# The classes of singular moduli in the generalized Jacobian

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## 1 Introduction

Let $j^*(\tau) = j(\tau) - 744 = 1/q + 0 + 196884q + \ldots$ be the unique modular function for the group $\mathrm{SL}_2(\mathbb{Z})$ which has a simple pole with residue $1$ at the cusp $\infty$ and has constant coefficient zero in its Fourier expansion. In [6], Don Zagier showed how the traces of singular values of $j^*(\tau)$ were related to the Fourier coefficients of a weakly holomorphic modular form of weight $\frac{3}{2}$ for the group $\Gamma_0(4)$.

Zagier's precise result is the following. Let $-D$ be a negative discriminant, and consider the sum of the values

$$\mathrm{Tr}(D) = \sum_{\mathrm{SL}_2(\mathbb{Z})\backslash \mathscr{Q}_D} j^*(z)/u(z) = \sum_{D=df^2} \sum_{\mathrm{SL}_2(\mathbb{Z})\backslash \mathscr{Q}_d^{pr}} j^*(z_d)/u(d).$$

Here the first sum is taken over the finite set of orbits of $\mathrm{SL}_2(\mathbb{Z})$ on the set $\mathscr{Q}_D$ of all positive definite binary quadratic forms of discriminant $-D$, $z$ is the (orbit of the) unique root in the upper half plane, and $u(z)$ is half the order of the stabilizer. The internal sum in the second expression is over the primitive positive definite binary quadratic forms of discriminant $-d$ and $z_d$ is a root in the upper half plane. We have $u(d) = 1$ unless $d = 3$ or $d = 4$, when $u(3) = 3$ and $u(4) = 2$ respectively. There are $h(d) = \#\,\mathrm{Pic}(O_d)$ terms in the internal sum, where $O_d$ is the imaginary quadratic order of discriminant $-d$, so the degree of the divisor in the sum for $\mathrm{Tr}(D)$ is equal to $H(D) = \sum_{D=df^2} h(d)/u(d)$, the Hurwitz class number.

The classical theory of complex multiplication shows that the values $j^*(z)$ are algebraic integers (which turn out to be divisible by $u(d)$). Moreover, the singular moduli $j^*(z_d)$ with a fixed primitive discriminant $-d$ form a complete set of conjugates over $\mathbb{Q}$. Hence $\mathrm{Tr}(D)$ is an integer, which is essentially the sum of the traces of singular moduli. Zagier [6, Theorem 1] proves that $-\mathrm{Tr}(D)$ is the $D^{th}$ Fourier coefficient of the modular form

$$g(\tau) = \eta(\tau)^2 E_4(\tau)/\eta(2\tau)\eta(4\tau)^6$$

of weight $\frac{3}{2}$ for the group $\Gamma_0(4)$. We recall that $\eta(\tau) = q^{1/24} \prod_{n\geq 1}(1-q^n)$ has weight $\frac{1}{2}$ and that $E_4(\tau) = 1 + 240 \sum_{n\geq 1}(\sum_{d|n} d^3)q^n$ is the Eisenstein series of weight $4$. The modular form $g(\tau)$ is holomorphic except at $\infty$; its full Fourier expansion is given by

$$g(q) = q^{-1} - 2 - \sum_D \mathrm{Tr}(D)q^D.$$

In this paper, I want to explain how Zagier's explicit result extends our earlier work [2] on the position of sums of Heegner points in the Jacobian of the modular curve $X_0(N)$. In this case, we need the generalized Jacobian of the modular curve $X_0(1)$, with respect to the modulus $\mathfrak{m} = 2(\infty)$. This generalized Jacobian is isomorphic to the additive group, and the sum of the Heegner points in the generalized Jacobian is equal to the negative $-\mathrm{Tr}(D)$ of the sum of traces of the singular moduli $j^*(z)$. This gives some insight into the appearance of the modified function $j^*(\tau)$, and suggests a problem for future work.

Gerard and I became good friends during our many (often futile) attempts to understand what Don had explained to us. So it seems appropriate to submit this short article to his birthday volume.

# 2 The generalized Jacobian

Let $X$ be a complete, non-singular algebraic curve over the field $k$, and let $y$ be a $k$-rational point on $X$. The Jacobian of $X$ is a commutative algebraic group over $k$ whose rational points correspond to elements of the quotient group $\mathrm{Div}^0(X)/P(X)$ [5]. Here $\mathrm{Div}^0(X)$ is the group of divisors of degree zero which are rational over $k$ and $P(X)$ is the subgroup of principal divisors of the form $(f) = \sum \mathrm{ord}_x(f)(x)$ with $f \in k(X)^*$ .

For $m \geq 1$ the generalized Jacobian $J_\mathfrak{m}$ corresponding to the modulus $\mathfrak{m} = m.(y)$ is a commutative algebraic group over $k$ whose rational points can be described as follows [5]. Let $\mathrm{Div}^0_\mathfrak{m}$ denote the group of divisors of degree zero supported on the points of $X - \{y\}$ which are rational over $k$, and let $P(X)_\mathfrak{m}$ denote the subgroup of principal divisors of functions $f \in k(X)^*$ with $f \equiv 1 \mod \mathfrak{m}$. If $O_y$ denotes the ring of integers in the completion $k(X)_y$, and $P_y$ is the maximal ideal, then we have an exact sequence of abelian groups

$$1 \to O_y^*/k^*.(1+P_y^m) \to J_\mathfrak{m}(k) \to J(k) \to 1.$$

Now assume that the curve $X$ has genus zero and that $\mathfrak{m} = 2.(y)$. Then $J(k) = 1$ and the group $O_y^*/k^*.(1+P_y^2) \simeq (1+P_y)/(1+P_y^2)$ is isomorphic to the additive group of $k$. We can fix an isomorphism of algebraic groups from $J_\mathfrak{m}$ to $\mathbb{G}_a$ by choosing a uniformizing parameter $\pi$ of the discrete valuation ring $O_y$. To the divisor $d = (f)$ in $\mathrm{Div}^0_\mathfrak{m}$ with unique generator $f \equiv 1 + a(d).\pi \pmod{\pi^2}$ we associate the element $a(d)$ in $\mathbb{G}_a(k)$.

We apply the above to the modular curve $X = X_0(1)$ which has genus zero over $\mathbb{Q}$. We take the the rational point $y$ to be the cusp at $\infty$, with the uniformizing parameter $q = e^{2\pi i\tau}$ given by the Tate curve.

The above argument gives an isomorphism of commutative algebraic groups from $J_{\mathfrak{m}} = J_0(1)_{2.\infty}$ to $\mathbb{G}_a$. At the level of rational points, the map $F(d) = a(d)$ defines an isomorphism of abelian groups

$$F : \mathrm{Div}^0(X)_{2.\infty}/P(X)_{2.\infty} \simeq \mathbb{Q}.$$

Using the parameter $q$, we can extend this to an isomorphism of abelian groups

$$F : \mathrm{Div}(X)_{2.\infty}/P(X)_{2.\infty} \simeq \mathbb{Q} \oplus \mathbb{Z}.$$

Let $d$ be an element of $\mathrm{Div}(X)$ with no support at $\infty$, and let $m(d) = \deg(d)$. The divisor $d_0 = d - m(d)(\infty)$ has degree zero, so is principal. We choose the unique generator $f$ of $d_0$ with the property that $q^{m(d)} f \equiv 1 + a(d).q \pmod{q^2}$ and define $F(d) = (a(d), m(d))$.

# 3 Heegner points

Let $N > 1$ and let $X_0(N)$ be the modular curve over $\mathbb{Q}$ which classifies elliptic curves with a cyclic $N$-isogeny. Let $P_D$ be the rational divisor on $X_0(N)$ over $\mathbb{Q}$ which is the weighted sum of all the Heegner points of discriminants dividing $D$ [1]:

$$R_D = \sum_{D=df^2} \sum x_d/u(d)$$

We note that the total number of Heegner points $x_d$ of exact discriminant $d$ on $X_0(N)$ is the product of the class number $h(d)$ by the number of invertible ideals $\mathfrak{n}$ in the ring $O_d$ where the quotient group $O_d/\mathfrak{n}$ is cyclic of order $N$. In particular, $R_D = 0$ unless the discriminant $-D$ is a square $(\mathrm{mod}\ 4N)$. Since the finite 2-group $W$ generated by the involutions $w_p$ for $p$ dividing $N$ permutes the Heegner points of a fixed discriminant, the divisor $R_D$ is fixed by $W$ and is pulled-back from a divisor $P_D$ on the quotient curve $X = X_0(N)/W$, which is relatively prime to the cusp $\infty$.

Let $P_D^0$ be the divisor $P_D - \deg(P_D)(\infty)$ of degree zero on $X$ and let $[P_D^0]$ denote the class of the divisor $P_D^0$ in the quotient $(\mathrm{Div}^0(X)/P(X)) \otimes \mathbb{Q} = J(\mathbb{Q}) \otimes \mathbb{Q}$. This is a rational vector space of finite dimension. In [2] we proved the following result, where Kohnen's $+$ space is defined in [4] as a subspace of the modular forms of weight $\frac{3}{2}$, using congruences on the Fourier coefficients.

**Proposition 1** *The series $\sum_D [P_D^0].q^D$ is the Fourier expansion of a holomorphic cusp form of weight $\frac{3}{2}$ in Kohnen's $+$ space for the group $\Gamma_0(4N)$, with coefficients in the rational vector space $J(\mathbb{Q}) \otimes \mathbb{Q}$.*

This means that for any $\mathbb{Q}$-linear map $f : J(\mathbb{Q}) \otimes \mathbb{Q} \to \mathbb{C}$, the series $\sum_D f(P_D^0).q^D$ is the Fourier expansion of a holomorphic modular form of this weight and level. In fact, we prove a more precise result of this type (which keeps track of the ideal $\mathfrak{n}$) using the Fourier expansion of Jacobi forms.

The degree of the divisor $P_D$ can be computed from the Hurwitz class number $H(D)$. For example, it is equal to $H(D)$ when all primes dividing $N$ are split in $O_D$ and equal to zero if there is a single prime dividing $N$ which is inert in $O_D$. Since these modified Hurwitz class numbers are the Fourier coefficients of the holomorphic Eisenstein series of weight $\frac{3}{2}$ in Kohnen's $+$ space on $\Gamma_0(N)$, we obtain a mild extension.

**Proposition 2** *Assume that $N > 1$. Then the series $\sum_D [P_D].q^D$ is the positive Fourier expansion of a holomorphic modular form of weight $\frac{3}{2}$ in Kohnen's $+$ space for the group $\Gamma_0(4N)$, with coefficients in the rational vector space $(\mathrm{Div}(X)/P(X)) \otimes \mathbb{Q} = (J(\mathbb{Q}) \otimes \mathbb{Q}) \oplus \mathbb{Q}$.*

# 4 Level $N = 1$

For $N = 1$, the curve $X = X_0(1)$ has genus zero and its Jacobian $J$ is trivial. Similarly, there are no holomorphic forms cusp of weight $\frac{3}{2}$ on $\Gamma_0(4)$. At this weight and level there is a weakly holomorphic form with a simple pole at the unique cusp:

$$g(\tau) = \eta(\tau)^2 E_4(\tau)/\eta(2\tau)\eta(4\tau)^6 = q^{-1} - 2 - \sum_D \mathrm{Tr}(D) q^D.$$

There is also a mock modular form

$$-\frac{1}{12} + \sum_D H(D) q^D$$

which is the holomorphic part of the Eisenstein series of weight $\frac{3}{2}$ [3]:

$$E(\tau) = -\frac{1}{12} + \sum_D H(D) q^D + y^{-1/2} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 y) q^{-n^2},$$

with

$$\beta(x) = \frac{1}{16\pi} \int_1^\infty u^{-3/2} e^{-xu}\, du.$$

We have the following result for the class $[P_D]$ in the generalized Jacobian.

**Proposition 3** *The series $\sum [P_D].q^D$ is the positive Fourier expansion of a modular form of weight $\frac{3}{2}$ in Kohnen's $+$ space for the group $\Gamma_0(4)$ with coefficients in the rational vector space $(\mathrm{Div}(X)_{2.\infty}/P(X)_{2.\infty}) \otimes \mathbb{Q}$.*

*More precisely, the series $\sum a(P_D).q^D$ is the positive Fourier expansion of the weakly holomorphic form $g(\tau)$, and the series $\sum m(P_D).q^D$ is the positive Fourier expansion of the mock Eisenstein series $E(\tau)$.*

Since the degree $m(P_D) = H(D)$, it suffices to check that $a(P_D) = -\operatorname{Tr}(D)$ is given by the negative of the weighted sum of traces of $j^*(z_d)$. The rational divisor $P_D - H(D)(\infty)$ has degree zero, so is principal. We take the generator

$$f(\tau) = \prod_{D=df^2} \prod (j^*(\tau) - j^*(z_d))^{1/u(d)}.$$

This is the unique generator with $q^{H(D)} f(\tau) \equiv 1 \pmod q$. In fact, we have

$$q^{H(D)} f(\tau) = 1 - \operatorname{Tr}(D) q \pmod{q^2}$$

so $a(P_D) = -\operatorname{Tr}(D)$. This completes the proof.

An interesting problem is to see if this result extends to the curve $X = X_0(N)/W$ when $N > 1$. Since the Heegner divisor $P_D$ is relatively prime to $\infty$ we can consider its class $[[P_D]]$ in the quotient space $(\mathrm{Div}(X)_{2.\infty}/P(X)_{2.\infty}) \otimes \mathbb{Q}$. Is the series $\sum [[P_D]].q^D$ the positive Fourier expansion of a modular form of weight $\frac{3}{2}$ in Kohnen's $+$ space for the group $\Gamma_0(4N)$ with coefficients in the rational vector space $(\mathrm{Div}(X)_{2.\infty}/P(X)_{2.\infty}) \otimes \mathbb{Q}$? When X has genus zero, the arguments in [6, %6] yield a proof.

# References


[1] B. Gross *Heegner points on $X_0(N)$*. In: Modular Forms. (Durham, 1983), pp. 87-105.

[2] B. Gross, W. Kohnen, D. Zagier *Heegner points and derivatives of $L$-series, II*. Math. Annalen **278** (1987), pp. 497-562.

[3] F. Hirzebruch, D. Zagier *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*. Inventiones math. **36** (1976), pp. 57-113.

[4] W. Kohnen *Newforms of half integral weight* Crelle J. **333** (1982), pp. 32-72.

[5] J.-P. Serre *Groupes algébriques et corps de classes* Hermann (1975).

[6] D. Zagier *Traces of singular moduli*. In: Motives, polylogarithms, and Hodge theory. International Press Lecture Series **3** (1998), pp. 211-244.