varphi'(f)$ in $\check{\mathfrak{g}}[-2]$. Then Springer shows that the element $N' + E'$ is both regular and semi-simple in $\check{\mathfrak{g}}$. This element is normalized by the semi-simple element $n' = \varphi'(h(e^{\pi i/d}))$, which has order either d or $2d$ in $\check{G}$.

Let M be the maximal torus which centralizes $N' + E'$. Then the image of n' in the Weyl group of M is a regular element of order d, in the sense of Springer [Sp]. It has $r + 2$ free orbits on the roots of $\check{\mathfrak{g}}$, where r is the rank of $\check{G}$. We tabulate these numerical invariants for our groups below, as well as the characteristic polynomial F of n' acting on the character group of M, as a product of cyclotomic polynomials $F(n)$.

$\check{G}$	m	d	$r+2$	F
G_2	3	3	4	$F(3)$
F_4	4	8	6	$F(8)$
E_6	3	9	8	$F(9)$
E_7	4	14	9	$F(14) \cdot F(2)$
E_8	6	24	10	$F(24)$

Define the subregular analog of ∇ as follows:

$$\nabla' = d + N'\frac{dt}{t} + E'dt.$$

Then ∇' has a regular singularity at $t = 0$, with monodromy the subregular unipotent element $u' = \exp(-2\pi i N')$. It has an irregular singularity at ∞, with slope $1/d$ and local inertia group $\langle \check{S}', n' \rangle$, where $\check{S}'$ is the subtorus of M on which n' acts by the primitive d^{th} roots of unity.

The connection ∇' is rigid, with differential Galois group given by the following table

$\check{G}$	$G_{\nabla'}$
G_2	SL_3
F_4	$Spin_9$
E_6	E_6
E_7	E_7
E_8	E_8

In the first two cases, we note that a subregular SL_2 in $\check{G}$ is a regular SL_2 in $\check{G}_{\nabla'}$. Hence the connection ∇' on $\check{G}$ is obtained from the rigid connection ∇ for the differential Galois group. In the three other cases, the connection ∇' is new; the third gives us a connection with differential Galois group of type E_6. In particular, the differential Galois group of $\mathbb{G}_m$ has any simple exceptional group as a rigid quotient.

We get two further rigid connections with differential Galois group E_8 from the two nilpotent classes that Springer lists in [Sp], Table 11. These have slopes $1/20$ and $1/15$ at ∞ respectively. Since there is a misprint in that table, we give the Dynkin labeling of these nilpotent classes below.

$$\begin{array}{ccccccccccccc} 2 & — & 2 & — & 0 & — & 2 & — & 0 & — & 2 & — & 2 \\ & & & & & & & & | & & & & \\ & & & & & & & & 2 & & & & \end{array}$$

$$\begin{tikzcd}
2 \arrow[r, dash] & 0 \arrow[r, dash] & 2 \arrow[r, dash] & 0 \arrow[r, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] & 2 \\
 & & & & 0 & &
\end{tikzcd}$$

In both cases, the action of the corresponding regular element in the Weyl group on the character group of the torus is by the primitive 20^{th} and 15^{th} roots of unity, respectively.

16. Example of the geometric Langlands correspondence with wild ramification

The Langlands correspondence for function fields has a counterpart for complex algebraic curves, called the geometric Langlands correspondence. As in the classical setting, there is a local and a global pictures. For simplicity we will restrict ourselves to the case when G is a simple connected simply-connected algebraic group, so that $\check{G}$ is a group of adjoint type.

The local geometric Langlands correspondence has been proposed in [FG] (see also [F1] for an exposition). According to this proposal, to each "local Langlands parameter" σ, which is a $\check{G}$-bundle with a connection on $D^\times = \operatorname{Spec} \mathbb{C}(\!(t)\!)$, there should correspond a category $\mathcal{C}_\sigma$ equipped with an action of the formal loop group $G(\!(t)\!)$. This correspondence should be viewed as a "categorification" of the local Langlands correspondence for the group $G(F)$, where F is a local non-archimedian field, $F = \mathbb{F}_q(\!(t)\!)$. This means that we expect that the Grothendieck group of the category $\mathcal{C}_\sigma$, equipped with an action of $G(\!(t)\!)$, attached to a Langlands parameter σ, should "look like" an irreducible smooth representation π of $G(F)$ attached to an ℓ-adic representation of the Weil group of F with the same properties as σ.

In particular, an object of $\mathcal{C}_\sigma$ should correspond to a vector in the representation π. Thus, the analogue of the subspace $\pi^{(K,\Psi)} \subset \pi$ of (K,Ψ)-invariant vectors in π (that is, vectors that transform via a character Ψ under the action of a subgroup K of $G(F)$) should be the category $\mathcal{C}_\sigma^{(K,\Psi)}$ of (K,Ψ)-equivariant objects of $\mathcal{C}_\sigma$.

Denote by $\operatorname{Loc}_{\check{G}}(D^\times)$ the set of isomorphism classes of flat $\check{G}$-bundles on $D^\times$. In [FG], a candidate for the category $\mathcal{C}_\sigma$ has been proposed for any $\sigma \in \operatorname{Loc}_{\check{G}}(D^\times)$. Namely, consider the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}$ of discrete modules over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ of critical level. The center of this category is isomorphic to the algebra of functions on the space $\operatorname{Op}_{\check{G}}(D^\times)$ of $\check{G}$-opers on $D^\times$ (see [F1]). Hence for each point $\chi \in \operatorname{Op}_{\check{G}}(D^\times)$ we have the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_\chi$ of those modules on which the center acts according to the character corresponding to χ.

Consider the forgetful map $p : \operatorname{Op}_{\check{G}}(D^\times) \to \operatorname{Loc}_{\check{G}}(D^\times)$. It was proved in [FZ] that this map is surjective. Given a local Langlands parameter $\sigma \in \operatorname{Loc}_{\check{G}}(D^\times)$, choose $\chi \in p^{-1}(\sigma)$. According to the proposal of [FG], the sought-after category $\mathcal{C}_\sigma$ should be equivalent to $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_\chi$ (these categories should therefore be equivalent to each other for all $\chi \in p^{-1}(\sigma)$). In particular, $\mathcal{C}_\sigma^{(K,\Psi)}$ should be equivalent to the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_\chi^{(K,\Psi)}$ of (K,Ψ)-equivariant objects in $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_\chi$.

Next, we consider the global geometric Langlands correspondence (see [FG, F2, F1] for more details). Let X be a smooth projective curve over $\mathbb{C}$. The Langlands parameters σ are now $\check{G}$-bundles on X with connections which are allowed to have poles at some points $x_1, \ldots, x_N$. Let $\mathrm{Bun}_{G,(x_i)}$ be the moduli stack of G-bundles on X with the full level structure at each point $x_i, i = 1, \ldots, N$ (that is, a trivialization of the G-bundle on the formal disc at x_i). The global correspondence should assign to σ the category $\mathrm{Aut}_{\sigma,(x_i)}$ of Hecke eigensheaves on $\mathrm{Bun}_{G,(x_i)}$ with eigenvalue σ.

The compatibility between the local and global correspondence should be that [FG]

$$\mathrm{Aut}_{\sigma,(x_i)} \simeq \bigotimes_{i=1}^{N} \mathcal{C}_{\sigma_{x_i}}, \tag{16.1}$$

where σ_{x_i} is the restriction of σ to the punctured disc at x_i. This equivalence should give rise to an equivalence of the equivariant categories. Let K_{x_i} be a subgroup of $G(\mathcal{O}_{x_i})$, where $\mathcal{O}_{x_i}$ is the completed local ring at x_i, and Ψ_{x_i} be its character. Then the equivariant category of $\mathrm{Aut}_{\sigma,(x_i)}$ is the category $\mathrm{Aut}_{\sigma,(x_i)}^{(K_{x_i},\Psi_{x_i})}$ of Hecke eigensheaves on $\mathrm{Bun}_{G,(x_i)}$ with eigenvalue σ which are (K_{x_i}, Ψ_{x_i})-equivariant. We should have an equivalence of categories

$$\mathrm{Aut}_{\sigma,(x_i)}^{(K_{x_i},\Psi_{x_i})} \simeq \bigotimes_{i=1}^{N} \mathcal{C}_{\sigma_{x_i}}^{(K_{x_i},\Psi_{x_i})}. \tag{16.2}$$

Now suppose that σ comes from an oper χ which is regular on $X\backslash\{x_1, \ldots, x_N\}$. Let χ_{x_i} be the restriction of χ to $D_{x_i}^{\times}$. Then we have the categories $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_{x_i}}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_{x_i}}^{(K_{x_i},\Psi_{x_i})}$, which we expect to be equivalent to $\mathcal{C}_{\sigma_{x_i}}$ and $\mathcal{C}_{\sigma_{x_i}}^{(K_{x_i},\Psi_{x_i})}$, respectively. The stack $\mathrm{Bun}_{G,(x_i)}$ may be represented as a double quotient

$$G(F)\backslash \prod_{i=1}^{N} G(F_{x_i}) / \prod_{i=1}^{N} G(\mathcal{O}_{x_i}),$$

where $F = \mathbb{C}(X)$ is the field of rational functions on X and F_{x_i} is its completion at x_i. A loop group version of the localization functor of Beilinson–Bernstein gives rise to the functors

$$\Delta : \bigotimes_{i=1}^{N} \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_{x_i}} \to \mathrm{Aut}_{\sigma,(x_i)}, \tag{16.3}$$

$$\Delta^{(K_{x_i},\Psi_{x_i})} : \bigotimes_{i=1}^{N} \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_{x_i}}^{(K_{x_i},\Psi_{x_i})} \to \mathrm{Aut}_{\sigma,(x_i)}^{(K_{x_i},\Psi_{x_i})}, \tag{16.4}$$

and it is expected [FG] that these functors give rise to the equivalences (16.1) and (16.2), respectively. This is a generalization of the construction of Beilinson and Drinfeld in the unramified case [BD1].

We now apply this to our situation, which may be viewed as the simplest example of the geometric Langlands correspondence with wild ramification (i.e., connections admitting an irregular singularity). We note that wild ramification has been studied by E. Witten in the context of S-duality of supersymmetric Yang–Mills theory [Wi].

Let χ be our $\check{G}$-oper on $\mathbb{P}^1$ with poles at the points 0 and ∞. Let σ denote the corresponding flat $\check{G}$-bundle. We choose K_0 to be the Iwahori subgroup I, with Ψ_0 the trivial character (we will therefore omit it in the formulas below), and K_∞ to be its radical I^0. Note that $I^0/[I^0,I^0] \simeq (\mathbb{G}_a)^{\mathrm{rank}(G)+1}$. We choose a non-degenerate additive character Ψ of I^0 as our Ψ_∞. Thus, we have the global categories $\mathrm{Aut}_{\sigma,(0,\infty)}$ and $\mathrm{Aut}_{\sigma,(0,\infty)}^{I,(I^0,\Psi)}$ on $\mathrm{Bun}_{G,(0,\infty)}$.

According to Section 3, there is a unique automorphic representation of $G(\mathbb{A})$ corresponding to an ℓ-adic analogue of our oper. Moreover, the only ramified local factors in this representation are situated at 0 and ∞. The former is the Steinberg representation, whose space of Iwahori invariant vectors is one-dimensional. The latter is the simple supercuspidal representation constructed in [GR]. Its space of (I^0,Ψ)-invariant vectors is also one-dimensional. We recall that the geometric analogue of the space of invariant vectors is the corresponding equivariant category. Hence the geometric counterpart of the multiplicity one statement of Section 3 is the statement (conjecture) that *the category* $\mathrm{Aut}_{\sigma,(0,\infty)}^{I,(I^0,\Psi)}$ *has a unique non-zero irreducible object.* (Here and below "unique" means "unique up to an isomorphism.")

The compatibility of the local and global correspondences gives us a way to construct this object. Namely, we have two local categories $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_0}^{I}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_\infty}^{(I^0,\Psi)}$ attached to the points 0 and ∞, respectively. The oper χ_0 on $D_0^\times$ has regular singularity and regular unipotent monodromy. Using the results of [FG, F1], one can show that the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_0}^{I}$ has a unique non-zero irreducible object $\mathbb{M}_{-\rho}(\chi_0)$, which is constructed as follows. It is the quotient of the Verma module

$$\mathbb{M}_{-\rho} = \mathrm{Ind}_{\mathrm{Lie}(I)\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\mathbb{C}_{-\rho})$$

over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ with highest weight $-\rho$, by the image of the maximal ideal in the center corresponding to the central character χ_0.

On the other hand, χ_∞ has irregular singularity with the slope $1/h$. To construct an object of the category of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_\infty}^{(I^0,\Psi)}$, we imitate the construction of [GR] (see Section 3). Define the affine Whittaker module

$$\mathbb{W}_\Psi = \mathrm{Ind}_{\mathrm{Lie}(I^0)\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(\Psi)$$

over $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ (here we denote by the same symbol Ψ the character of the Lie algebra $\mathrm{Lie}(I^0)$ corresponding to the above character Ψ of the group I^0). Let $\mathbb{W}_\Psi(\chi_\infty)$ be the quotient of $\mathbb{W}_\Psi$ by the image of the maximal ideal in the center corresponding to the central character χ_∞. By construction, it is an (I^0,Ψ)-equivariant $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-module and hence it is indeed an object of our local category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi_\infty}^{(I^0,\Psi)}$. Applying the localization functor $\Delta^{I,(I,\Psi)}$ of (16.4) to $\mathbb{M}_{-\rho}(\chi_0)\otimes\mathbb{W}_\Psi(\chi_\infty)$, we obtain an object of the category $\mathrm{Aut}_{\sigma,(0,\infty)}^{I,(I^0,\Psi)}$. It is natural to conjecture that this is the unique non-zero irreducible object of this category.

One can show (see [FF], Lemma 5) that the image of the center of the completed enveloping algebra in $\operatorname{End} \mathbb{W}_\Psi$ is the algebra of functions on the space of opers which have representatives of the form

$$d - N\frac{ds}{s} - E\frac{ds}{s^2} + \mathbf{v}\frac{ds}{s},$$

where $\mathbf{v} \in \mathfrak{b}[[s]]$. Since our oper χ_∞ belongs to this space, we obtain that the quotient $\mathbb{W}_\Psi(\chi_\infty)$ is non-zero. This provides supporting evidence for the above conjecture describing an example of the geometric Langlands correspondence with wild ramification.

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