On the centralizer of a regular, semi-simple, stable conjugacy class

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Let k be a field, and let G be a semi-simple, simply-connected algebraic group, which is quasi-split over k. The theory of semi-simple conjugacy classes in G is well-understood, from work of Steinberg [S] and Kottwitz [K]. Any semi-simple conjugacy class s which is defined over k is represented by a semi-simple element γ in $G(k)$. The centralizer G_γ of γ in G is connected and reductive. It is determined by the stable class s up to inner twisting, and one can choose a representative γ so that G_γ is quasi-split over k.

In this paper, we will only consider the case when the semi-simple stable class s is regular. Then G_γ is a maximal torus in G, whose k-isomorphism class depends only on the class s. Our aim is to determine the isomorphism class of this torus, which we denote T_s over k, from the data specifying s in the variety of semi-simple stable conjugacy classes.

We will first give an abstract description of the character group $X(T_s)$, as an integral representation of the Galois group of k. We will then describe T_s concretely, in some special cases. In particular, for a simple, split group G which is not simply-laced, we use a semi-direct product decomposition of the Weyl group to reduce the problem to a semi-simple, quasi-split subgroup H_s containing T_s and the long root subgroups of G.

The concrete description of T_s allows one to compute the terms corresponding to regular classes s in the stable trace formula (cf. [G-P]). For the general semi-simple class, one would like to have a description of the motive $M(G_\gamma)$ of the centralizer.

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1. The extended Weyl group

We recall that G is assumed quasi-split over k. Let B be a Borel subgroup, and let T be a Levi factor of B — which is a maximal torus in G.

Let k^s denote a separable closure of k, and let $\Gamma = \mathrm{Gal}(k^s/k)$. Let $X(T) = \mathrm{Hom}_{k^s}(T, \mathbb{G}_m)$ be the character group of T over k^s, which is an integral representation of Γ. Let E be the fixed field of the kernel of this action, so the quotient $\Gamma_E = \mathrm{Gal}(E/k)$ acts faithfully on $X(T)$. Both the torus T and the group G are split by the finite Galois extension E of k.

Let $\underline{W} = N_G(T)/T$ be the Weyl group of T in G. This is a finite, étale group scheme over k, which is pointwise rational over E. We put $W = \underline{W}(E)$. The Galois group Γ_E acts on W, and the semi-direct product $W.\Gamma_E$ acts on $X(T)$ via the reflection representation

$$r : W.\Gamma_E \to GL(X(T)).$$

We call $W.\Gamma_E$ the extended Weyl group.

The roots of α of T are the non-zero elements of $X(T) = \mathrm{Hom}_E(T, \mathbb{G}_m)$ which occur in the action of T on $\mathrm{Lie}(G)$ over E. They are permuted under the reflection action of the extended Weyl group $W.\Gamma_E$. Associated to each root α is a co-root $\alpha^\vee$ in $\mathrm{Hom}_E(\mathbb{G}_m, T)$, as well as a reflection r_α in W, whose action on $X(T)$ is given by $r_\alpha(x) = x - \langle x, \alpha^\vee \rangle \cdot \alpha$.

Let $\{\alpha_1, \ldots, \alpha_n\}$ be the simple roots of T determined by B. These simple roots are permuted by the action of Γ_E on $X(T)$, and the simple reflections r_{α_i} generate W.

2. The variety of semi-simple classes

The simple co-roots $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ form a basis of $\mathrm{Hom}_E(\mathbb{G}_m, T)$, as G is simply-connected. Let $\{\omega_1, \ldots, \omega_n\}$ be the dual basis of $X(T)$. This basis is permuted by the action of Γ_E. If $\sigma \in \Gamma_E$, we write $\underline{\sigma}$ for the associated permutation of the ω_i.

Let V_i denote the irreducible representation of G over E with highest weight ω_i for B. Then, for all σ in Γ_E, we have

$$V_i^\sigma \simeq V_{\underline{\sigma}(i)}.$$

In particular, if γ is any element in $G(k)$, we have

$$\mathrm{Tr}(\gamma|V_i)^\sigma = \mathrm{Tr}(\gamma|V_{\underline{\sigma}(i)}) \quad \text{in} \quad E.$$

Let S be the twisted form of affine n-space over k, given by the permutation representation $\underline{\sigma}$ of Γ_E on the coordinates:

$$S(k) = \{(x_1, \ldots, x_n) \in E^n : x_i^\sigma = x_{\underline{\sigma}(i)}\}.$$

If γ is an element in $G(k)$, then

$$x(\gamma) = (\mathrm{Tr}(\gamma|V_1), \ldots, \mathrm{Tr}(\gamma|V_n))$$

is a point of $S(k)$, which depends only on the stable conjugacy class of γ in $G(k^s)$.

The following fundamental result is due to Steinberg [S].

Proposition. *If s is any point in $S(k)$, there is a semi-simple element γ in $G(k)$ with $x(\gamma) = s$. The element γ is well-defined up to conjugacy in $G(k^s)$. The map that assigns to each semi-simple element γ the point $x(\gamma)$ identifies S with the variety of semi-simple stable conjugacy classes in G.*

3. The discriminant locus

Steinberg constructs the variety S as a quotient:

$$T \to T/\underline{W} = S.$$

This covering is étale over the complement of a divisor $D \subset S$.

Over the extension E, $S = \mathbb{A}^n = T/W$ and the divisor D is given by the zero locus of a polynomial $D(x_1, \ldots, x_n)$. As a W-invariant function on T, $D(t)$ can be given by the formula

$$D(t) = (-1)^N \cdot \prod_{\alpha} (t^\alpha - 1).$$

Here the product is taken over all the roots, and N is the number of positive roots.

For example, when $G = \mathrm{SL}_2$, we have $x = t + t^{-1}$, and

$$\begin{aligned} D(t) &= (-1)(t^2 - 1)(t^{-2} - 1) \\ &= x^2 - 4. \end{aligned}$$

The square-root Δ of D is the usual denominator in the Weyl character formula:

$$\Delta(t) = \prod_{\alpha > 0} \left(t^{\alpha/2} - t^{-\alpha/2} \right).$$

This function on T satisfies $\Delta(wt) = \mathrm{sign}(w)\Delta(t)$.

Since $D(t)$ is also invariant under the action of Γ_E, it defines a divisor D on S over k. The complement $S' = S - D$ defines the variety of regular, semi-simple, stable conjugacy classes in G. If s is a point of $S'(k)$, there is a regular, semi-simple conjugacy class γ in $G(k)$ with $x(\gamma) = s$. The centralizer G_γ of γ in G is a maximal torus, whose isomorphism class T_s over k depends only on s.

4. The character group $X(T_s)$

Since the covering $T \to S$ is Galois over $S' = S - D$, it gives rise to a homomorphism of the fundamental group

$$\rho : \pi_1(S') \to W.\Gamma_E$$

well-defined up to conjugacy by W. The subgroup $\pi_1^{\mathrm{geom}}(S')$ maps to W, and the resulting homomorphism from the quotient $\Gamma = \pi_1/\pi_1^{\mathrm{geom}}$ to Γ_E is the standard projection.

Specializing ρ to the point s in $S'(k)$, we obtain a homomorphism

$$\rho_s : \Gamma \to W.\Gamma_E,$$

well-defined up to conjugation by W, such that the resulting map $\Gamma \to \Gamma_E$ is the standard projection. In particular, the normal subgroup $\mathrm{Gal}(K^s/E)$ maps into W.

The following result gives an abstract determination of the torus T_s over k, via a description of its character group $X(T_s)$.

Proposition. *The character group $X(T_s)$ is isomorphic to the free $\mathbb{Z}$-module $X(T)$, with Galois action given by the composite homomorphism*

$$\Gamma \underset{\rho_s}{\longrightarrow} W.\Gamma_E \underset{r}{\longrightarrow} \mathrm{GL}(X(T)),$$

where r is the reflection representation.

Proof. We give the argument in the split case, when $E = k$. Let γ be a regular, semi-simple class in $G(k)$ which maps to s in $S'(k)$, and let t be an element in $T(k^s)$ which lies above s in the covering $T \to S = T/W$.

Since γ and t have the same image in $S(k)$, they are conjugate in $G(k^s)$: $g\gamma g^{-1} = t$. Conjugation by g gives an isomorphism of their centralizers, which is defined over k^s:

$$\varphi : G_\gamma \to T.$$

The fiber over s in the covering $T \to S$ can be identified with the orbit Wt in $T(k^s)$. In particular, since s is defined over k, $t^\sigma = w_\sigma(t)$ for every $\sigma \in \Gamma$. The map $\sigma \mapsto w_\sigma$ is the homomorphism $\rho_s : \Gamma \to W$. In particular, the isomorphism φ^σ (which is conjugation by g^σ) is equal to $\rho_s(\sigma) \circ \varphi$. Hence the 1-cocycle $\sigma \mapsto \varphi^{\sigma-1}$ defining G_γ as a twist of T over k is given by the homomorphism $\rho_s : \Gamma \to W \subset \mathrm{Aut}_k(T)$. It follows that the action of Γ, on $X(T_s) = X(G_\gamma)$ is given by the composition of ρ_s with the reflection representation.

The quasi-split case is similar, but the 1-cocycle $\sigma \mapsto w_\sigma$ is not a homomorphism, as Γ_E acts nontrivially on W. This can be converted to a homomorphism $\rho_s(\sigma) = w_\sigma \times \sigma_E$ from Γ to the extended Weyl group $W.\Gamma_E$ [Se, pg 43]. The rest of the argument is similar.

Note. The above argument shows that the class of the 1-cocycle ρ_s in $H^1(k, \underline{W})$ is in the image of $ker : (H^1(k, N(T)) \to H^1(k, G))$. Any cocycle with this property (or the equivalent homomorphism from Γ to the extended Weyl group $W.\Gamma_E$) arises from a stable, regular, semi-simple class in G.

5. Long and short roots

In this section, we assume that G is quasi-simple and split, and that W has two orbits on the set of roots in $X(T)$. These are the long and short roots; the long roots are in the orbit of the highest root (the highest weight of B on $\mathrm{Lie}(G)$).

Let W_ℓ denote the normal subgroup of W generated by the reflections in the long roots. Let W_{ss} denote the subgroup of W generated by the reflections in the short *simple* roots (relative to B).

Proposition. *W is isomorphic to the semi-direct product*

$$W = W_\ell . W_{ss}.$$

The group W_ℓ is the Weyl group of the sub-root system (of the same rank) of long roots. The group W_{ss} isomorphic to the symmetric group S_m, where $(m-1)$ is the number of short simple roots.

Proof. The following argument was shown to me by Mark Reeder. Let P be the set of positive roots for G, relative to B, and let P_ℓ be the set of long positive roots. Then P_ℓ is a positive system for the sub-root system of long roots.

If α is a simple root, then $r_\alpha(P) = P - \{\alpha\} \cup \{-\alpha\}$. Hence every element in W_{ss} stabilizes P_ℓ. Since W_ℓ acts simply-transitively on the positive systems of long roots, $W_\ell \cap W_{ss} = 1$.

To show $W = W_\ell . W_{ss}$, let w be an arbitrary element of W. Then

$w(P) = P'$ is another system of positive roots, and $w(P_\ell) = P'_\ell$ is another system of positive long roots. Hence there is a unique element w_ℓ in W_ℓ with $P'_\ell = w_\ell(P_\ell)$. The element $v = w_\ell^{-1} \cdot w$ then stabilizes P_ℓ.

To show that v is in W_{ss}, we use an argument familiar in Lie theory. Let J be the subset of short simple roots. In the coset $v \cdot W_{ss}$, choose an element u of shortest length. (In fact, the element u is unique.) Then $u(J) \subset P$, for if $u(\alpha) < 0$ for any $\alpha \in J$, u would have a reduced expression ending in r_α, contradicting the minimality of its length. Since $u(P_\ell) = P_\ell$ and $u(J) \subset P$, the element u of W stabilizes P. Hence $u = 1$, so v is in W_{ss}, and $w = w_\ell \cdot v = w_\ell \cdot w_{ss}$ as claimed.

Here is a table of the cases

W of type	W_ℓ of type	W_{ss} isomorphic to
B_n	D_n	S_2
C_n	$(A_1)^n$	S_n
G_2	A_2	S_2
F_4	D_4	S_3

In this case, the discriminant divisor $D \subset S$ reducible, as $D(x) = D_\ell(x)D_s(x)$ with

$$\begin{aligned} D_\ell(t) &= (-1)^{N_\ell} \prod_{\alpha \text{ long}} (t^\alpha - 1) \\ D_s(t) &= (-1)^{N_s} \prod_{\alpha \text{ short}} (t^\alpha - 1). \end{aligned}$$

Here N_ℓ and N_s are half the number of long and short roots, respectively.

Now let s be a regular, semi-simple, stable class in G, and fix an isomorphism $W_{ss} \simeq S_m$. (When $m \neq 6$, this is unique up to inner automorphism. When $m = 6$ it can be fixed by having W_{ss} act on the $\pm$ weight spaces in the standard representation of $G = \mathrm{Sp}_{12}$.) The composite homomorphism

$$\delta : \Gamma \underset{\rho_s}{\longrightarrow} W \underset{\text{proj.}}{\longrightarrow} W_{ss} \simeq S_m$$

(up to conjugacy) defines an étale k-algebra K of rank m: the algebra K is the twist of k^m by the 1-cocycle δ [Se2, pg. 652]. Let E be the finite Galois extension of k, fixed by the kernel of δ. Then E is the "Galois closure" of K, and $\Gamma_E \subset S_m$ is the image of δ. The automorphism group of the algebra K over k is the centralizer of the subgroup Γ_E in S_m.

We may view ρ_s as a homomorphism

$$\rho_s : \Gamma \to W_\ell.\Gamma_E$$

corresponding to a stable class in the quasi-split subgroup $H_s \subset G$ with root system the long roots and Weyl group W_ℓ. This subgroup is split by E, and $T_s \subset H_s \subset G$. This approach often simplifies the computation of T_s, as we will see in §7 and §8.

One caveat — several distinct stable classes s' in H_s may become fused (i.e., conjugate) with s in G. Indeed, if we conjugate ρ_s above with any element of $W_{ss} \simeq S_m$ which centralizes Γ_E, we get a homomorphism $\rho_{s'}$ corresponding to a *different* stable class in H_s which is stably conjugate to s

in G. This corresponds to the fact that the finite group $\mathrm{Aut}_k(K)$ normalizes the subgroup H_s in G.

6. Linear and unitary groups

The description of T_s in §4 is fairly abstract. For some classical groups G, we can give a more concrete realization of T_s, using the characteristic polynomial of the standard representation (cf. [S-S], and [G-Mc, Appendix]). We will describe the group of points $T_s(k)$, and when k is local or global the Artin L-function of the Galois representation $X(T_s)$,

Consider the split group $G = SL(V)$ with $n = \dim(V) \geq 2$. The fundamental representations V_i are given by the exterior powers $\overset{i}{\Lambda} V$, for $i = 1, 2, \ldots, n-1$. Giving the point $s = (x_1, \ldots, x_n)$ in S with $x_i = \mathrm{Tr}(\gamma | \overset{i}{\Lambda} V)$ is equivalent to specifying the characteristic polynomial of γ on V:

$$\begin{aligned} f(z) &= \det(z \cdot 1 - \gamma | V) \\ &= z^n - x_1 z^{n-1} + x_2 z^{n-2} - \cdots + (-1)^n. \end{aligned}$$

The discriminant $D(s)$ is equal to disc $f(z)$, so s lies in S' if and only if the characteristic polynomial $f(z)$ is separable (by which we mean that $f(z)$ has distinct roots in an algebraic closure of k).

Assume s lies in $S'(k)$. The k-algebra $K = k[z]/(f(z))$ is then étale of rank n. The permutation action of Γ on the finite set $\mathrm{Hom}(K, k^s)$ gives a homomorphism $\delta : \Gamma \to S_n$. If we identify W with S_n, by having it permute the weight spaces for T on V, then δ is conjugate to ρ_s. The torus T_s has points

$$T_s(k) = \{t \in K^* : \mathbb{N}t = 1\} \subset G(k) = SL(K).$$

The L-function of the character group is given by $\zeta_K(s)/\zeta_k(s)$.

Now consider the quasi-split unitary group $G = SU(V)$, associated to a (quasi-split) Hermitian space V with $n = \dim(V) \geq 3$ over the separable quadratic field extension E. Let $\beta \mapsto \overline{\beta}$ denote the nontrivial automorphism of E over k.

Again, the fundamental representations of G are the $\overset{i}{\Lambda} V$. These are defined over E, and for $\gamma \in G(k)$, $x_i = \mathrm{Tr}(\gamma | \overset{i}{\Lambda} V)$ is conjugate to x_{n-i}. Furthermore, if $n = 2m$, x_m lies in k. Giving the point $s = (x_1, \ldots, x_{n-1})$ in $S(k)$ is equivalent to specifying the characteristic polynomial of γ on V:

$$f(z) = z^n - x_1 z^{n-1} + x_2 z^{n-2} - \cdots + (-1)^n.$$

Again we have $D(s) = \mathrm{disc}(f)$, so s lies in $S'(k)$ precisely when $L = E[z]/(f(z))$ is an étale E-algebra of rank n.

Since $\overline{f}(z) = (-z)^n f(1/z)$, the involution $\beta \mapsto \overline{\beta}$ of E extends to an involution $z^\tau = 1/z$ of L. Let M be the fixed algebra of τ. If

$$f(z) \cdot \overline{f}(z) = z^n g(z + 1/z),$$

then $g(y)$ is separable of degree n, and $M \simeq k[y]/(g(y))$. Here is a diagram of étale k-algebras:

$$
\begin{tikzcd}
& L \arrow[dl, dash, "2"'] \arrow[ddr, dash, "n"] & \\
M \arrow[ddr, dash, "n"'] & & \\
& & E \arrow[dl, dash, "2"] \\
& k &
\end{tikzcd}
$$

The Hermitian form $\varphi(x, y) = \mathrm{Tr}_{L/E}(cxy^{\tau})$ on L/E is nondegenerate provided $c \in M^*$. For some choice of c, this space is quasi-split. The torus T_s has points

$$
\begin{array}{ccl}
T_s(k) & = & \{t \in L^* : \mathbb{N}_M t = \mathbb{N}_E t = 1\} \\
\cap & & \\
G(k) & = & SU(L, \varphi)
\end{array}
$$

When k is local or global, the L-function of the character group is

$$\zeta_L(s)\zeta_k(s)/\zeta_M(s)\zeta_E(s).$$

7. Symplectic groups

In this section, we consider the split group $G = Sp(V)$, where V is a non-degenerate symplectic space over k of dimension $2n$. We use the method of §5 to reduce to the quasi-split subgroup $H_s = \mathrm{Res}_{K/k}\mathrm{SL}_2$, where K is an étale k-algebra of rank n.

The fundamental representations of G are the virtual modules $\overset{i}{\Lambda}V - \overset{i-2}{\Lambda}V$, for $1 \leq i \leq n$, when k has characteristic zero. In general, they are always a virtual sum of the $\overset{i}{\Lambda}V$. Hence the point $s = (x_1, \ldots, x_n)$ in S, with $x_i = \mathrm{Tr}(\gamma|V_i)$ determines, and is determined by, the characteristic polynomial of γ on V:

$$\begin{aligned} f(z) &= \det(z \cdot 1 - \gamma|V) \\ &= z^{2n} - x_1 z^{2n-1} + \cdots - x_1 z + 1. \end{aligned}$$

This polynomial is palindromic:

$$f(z) = z^{2n} f(1/z)$$

as $\overset{m}{\Lambda}V$ is isomorphic to $\overset{2n-m}{\Lambda}V$. Hence

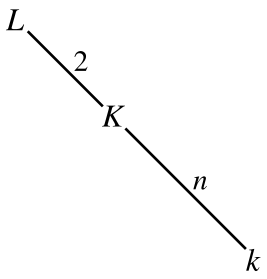
$$f(z) = z^n g(z + 1/z)$$

with $g(y) = y^n - x_1 y + \cdots$ of degree n.

We have the formulae:

$$
\begin{aligned}
D &= D_\ell \cdot D_s \\
D_\ell(s) &= (-1)^n f(1) f(-1) \\
D_s(s) &= \mathrm{disc}(g) \\
D_\ell \cdot D_s^2 &= \mathrm{disc}(f).
\end{aligned}
$$

Hence γ is regular if and only if $f(z)$ is separable. If $s \in S'(k)$, we let K be the étale k-algebra of rank n defined by $K = k[y]/(g(y))$, and L be the étale k-algebra of rank $2n$ defined by $L = k[z]/(f(z)) = K[z]/(z^2 - yz + 1)$. Here is an algebra diagram



Let τ be the nontrivial involution of L over K, defined by $z^\tau = 1/z$.

The homomorphism

$$\rho_s : \Gamma \to W = \langle \pm 1 \rangle^n \cdot S_n = W_\ell \cdot W_{ss}$$

is given by the action of Γ on the covering of finite sets

$\mathrm{Hom}(L, k^s) \longrightarrow \mathrm{Hom}(K, k^s)$. The projection

$$\delta : \Gamma \underset{\rho_s}{\longrightarrow} W \to W_{ss} = S_n$$

is given by the étale algebra K, with Galois closure E. The subgroup $H_s = \mathrm{SL}_2(K)$ is defined by the long roots, and T_s is the maximal torus in H_s defined by the quadratic extension L of K:

$$T_s(k) = \{t \in L^* : t^{1+\tau} = 1\} \subset \mathrm{SL}_2(K).$$

When k is local or global, the L-function of $X(T_s)$ is equal to $\zeta_L(s)/\zeta_K(s)$.

8. The group G_2

We now use the method of §5 to treat the split group of type G_2, the automorphisms of a split octonion algebra over k. Let V_1 denote the 7-dimensional representation of G on the octonions of trace 0; this is irreducible and fundamental provided that $\mathrm{char}(k) \neq 2$. Let V_2 denote the 14-dimensional adjoint representation; this is irreducible and fundamental provided that $\mathrm{char}(k) \neq 3$.

In general, let $x_1 = \mathrm{Tr}(\gamma|V_1)$ and $x_2 = \mathrm{Tr}(\gamma|V_2)$. These elements of k determine the stable conjugacy class of a semi-simple element γ. The characteristic polynomial of γ on the 7-dimensional representation V_1 has the form $(z-1)f(z)$, with

$$\begin{aligned}
f(z) &= z^6 - Az^5 + Bz^4 - Cz^3 + Bz^2 - Az + 1 \\
A &= x_1 - 1 \\
B &= x_2 + 1 \\
C &= x_1^2 - 2x_2 + 1 = A^2 + 2A - 2B + 2.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
D_\ell &= -4x_1^3 + x_2^2 + 10x_1x_2 + x_1^2 + 2x_1 + 10x_2 - 7 \\
D_s &= x_1^2 + 2x_1 - 4x_2 - 7.
\end{aligned}$$

Assume that the stable class defined by γ is regular. Let

$$\begin{aligned} h(\beta) &= \beta^2 - A\beta + (B - A) \\ &= \beta^2 - (x_1 - 1)\beta + (x_2 - x_1 + 2). \end{aligned}$$

Since

$$\mathrm{disc}(h) = D_s,$$

the quadratic algebra $K = k[\beta]/(h(\beta))$ is étale. This is the étale algebra defined by the projection $\rho_s : \Gamma \to W = W_\ell \cdot W_{ss} \to W_{ss} = S_2$. Its Galois closure E is either K (if K is a field) or k (if $K \simeq k + k$). In both cases, $\mathrm{Aut}_k(K) = S_2$.

Over the algebra K, we have the factorization

$$f(z) = (z^3 - \beta z^2 + \overline{\beta} z - 1)(z^3 - \overline{\beta} z^2 + \beta z - 1),$$

where $\alpha \mapsto \overline{\alpha}$ denotes the nontrivial automorphism of K. The quasi-split group H_s defined by the long roots is $\mathrm{SU}_3(K)$ (which is isomorphic to the split group SL_3 when $K = k+k$), and s is the stable class in H_s with separable characteristic polynomial $z^3 - \beta z^2 + \overline{\beta} z - 1$. The class s' with polynomial $z^3 - \overline{\beta} z^2 + \beta z - 1$ is fused with s in $G = G_2$.

Let $L = K[z]/(z^3 - \beta z^2 + \overline{\beta} z - 1) = k[z]/f(z)$. The involution $z^\tau = 1/z$ of L induces conjugation $\alpha \mapsto \overline{\alpha}$ on K/k. Let M be the fixed algebra. Then

$M = k[y]/(g(y))$ with

$$\begin{aligned} g(y) &= y^3 - (x_1 - 1)y^2 + (x_2 - 2)y - (x_1^2 - 2x_2 - 2x_1 + 1) \\ f(z) &= z^3 g(z + 1/z) \\ \mathrm{disc}(g) &= D = D_\ell \cdot D_s. \end{aligned}$$

Here is a diagram of the étale k-algebras in question:

$$\begin{tikzcd} & L \arrow[dl, dash, "2"'] \arrow[dr, dash, "3"] & \\ M \arrow[dr, dash, "3"'] & & K \arrow[dl, dash, "2"] \\ & k & \end{tikzcd}$$

The torus T_s has points

$$T_s(k) = \{t \in L^* : \mathbb{N}_K t = \mathbb{N}_M t = 1\}.$$

If k is local or global, the L-function of $X(T)$ is $\zeta_L(s)\zeta_k(s)/\zeta_K(s)\zeta_M(s)$. If $K = k + k$, the L-function is simply $\zeta_M(s)/\zeta_k(s)$, and the torus T_s has points $T_s(k) = \{t \in M^* : \mathbb{N}_k t = 1\}$.

A similar method works for the stable regular semi-simple classes s in the group $G = F_4$. Here the projection of ρ_s to $W_{ss} \to S_3$ determines an étale cubic algebra K, and the characteristic polynomial of s on the 26-dimensional representation of G factors over K as $(z-1)^2 h_8(z) g_{16}(z)$. This allows one to reduce the calculation of T_s to tori in the quasi-split long root subgroup $H_s = \mathrm{Spin}_8^K$.

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