# From Laplace to Langlands via representations of orthogonal groups

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## 1 Introduction

In the late 1960s, Robert Langlands proposed a new and far-reaching connection between the representation theory of Lie groups over real and $p$-adic fields, and the structure of the Galois groups of these fields [24]. Even though this local Langlands correspondence remains largely conjectural, the relation that it predicts between representation theory and number theory has profoundly changed our views of both fields. Moreover, we now know enough about the correspondence to address, and sometimes solve, traditional problems in representation theory that were previously inaccessible.

Roughly speaking, the local Langlands correspondence predicts that complex irreducible representations of a reductive group $G$ over a local field $k$ should be

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parametrized by certain homomorphisms of the Galois group (or more generally the Weil group) of $k$ into a complex Lie group ${}^LG$ which is dual to $G$, in a sense that will be explained below.

In this article, our aim is two-fold: First, we want to introduce the Langlands correspondence for reductive groups over local fields, giving many examples along the way. Second, we want to show how the Langlands correspondence, combined with some ideas from number theory, can be used to study the classical problem of restricting irreducible discrete series representations from $SO_{2n+1}$ to $SO_{2n}$.

The conjectures of Gross-Prasad [14] describe these restrictions in terms of symplectic local root numbers attached to the Langlands parameters of representations of orthogonal groups. These root numbers are number-theoretic invariants of the parameters, but to test the conjectures one needs to construct the corresponding irreducible representations. This has been done recently, for some interesting discrete series parameters, in [9] and [22]. Specializing this constuction to orthogonal groups, we verify the Gross-Prasad conjectures for these parameters.

Since the restriction problem we consider has roots in the very origins of representation theory, we have tried to place our results on orthogonal groups in an historical context. Moreover, as representation theory is now used in many areas of mathematics, we have also tried to make much of this paper accessible to a wide audience, by raising the prerequisites as gradually as possible. This aim has forced us to omit much recent work on the local Langlands correspondence, such as [18], [20] for $GL_n$, and [26], [27] on unipotent representations.

We begin with a discussion of spherical harmonics for the compact real group $SO(3)$ of rotations in $\mathbb{R}^3$. We will see that each irreducible representation of $SO(3)$ is naturally paired with a discrete series representation of the noncompact orthogonal group $SO(2,1)$. This is our first example of an $L$-packet of representations attached to a Langlands parameter. We then introduce the $p$-adic versions of $SO(3)$ and $SO(2,1)$, and arrive at analogous $L$-packets of discrete series representations. In both cases, we discuss the restriction of the representations to the subgroup $SO(2)$.

Next, we return to real groups, in a more general setting. After a review of the fundamental results on discrete series for real Lie groups, due to Harish-Chandra and Schmid, we partition their representations into $L$-packets, with illustrations in the orthogonal case. This permits us to formulate the Gross-Prasad conjecture on restriction from $SO_{2n+1}$ to $SO_{2n}$, which in the real case generalizes classical branching laws.

In section 6 we turn to a general description of Langlands parameters, introducing the Weil group of a local field and the complex Lie group ${}^LG$ mentioned above. Here the prerequisites increase; to help the reader we give some background on root data and quasi-split groups. We then show how the $L$-packets constructed by Harish-Chandra and Schmid correspond to real Langlands parameters, and we illustrate these parameters for orthogonal groups in more detail.

The remainder of the paper is devoted to the $p$-adic case, where much of the Langlands correspondence remains conjectural. We focus on parameters which we call "tame regular discrete", and which are analogous to discrete series parameters in the real case. In section 11 we briefly outline the recent construction of the $L$-packets of irreducible representations associated to these parameters. After a few words about Bruhat-Tits theory, we then give more details about this construction, in section 13.

We then turn to the parameters for $p$-adic orthogonal groups, and introduce local epsilon factors from number theory, in order to state the Gross-Prasad conjecture on restriction from $SO_{2n+1}$ to $SO_{2n}$ in the $p$-adic setting. The rest of the paper is devoted to verifying this conjecture, for tame regular discrete Langlands parameters. The proof invokes a result in [28] on the restriction of Deligne-Lusztig characters for finite orthogonal groups.

There is no denying that the Langlands correspondence throws many technical barriers in the way of the interested mathematician, neophyte and expert alike! We hope that the mixture of general theory with explicit examples will enable many readers to hurdle, or at least to see over these barriers, in order to appreciate this fruitful interaction between representation theory and number theory.

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# 2 Spherical Harmonics

One of the earliest results in harmonic analysis for a nonabelian group was the decomposition of square-integrable functions on the sphere $S^2$:

$$L^2(S^2) = \widehat{\bigoplus_{m\geq 0}} \pi_m \qquad \text{(Hilbert direct sum)} \tag{1}$$

into eigenspaces of the spherical Laplacian. The eigenspace denoted $\pi_m$ has dimension $2m + 1$, and has associated eigenvalue $\lambda = -m(m + 1)$. It affords an irreducible representation of the spherical rotation group $SO(3)$.

Every irreducible complex representation of the compact group $G = SO(3)$ is isomorphic to some $\pi_m$. Let $H$ be the subgroup of $G$ fixing a point on $S^2$. Then $H$ is isomorphic to the rotation group in the plane orthogonal to the line through the fixed-point and its antipode. Thus, we may identify

$$H = SO(2) = \{z \in \mathbb{C}^\times : \ |z| = 1\}.$$

For $k \in \mathbb{Z}$, let $\chi_k : H \to \mathbb{C}^\times$ be the unitary character defined by $\chi_k(z) = z^k$. Then the restriction of the representation $\pi_m$ from $G$ to $H$ decomposes as a direct sum of irreducible representations:

$$\mathrm{Res}_H(\pi_m) = \bigoplus_{|k|\leq m} \chi_k. \tag{2}$$

For all of the above see, for example, [43, III.9]

The decompositions (1) and (2) were used by theoretical physicists to model energy levels of the hydrogen atom [45]. Physicists also initiated the study of representations of the non-compact form $G' = SO(2, 1)$ of $G$; see [1]. This is the group of orientation-preserving isometries of the quadratic form $Q = x^2 + y^2 - z^2$, and is isomorphic to $PGL_2(\mathbb{R})$.

The irreducible complex representations of a Lie group $\mathcal{G}$ which occur as closed subspaces of $L^2(\mathcal{G})$ are called **discrete series**. If $\mathcal{G}$ is compact then every irreducible representation is in the discrete series.

For $G' = SO(2,1)$ the discrete series are representations $\pi'_m$, parametrized by integers $m \geq 0$. The subgroup $H \subset G'$ fixing a vector in $\mathbb{R}^3$ whose orthogonal complement is definite is isomorphic to $SO(2)$. The representation $\pi'_m$ can be characterized by its restriction to $SO(2)$:

$$\mathrm{Res}_H(\pi'_m) = \widehat{\bigoplus_{|k|>m}} \chi_k. \tag{3}$$

To summarize, for each integer $m \geq 0$ we have a set

$$\Pi_m = \{\pi_m, \pi'_m\} \tag{4}$$

of irreducible representations of $G = SO(3)$ and $G' = SO(2,1)$. Every irreducible representation $\chi_k$ of $SO(2)$ occurs once in the restriction of either $\pi_m$ or $\pi'_m$ for every $m$, but not in both. We will see that this appears to be a general phenomenon for $L$-packets (of which $\Pi_m$ is an example) of representations of orthogonal groups over real and $p$-adic fields.

# 3 $p$-adic $SO(3)$

A similar result holds for representations of the rotation group $SO(V)$, for 3-dimensional orthogonal spaces $V$ over $p$-adic fields. Let $k = \mathbb{Q}_p$ be the field of $p$-adic numbers, let $\mathfrak{o} = \mathbb{Z}_p$ be the ring of integers in $k$, and let $\mathfrak{f} = \mathfrak{o}/p\mathfrak{o} \simeq \mathbb{Z}/p\mathbb{Z}$ be the residue field. Let $k_2$ be the unramified quadratic extension of $k$, with ring of integers $\mathfrak{o}_2$, and residue field $\mathfrak{f}_2 = \mathfrak{o}_2/p\mathfrak{o}_2$, with $p^2$ elements. If $p > 2$ then $k_2 = k(\sqrt{u})$ and $\mathfrak{o}_2 = \mathfrak{o}[\sqrt{u}]$ for $u \in \mathfrak{o}^\times$ a unit which is not a square. Let $\tau$ be the nontrivial automorphism of $k_2$ which is trivial on $k$; on $\mathfrak{o}_2$ we have the formula

$$\tau(a) \equiv a^p \mod p\mathfrak{o}_2.$$

We begin with two-dimensional spaces which are analogous to the complex plane. The space $W = k_2$ is an orthogonal space over $k$ of rank two, whose quadratic form $a \mapsto a \cdot \tau(a)$ is the norm from $k_2$ to $k$. The rotation group of $W$ is

$$H = SO(W) = \{a \in k_2^\times : \tau(a) = a^{-1}\} = \{a \in \mathfrak{o}_2^\times : \tau(a) = a^{-1}\}.$$

There is extra structure that did not appear in the real case: we also have a quadratic $\mathfrak{f}$-space $\bar{W} = \mathfrak{f}_2$ with quadratic form $\alpha \mapsto \alpha^{p+1}$. Reduction modulo $p$ from $\mathfrak{o}_2$ to $\mathfrak{f}_2$ gives a surjective group homomorphism

$$H \longrightarrow \bar{H} = SO(\bar{W}) = \{\alpha \in \mathfrak{f}_2^\times : \alpha^p = \alpha^{-1}\}.$$

The group $\bar{H}$ is cyclic of order $p+1$. We say a character $\chi$ of $H$ is **tame** if it factors through $\bar{H}$, and is **regular** if $\chi^2 \neq 1$.

In three dimensions, there are exactly two distinct orthogonal spaces of rank three over $k$, up to equivalence and scaling [32, Chap. IV]. We denote them by $(V, Q)$ and $(V', Q')$. They both contain $W$ and have orthogonal decompositions

$$\begin{aligned} V &= W \oplus \langle e \rangle, & Q(e) &= p, \\ V' &= W \oplus \langle e' \rangle, & Q'(e') &= 1. \end{aligned}$$

In the first case, $-Q(e) = -p$ is not a norm from $k_2^\times$, $Q$ does not represent zero nontrivially on $V$, and $G = SO(V)$ is compact. In the second case, $-Q'(e') = -1$ is a norm from $k_2^\times$, $Q'$ has nontrivial zeros on $V'$, and $G' = SO(V')$ is noncompact. The group $H = SO(W)$ is a subgroup of both $G$ and $G'$, fixing the vectors $e$ and $e'$, respectively.

We will now construct irreducible representations $\pi_\chi$ and $\pi'_\chi$ of $G$ and $G'$, starting with tame regular characters $\chi$ of $H$.

The group $G$ preserves the lattice $L = \mathfrak{o}_2 \oplus \mathfrak{o}e$ in $V$, and the form induced by $Q$ on $L/pL$ has one dimensional radical. This implies a surjection $G \to O(\bar{W})$. Hence $G$ has a normal subgroup $K$ of index two, which surjects onto $\bar{H} = SO(\bar{W})$. We view $\chi$ as a character of $K$, and define $\pi_\chi$ as the induced representation:

$$\pi_\chi = \operatorname{Ind}_K^G \chi. \tag{5}$$

More precisely, induced representations are defined as follows. If $K$ is a finite-index subgroup of a group $G$ and $\rho : K \to GL(V_\rho)$ is a representation of $K$, then

$$\operatorname{Ind}_K^G \rho = \{ f : G \to V_\rho \ \text{ such that } \ f(kg) = \rho(k) f(g), \ \text{ for all } k \in K, \ g \in G \}, \tag{6}$$

and the group $G$ acts on $\operatorname{Ind}_K^G \rho$ by $(g \cdot f)(x) = f(xg)$ for $x, g \in G$. We have $\dim \operatorname{Ind}_K^G \rho = [G : K] \cdot \dim(V_\rho)$. We will use various elementary properties of induced representations without further comment. See, for example, [34, chap. 7].

Returning to $\pi_\chi$, we have $\dim(\pi_\chi) = 2$. Since $\chi$ is conjugate to $\chi^{-1}$ in $O(W) \subset G$, it follows that $\pi_\chi$ is isomorphic to $\pi_{\chi^{-1}}$. Restricting $\pi_\chi$ to $H$, we find

$$\operatorname{Res}_H(\pi_\chi) = \chi \oplus \chi^{-1}. \tag{7}$$

.

To define $\pi'_\chi$, it is convenient to view the noncompact group $G'$ as $PGL_2(k)$, whose Lie algebra is a three-dimensional quadratic space isometric to $(V', Q')$. This group has a maximal compact subgroup

$$K' = PGL_2(\mathfrak{o}) \subset G'.$$

In turn, $K'$ maps surjectively onto $\bar{G}' = PGL_2(\mathfrak{f})$. The latter is a finite group of order $p(p^2-1)$. The group $\bar{H}$ embeds as a subgroup of $\bar{G}'$, and a cohomological form of induction, due to Drinfeld in this case and extended by Deligne and Lusztig [10], associates to the data $(\bar{H}, \chi)$ an irreducible representation $R(\chi)$ of $\bar{G}'$, of dimension $p-1$. For the particular group $\bar{G}' = PGL_2(\mathfrak{f})$, the character of the representation $R(\chi)$ was discovered by Frobenius [11, section 9] in one of his first papers on representation theory.

The irreducible representation $R(\chi)$ of $\bar{G}'$ is characterized by its restriction to $\bar{H}$:

$$\mathrm{Res}_{\bar{H}}\,(R(\chi)) = \bigoplus_{\substack{\eta \in \mathrm{Irr}(\bar{H}) \\ \eta \neq \chi, \chi^{-1}}} \eta.$$

We pull $R(\chi)$ back to a representation of $K'$ via the homomorphism $K' \to PGL_2(\mathfrak{f})$, and define

$$\pi'_\chi = \mathrm{Ind}_{K'}^{G'} R(\chi). \tag{8}$$

The group $K'$ has infinite index in $G'$. Induction is defined as in (6), but now we require the functions $f \in \mathrm{Ind}_{K'}^{G'} R(\chi)$ to have compact support, and to be invariant under a compact open subgroup of $G'$.

The representation $\pi'_\chi$ is irreducible and belongs to the discrete series of $G'$. All characters of $H$ appear in the restriction $\mathrm{Res}_H(\pi'_\chi)$ with multiplicity one, except for $\chi$ and $\chi^{-1}$. See, for example, [41].

In summary, as in the real case, the representations $\pi_\chi$ of $G$ and $\pi'_\chi$ of $G'$, defined in (5) and (8), form an $L$-packet

$$\Pi_\chi = \{\pi_\chi, \pi'_\chi\}$$

with the property that every character of $H$ appears with multiplicity one in $\pi_\chi$ or $\pi'_\chi$, but not both.

# 4 Discrete series for real groups

It would now be helpful if the reader had some familiarity with the basic structure theory of semisimple Lie groups (cf. [42]). It may also help to glance at the next section, where the general ideas below are illustrated for orthogonal groups.

A real semisimple Lie group $G$ has discrete series representations precisely when it contains a compact maximal torus. The discrete series of such groups were constructed in a uniform way, first by Harish-Chandra [17] on the level of characters, later by Schmid [30] on the level of vector spaces.

There are many excellent treatments of discrete series for real groups in the literature, for example [31]. The following sketch has the advantage of showing how discrete series are naturally grouped into $L$-packets containing representations of different inner forms of a compact Lie group. We also avoid certain technicalities by restricting the class of Lie groups under discussion. Our treatment applies to all real orthogonal groups that have discrete series, like $SO(2,1) = PGL_2(\mathbb{R})$, as well as to all semisimple Lie groups with trivial center, but it omits other Lie groups with discrete series, like $SL(2,\mathbb{R})$. The Lie groups we consider are those obtained from compact Lie groups $G$ by twisting inside the complexification $G_{\mathbb{C}}$, as will be explained below.

Let $G$ be a compact connected Lie group with finite center. Up to conjugacy, $G$ contains a unique maximal torus $S \simeq (S^1)^n$, a direct product of copies of the circle group $S^1$. The number $n$ is called the **rank** of $G$.

Associated to $G$ is its **complexification** $G_{\mathbb{C}}$ which is a connected complex Lie group containing $G$ as a maximal compact subgroup. For example, if $G = SO(2n+1)$ is the rotation group of a positive definite quadratic form $Q$ on $\mathbb{R}^{2n+1}$, then $G_{\mathbb{C}} = SO(2n+1,\mathbb{C})$ is the subgroup of $SL(2n+1,\mathbb{C})$ preserving the extension of $Q$ to $\mathbb{C}^{2n+1}$.

We need to recall some structure theory of complex Lie groups. The group $G_{\mathbb{C}}$ contains the complexification $S_{\mathbb{C}} \simeq (\mathbb{C}^\times)^n$ of $S$. We choose a Borel subgroup (i.e., a maximal connected solvable subgroup) of $G_{\mathbb{C}}$ containing $S_{\mathbb{C}}$.

Borel subgroups can also be characterized as follows. There is a unique (up to isomorphism) complex projective variety $\mathcal{B}$ of maximal dimension, on which $G_{\mathbb{C}}$ acts transitively by holomorphic maps. The variety $\mathcal{B}$ is called the **flag variety** of $G_{\mathbb{C}}$. The Borel subgroups in $G_{\mathbb{C}}$ are the stabilizers of points in the flag variety $\mathcal{B}$. In fact, Borel subgroups are their own normalizers, so we can think of $\mathcal{B}$ as the variety of Borel subgroups of $G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ acts by conjugation. The complex torus $S_{\mathbb{C}}$ has only finitely many fixed points in $\mathcal{B}$. By choosing $B_{\mathbb{C}}$ to contain $S_{\mathbb{C}}$, we are choosing $B_{\mathbb{C}} \in \mathcal{B}$ to be one of these $S_{\mathbb{C}}$-fixed points. We can then identify $\mathcal{B} = G_{\mathbb{C}}/B_{\mathbb{C}}$.

The flag varieties of classical groups are familiar objects from algebraic geometry. For example, a Borel subgroup in $SO(2n+1,\mathbb{C})$ is the stabilizer of an isotropic flag in $\mathbb{C}^{2n+1}$. The latter are nested chains of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{2n+1},$$

with $\dim V_i = i$, on which the quadratic form $Q$ vanishes identically. In particular, the flag variety of $SO(3,\mathbb{C})$ is the quadric in $\mathbb{CP}^2$ defined by $Q$, and is isomorphic to $\mathbb{CP}^1$.

We now recall a bit more structure theory. The group $S_{\mathbb{C}}$ acts on the Lie algebras $\mathfrak{g}$ and $\mathfrak{b}$ of $G_{\mathbb{C}}$ and $B_{\mathbb{C}}$, respectively. Hence $S_{\mathbb{C}}$ acts on the quotient $\mathfrak{g}/\mathfrak{b}$. The **positive roots** of $S_{\mathbb{C}}$ are the characters of $S_{\mathbb{C}}$ appearing in its action on $\mathfrak{g}/\mathfrak{b}$. We let $\Phi^+$ denote the set of positive roots. Within $\Phi^+$ there is a unique subset $\Delta$ of **simple roots**, with the property that every root in $\Phi^+$ can be written uniquely as a non-negative integral combination of roots in $\Delta$. Each positive root appears in $\mathfrak{g}/\mathfrak{b}$ with multiplicity one. Since $\mathfrak{g}/\mathfrak{b}$ is the tangent space to $\mathcal{B}$ at $B_{\mathbb{C}}$, we have

$$\dim \mathcal{B} = |\Phi^+|.$$

The **negative roots** $\Phi^-$ are those nontrivial characters of $S_{\mathbb{C}}$ appearing in $\mathfrak{b}$ (this seemingly odd convention is standard in the theory).

We have not forgotten the compact group $G$ that we started with. There is an analogue of complex conjugation, which is an automorphism of order two of the abstract group $G_{\mathbb{C}}$, denoted $g \mapsto \bar{g}$. This automorphism preserves $S_{\mathbb{C}}$, and we have

$$G = \{g \in G_{\mathbb{C}} :\ \bar{g} = g\}, \qquad S = \{s \in S_{\mathbb{C}} :\ \bar{s} = s\}.$$

Since the automorphism $g \mapsto \bar{g}$ preserves $S_{\mathbb{C}}$, it permutes the roots. In fact, it interchanges $\Phi^+$ and $\Phi^-$.

We will twist this automorphism to obtain various non-compact real groups inside $G_{\mathbb{C}}$, as follows. Let $S_2 = \{s \in S :\ s^2 = 1\}$ be the 2-torsion subgroup of $S$. Then

$$S_2 \simeq \{\pm 1\}^n$$

is an elementary abelian two-group of rank $n$. For each $s \in S_2$, we define a subgroup $G_s \subset G_{\mathbb{C}}$ by

$$G_s := \{g \in G_{\mathbb{C}} :\ \bar{g} = sgs\}.$$

(Recall that $s = s^{-1}$.) In other words, $G_s$ consists of the fixed-points of the nontrivial involution $g \mapsto s\bar{g}s$. The group $G_s$ is a real Lie group, which is usually

non-compact. The centralizer $K_s$ of $s$ in $G$ is a maximal compact subgroup of $G_s$, and
$$S \subset K_s \subset G_s. \tag{9}$$
When $s = 1$, we have $K_s = G_s = G$.

Since the automorphism $g \mapsto \bar{g}$ interchanges $\Phi^+$ and $\Phi^-$, it follows that

$$G_s \cap B_{\mathbb{C}} = S,$$

for any $s \in S_2$. Therefore, the homogeneous space $G_s/S$ can be identified with the $G_s$-orbit of $B_{\mathbb{C}}$ in $\mathcal{B}$. This orbit is open in $\mathcal{B}$, so the manifold $G_s/S$ acquires a complex structure, via its identification with the domain

$$\mathcal{D}_s = G_s B_{\mathbb{C}}/B_{\mathbb{C}} \subset G_{\mathbb{C}}/B_{\mathbb{C}} = \mathcal{B}.$$

The homogenous space $K_s/S$ also has a complex structure; it is the flag variety of the complexification of $K_s$. In fact, $K_s/S$ is a maximal compact complex submanifold of the domain $\mathcal{D}_s = G_s/S$. The fibration

$$K_s/S \hookrightarrow \mathcal{D}_s = G_s/S \longrightarrow G_s/K_s$$

of real manifolds shows that $\mathcal{D}_s$ can be thought of as the family of flag varieties of maximal compact subgroups of $G_s$. For $G = SO(3)$, we have $\mathcal{D}_1 = \mathcal{B} = \mathbb{CP}^1$ and $\mathcal{D}_{-1} = \mathbb{CP}^1 - \mathbb{RP}^1$ is the union of two copies of the unit disk.

The existence and parametrization of the discrete series was obtained by Harish-Chandra, in terms of their characters. The vector space realization of the discrete series, in terms of cohomology of line bundles on the domain $\mathcal{D}_s$, is due to Schmid, and was in part conjectured by Langlands. We now sketch Schmid's construction.

Our line bundles will be restrictions to $\mathcal{D}_s$ of $G_{\mathbb{C}}$-equivariant holomorphic line bundles on $\mathcal{B}$. The latter correspond bijectively to (algebraic) characters of $B_{\mathbb{C}}$, or even to characters of $S_{\mathbb{C}}$, because $B_{\mathbb{C}}$ is the semidirect product $B_{\mathbb{C}} = S_{\mathbb{C}} \ltimes U_{\mathbb{C}}$ of $S_{\mathbb{C}}$ by the commutator subgroup $U_{\mathbb{C}}$ of $B_{\mathbb{C}}$, so we may identify the characters of $B_{\mathbb{C}}$ and $S_{\mathbb{C}}$.

Given a character $\chi : S_{\mathbb{C}} \to \mathbb{C}^\times$, we pull $\chi$ back to $B_{\mathbb{C}}$ and define

$$\mathcal{L}(\chi) = G_{\mathbb{C}} \times_{B_{\mathbb{C}}} \mathbb{C}_\chi,$$

which is the quotient of $G_{\mathbb{C}} \times \mathbb{C}$ by the $B_{\mathbb{C}}$-action: $b \cdot (g, z) = (gb^{-1}, \chi(b)z)$. Projection onto the left factor makes $\mathcal{L}(\chi)$ a holomorphic line bundle over $\mathcal{B}$ on which $G_{\mathbb{C}}$ acts by holomorphic vector bundle maps.

Written additively, the character group $X(S_{\mathbb{C}})$ of $S_{\mathbb{C}}$ is isomorphic to $\mathbb{Z}^n$; to $(\chi_1, \ldots, \chi_n) \in \mathbb{Z}^n$ corresponds the character $\chi : S_{\mathbb{C}} \to \mathbb{C}^\times$ given by

$$\chi(z_1, \ldots, z_n) = \prod_{k=1}^{n} z_k^{\chi_k}.$$

So the line bundles $\mathcal{L}(\chi)$ constructed above are parametrized by points $\chi$ in the lattice $X(S_{\mathbb{C}})$. It turns out to be helpful to shift this lattice by the element

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Note that $\rho$ does not necessarily belong to $X(S_{\mathbb{C}})$, but $2\rho$, being a sum of roots, does belong to $X(S_{\mathbb{C}})$. Since $2\rho$ is the sum of the characters of $S_{\mathbb{C}}$ in the tangent space $\mathfrak{g}/\mathfrak{b}$ to $\mathcal{B}$ at $B_{\mathbb{C}}$, it follows that $\mathcal{L}(-2\rho)$ is the canonical bundle of $\mathcal{B}$, whose sections are the top-dimensional holomorphic differential forms on $\mathcal{B}$.

We are going to construct representations of $G_s$ parametrized by those elements of the shifted lattice $\rho + X(S_{\mathbb{C}})$ which satisfy a certain positivity property. We call elements of $\rho + X(S_{\mathbb{C}})$ "characters" although they may not be characters of $S_{\mathbb{C}}$, but rather of a two-fold cover of $S_{\mathbb{C}}$.

To state the positivity property, we must first recall the notion of co-root. For each root $\alpha \in \Phi^+$, there is a homomorphism

$$\psi_\alpha : SL_2(\mathbb{C}) \longrightarrow G_{\mathbb{C}}$$

with differential $d\psi_\alpha$, sending the diagonal matrices of $SL_2(\mathbb{C})$ into $S_{\mathbb{C}}$ and such that $d\psi_\alpha\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \in \mathfrak{g}$ projects to a non-zero vector in the $\alpha$-eigenspace of $S_{\mathbb{C}}$ in $\mathfrak{g}/\mathfrak{b}$. The 1-parameter subgroup

$$\check{\alpha} : \mathbb{C}^\times \longrightarrow S_{\mathbb{C}}, \qquad z \mapsto \psi_\alpha\left(\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}\right)$$

is called the **co-root** of $\alpha$. Each co-root $\check{\alpha}$ defines a homomorphism

$$\langle\,\cdot\,, \check{\alpha}\rangle : X(S_{\mathbb{C}}) \to \mathbb{Z} \quad \text{such that} \quad \chi\left(\check{\alpha}(z)\right) = z^{\langle \chi, \check{\alpha}\rangle}.$$

The character $\rho$ has the property that

$$\langle 2\rho, \check{\alpha}\rangle = 2 \tag{10}$$

for all simple roots $\alpha \in \Delta$. It follows that $\langle \chi, \check{\alpha} \rangle \in \mathbb{Z}$ for every character $\chi \in \rho + X(S_{\mathbb{C}})$. We say that $\chi \in \rho + X(S_{\mathbb{C}})$ is **positive** if $\langle \chi, \check{\alpha} \rangle \geq 1$ for all $\check{\alpha} \in \Phi^+$. By (10), the character $\chi = \rho$ is positive, and is the least positive of all positive characters in $\rho + X(S_{\mathbb{C}})$.

To every positive character $\chi \in \rho + X(S_{\mathbb{C}})$, we will associate $2^n$ irreducible discrete series representations $\pi(\chi, s)$, indexed by the elements $s \in S_2$. Take $s \in S_2$, giving rise, as above, to a domain $\mathcal{D}_s \subset \mathcal{B}$, with $\mathcal{D}_s \simeq G_s/S$. Since $G_s$ acts on $\mathcal{D}_s$ and on the restriction of the equivariant line bundles $\mathcal{L}(\cdot)$, the group $G_s$ acts on the holomorphic sheaf cohomology groups

$$H^i(\mathcal{D}_s, \mathcal{L}(-\chi - \rho)).$$

Schmid proves that $H^i(\mathcal{D}_s, \mathcal{L}(-\chi - \rho))$ is nonzero only in degree $i = q(s)$, where $q(s)$ is the complex dimension of $K_s/S$. This nonvanishing cohomology group

$$\pi(\chi, s) := H^{q(s)}(\mathcal{D}_s, \mathcal{L}(-\chi - \rho))$$

is a Fréchet space affording an irreducible representation $\pi(\chi, s)$ of $G_s$, which Schmid proves is in the discrete series (he shows that the canonical map from $L^2$-cohomology to sheaf cohomology is injective with dense image).

When $s = 1$, the group $G = G_s = K_s$ is compact, and $\pi(\chi, s)$ is the finite dimensional representation of $G$ with highest weight $\chi - \rho$ for $S$, by the Borel-Weil-Bott theorem. In general, the cohomological restriction map

$$H^{q(s)}(G_s/S, \mathcal{L}(-\chi - \rho)) \longrightarrow H^{q(s)}(K_s/S, \mathcal{L}(-\chi - \rho))$$

is surjective, so that the irreducible $K_s$-representation $H^{q(s)}(K_s/S, \mathcal{L}(-\chi - \rho))$, of highest weight $\chi - \rho_s$ appears in $\pi(\chi, s)$, where

$$\rho_s = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha(s)\alpha. \tag{11}$$

(Note that $\alpha(s) = \pm 1$ for any $\alpha \in \Phi^+$, since $s^2 = 1$.) Schmid proves, moreover, that $\pi(\chi, s)$ is the unique irreducible representation of $G_s$ containing the $K_s$-representation of highest weight $\chi - \rho_s$ with multiplicity one and no other $K_s$-representations of the form $\chi - \rho_s + A$, where $A$ is a non-empty sum of roots in $\Phi^+$ which are trivial on $s$. Thus, $\pi(\chi, s)$ is completely characterized by its restriction to the maximal compact subgroup $K_s$ of $G_s$.

To summarize: For every positive character $\chi \in \rho + X(S_{\mathbb{C}})$, we have constructed $2^n$ discrete series representations $\pi(\chi, s)$, one for each group $G_s$, with $s \in S_2$. We call the set of these representations

$$\Pi(\chi) := \{\pi(\chi, s) : \ s \in S_2\}$$

an **L-packet**.

We still have a few loose ends to tie up. The first point is that many of the groups $G_s$ will be isomorphic to one another, so we are really getting different representations of the same group. This is because of the action of the **Weyl group** $W := N_G(S)/S$, the quotient of the normalizer of $S$ in $G$ by $S$. The Weyl group acts on $S$ by conjugation (since $S$ is abelian), preserving the finite subgroup $S_2$. If $s'$ is in the $W$-orbit of $s$ in $S_2$, then $s' = s^n = n^{-1}sn$ for some $n \in N_G(T)$. We find that $G_{s'} = n^{-1}G_s n$ in $G_{\mathbb{C}}$ and $K_{s'} = n^{-1}K_s n$ in $G$. Hence conjugation by $n$ gives isomorphisms

$$\begin{array}{ccccc} S & \hookrightarrow & K_s & \hookrightarrow & G_s \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ S & \hookrightarrow & K_{s'} & \hookrightarrow & G_{s'}. \end{array} \tag{12}$$

The choice of $n$ such that $s' = s^n$ is not unique. However, two choices of $n$ differ by an element of $N_G(S) \cap K_s$. This implies that if $\pi : G_{s'} \to GL(V)$ is a representation of $G_s$ on a Hilbert space $V$, then the isomorphism class of the representation $\pi^n : G_s \longrightarrow GL(V)$ given by $\pi^n(g) = \pi(ngn^{-1})$ is independent of the choice of $n$. Hence, we have two representations

$$\pi(\chi, s) \quad \text{and} \quad \pi(\chi, s')^n$$

of $G_s$. It turns out that these two representations of $G_s$ are isomorphic if and only if $s = s'$. Thus, we have constructed exactly $|W \cdot s|$ distinct representations of $G_s$. If we choose representatives $s_1, \ldots s_m$ of the $W$-orbits in $S_2$, then our $L$-packet $\Pi(\chi)$ is partitioned as

$$\Pi(\chi) = \coprod_{p=1}^{m} \Pi(\chi, p), \tag{13}$$

where $\Pi(\chi, p)$ consists of $[W : W_{s_p}]$ distinct representations of $G_{s_p}$.

There is one more subtle point: it could even happen that $G_s \simeq G_{s'}$ with $s, s'$ belonging to different $W$-orbits in $S_2$. In fact, $G_s \simeq G_{s'}$ if and only if there is $w \in W$ such that $s^w = zs'$, for some $z$ in the center of $G$. Thus, it can happen that the same representation of the same group appears more than once in our $L$-packet $\Pi(\chi)$. This phenomenon cannot happen for odd orthogonal groups, since these have trivial center. However, it does happen for even orthogonal groups, as will be illustrated in the next section.

# 5 Discrete series for real orthogonal groups and branching laws

We now illustrate the previous general theory of discrete series in the case of real orthogonal groups in higher dimensions.

Once again, we begin with spherical harmonics. Assume that $d \geq 2$. The square-integrable functions on the sphere $S^d$ behave like (1): there is a decomposition

$$L^2(S^d) = \hat{\bigoplus_{m \geq 0}} \pi_m, \tag{14}$$

into eigenspaces of the Laplacian, where $\pi_m$ is the eigenspace with eigenvalue $-m(m+d-1)$. The space $\pi_m$ is the restriction to $S^d$ of the harmonic polynomials of degree $m$ on $\mathbb{R}^{d+1}$; it affords an irreducible representation of the rotation group $SO(d+1)$, of dimension

$$\dim \pi_m = \binom{d+m}{d} - \binom{d+m-2}{d}.$$

For all of the above, see, for example, [21, p.17, Thm. 3.1].

For $d \geq 3$, not all irreducible representations of $SO(d+1)$ appear in $L^2(S^d)$. A representation $\pi$ appears in $L^2(S^d)$ precisely when $\pi$ contains a nonzero vector invariant under $SO(d)$. Equivalently, the representations $\pi_m$ occurring in (14) are precisely the irreducible representations of $SO(d+1)$ whose restriction to $SO(d)$ contains the trivial representation.

The decomposition of an irreducible representation of $SO(d+1)$ when restricted to $SO(d)$ is known classically as a **branching law**. Here we will generalize branching laws to the restriction of discrete series representations of both compact and non-compact orthogonal groups.

We begin by describing the discrete series parameters for the group $G = SO(2n+1)$. Here $S = SO(2)^n \simeq (S^1)^n$, and the character group of $S_{\mathbb{C}} = (\mathbb{C}^\times)^n$ is

$$X(S_{\mathbb{C}}) = \bigoplus_{k=1}^{n} \mathbb{Z} e_k,$$

where $e_k : S_{\mathbb{C}} \to \mathbb{C}^\times$ is projection onto the $k^{th}$ factor of $S_{\mathbb{C}}$. The Weyl group $W_n$ is the group of the $n$-cube, of order $2^n n!$. Viewed as a group of permutations of $X(S_{\mathbb{C}})$, the group $W_n$ is the semidirect product of the symmetric group $\mathfrak{S}_n$ of permutations of the $e_i$, with the normal subgroup $\{\pm 1\}^n$ of sign changes in the $e_i$. We may choose our Borel subgroup $B_{\mathbb{C}}$ so that

$$\Phi^+ = \{e_i \pm e_j :\ 1 \le i < j \le n\} \cup \{e_k :\ 1 \le k \le n\}, \tag{15}$$

and

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{k=1}^{n} (n - k + \tfrac{1}{2}) e_k = (n - \tfrac{1}{2}, n - \tfrac{3}{2}, \dots, \tfrac{1}{2}).$$

The positive characters $\chi \in \rho + X(S_{\mathbb{C}})$ are given by $\chi = \sum \chi_k e_k$ with each $\chi_k \in \frac{1}{2} + \mathbb{Z}$ and

$$\chi_1 > \chi_2 > \cdots > \chi_n > 0.$$

We have $S_2 = \{\pm 1\}^n \subset SO(2)^n$, and the $W_n$-orbit of $s \in S_2$ is determined by the number of components $k$ such that $e_k(s) = -1$. If $s \in S_2$ has $p$ such components equal to $-1$ and $p + q = n$ then

$$G_s \simeq SO(2p, 2q + 1)$$

is the special orthogonal group of a real quadratic form with signature $(2p, 2q+1)$.

The stabilizer of $s$ in $W_n$ is $W_p \times W_q$, so the $W$-orbit of $s$ has size $\binom{n}{p}$. This is the number of distinct discrete series representations of $G_s$ in the $L$-packet $\Pi(\chi) = \Pi_G(\chi)$.

For the $L$-packet $\{\pi_m, \pi'_m\}$ defined in (4) for $n = 1$, we have $\chi = (m + \frac{1}{2}) e_1$. The line bundle $\mathcal{L}_m = \mathcal{L}(-\chi - \rho)$ on $\mathbb{CP}^1$ is the one traditionally denoted by $\mathcal{O}(-2-2m)$. The representations $\pi_m,\ \pi'_m$ are realized on the cohomology groups

$$\pi_m = H^1(\mathbb{CP}^1, \mathcal{L}_m), \quad \text{and} \quad \pi'_m = H^0(\mathbb{CP}^1 - \mathbb{RP}^1, \mathcal{L}_m).$$

The group $H = SO(2n)$ contains the same maximal torus as $SO(2n + 1)$, so we again have $X(S_{\mathbb{C}}) = \oplus_{k=1}^n \mathbb{Z} e_k$. The Weyl group $W$ is the subgroup of index two in $W_n$, generated by $\mathfrak{S}_n$ and an even number of sign changes. We may choose our Borel subgroup $B_{\mathbb{C}}$ so that

$$\Phi^+ = \{e_i \pm e_j :\ 1 \le i < j \le n\}, \tag{16}$$

and

$$\rho = \sum_{k=1}^{n} (n-k)e_k = (n-1, n-2, \dots, 1, 0).$$

The positive characters $\eta \in \rho + X(S_{\mathbb{C}})$ are those of the form $\eta = \sum_k \eta_k e_k$ with $\eta_k \in \mathbb{Z}$ and

$$\eta_1 > \eta_2 > \cdots > \eta_{n-1} > |\eta_n|.$$

If $t \in S_2$ has $p$ components $k$ such that $e_k(t) = -1$ and $p + q = n$ then $H_t \simeq SO(2p, 2q)$. For each positive character $\eta$ for $H$, the construction of the previous section gives us an irreducible discrete series representation of $H_t$, which we will denote by $\sigma(\eta, t)$, so as not to confuse it with the analogous representation $\pi(\chi, s)$ of $G_s$. We then have a discrete series $L$-packet $\Pi_H(\eta) = \{\sigma(\eta, t) : \ t \in S_2\}$.

The $W$-orbit of $t$ again has size $\binom{n}{p}$. Since $SO(2p, 2q) \simeq SO(2q, 2p)$, we see that there are two $W$-orbits in $S_2$ giving rise to the isomorphic groups $H_t \simeq H_{-t}$, when $p \neq q$. These discrete series representations occur twice in $\Pi_H(\eta)$, once for $H_t \simeq SO(2p, 2q)$, and once for $H_{-t} \simeq SO(2q, 2p)$.

For the one-dimensional representation $\chi_m$ of $SO(2)$, we have $\eta = me_1$. The $L$-packet $\Pi_H(\eta)$ consists of the two occurrences of $\chi_m$, once for $SO(2, 0)$ and once for $SO(0, 2)$.

Let $\chi$ and $\eta$ be positive characters for $G$ and $H$ respectively, giving rise to $L$-packets $\Pi_G(\chi)$ and $\Pi_H(\eta)$, as above.

In [14], a branching law was proposed for the multiplicity of representations in $\Pi_H(\eta)$ in the restriction of representations in $\Pi_G(\chi)$. According to this conjecture, the characters $\chi = \sum \chi_k e_k$ and $\eta = \sum \eta_k e_k$ determine elements $s, t \in S_2$ by the conditions

$$e_k(s) = (-1)^{\nu(k)}, \qquad e_k(t) = (-1)^{\mu(k)}, \tag{17}$$

where $\nu(k)$ is the total number of $i, j$ such that $\chi_i, \eta_j \leq \chi_k$, and $\mu(k)$ is the total number of $i, j$ such that $\chi_i, \eta_j \geq \eta_k$.

The coordinates of $s$ and $t$ have the same number $p$ of coordinates equal to $-1$, so

$$\pi := \pi(\chi, s) \quad \text{is a representation of } G_s \simeq SO(2p, 2q+1)$$

and

$$\sigma := \sigma(\eta, t) \quad \text{is a representation of } H_t \simeq SO(2p, 2q).$$

The prediction is that $\sigma$ occurs in the restriction of $\pi$, with multiplicity one, i.e., that

$$\dim \operatorname{Hom}_{H_t}(\pi, \sigma) = 1, \tag{18}$$

and that no other representation in the $L$-packet $\Pi_H(\eta)$ should appear in the restriction of a representation in the $L$-packet $\Pi_G(\chi)$.

Formula (18) was proved in [15], in the cases where the restriction of $\pi$ to $SO(2p, 2q)$ decomposes discretely as a Hilbert direct sum. (In most cases, the restriction will have continuous spectrum.) This discrete decomposition occurs when $\chi = \sum \chi_k e_k$ and $\eta = \sum \eta_k e_k$ interlace in the following way: For some integer $0 \leq p \leq n$ we have

$$\eta_1 > \chi_1 > \cdots > \eta_p > \chi_p, \qquad \chi_{p+1} > \eta_{p+1} > \cdots > \chi_n > |\eta_n|. \tag{19}$$

Then $s = t$ and we have

$$e_k(s) = e_k(t) = \begin{cases} -1 & \text{if} \quad 1 \leq k \leq p \\ +1 & \text{if} \quad p < k \leq n. \end{cases}$$

The representations $\pi = \pi(\chi, s)$ and $\sigma = \sigma(\eta, s)$ are "small" discrete series: their restriction to the compact subgroup $SO(2p) \subset H_s \subset G_s$ contains each irreducible representation of $SO(2p)$ with finite multiplicity.

For example, if

$$\chi_1 > \eta_1 > \chi_2 > \eta_2 > \cdots > \chi_n > |\eta_n|, \tag{20}$$

then $p = 0$, and $s = t = 1$. In this case, $G = SO(2n+1)$ and $H = SO(2n)$ are compact, and we recover a classical branching law (see [2]) for the finite dimensional representation $\pi(\chi, 1)$:

$$\mathrm{Res}_H\left(\pi(\chi, 1)\right) \simeq \bigoplus_{\eta} \sigma(\eta, 1),$$

where we sum over all positive characters $\eta$ for $SO(2n)$ satisfying (20).

The trivial representation of $SO(2n)$ is $\pi(\rho, 1)$, where we recall that $\rho = \rho_H = \sum(n-k)e_k$ for $H = SO(2n)$. For $\eta_k = n - k$, the characters $\chi = \sum \chi_k e_k$ satisfying (20) are those of the form

$$\chi = me_1 + \rho, \qquad m \geq 0,$$

where we recall that $\rho = \rho_G = \sum_{k=1}^{n}(n-k+\frac{1}{2})e_k$ for $G = SO(2n+1)$. The representation $\pi(me_1+\rho, 1)$ is the representation $\pi_m$ appearing in the decomposition (14) of functions on the sphere $S^d$.

At the other extreme, if

$$\eta_1 > \chi_1 > \eta_2 > \chi_2 > \cdots > \eta_n > \chi_n, \tag{21}$$

we have $p = n$, $G_s = SO(2n, 1)$ and $H_s = SO(2n)$ has index two in the maximal compact subgroup $K_s = O(2n)$ of $G_s$. We recover another known branching law, which is a special case of Blattner's formula, proved by Hecht and Schmid [19]:

$$\mathrm{Res}_{H_s}(\pi(\chi, s)) = \hat{\bigoplus_{\eta}} \sigma(\eta, s),$$

a Hilbert direct sum over all $\eta = \sum \eta_k e_k$ satisfying (21).

# 6 Introduction to Langlands parameters

All of the results on branching laws for orthogonal groups which we have discussed so far fit nicely into a general conjecture [14], using the language of Langlands parameters for irreducible representations of reductive groups $G$ over local fields $k$. The rough idea is that irreducible representations of these groups should be parametrized by homomorphisms from the Weil group of $k$ into a group ${}^LG$ which is dual to $G$ and encodes the arithmetic structure of $G$ as a group over $k$. Then properties of irreducible representations, such as branching laws, should be determined by number-theoretic invariants of the corresponding parameters.

Historically, and in this paper so far, this connection between number theory and representation theory was not evident in the real case, because the Galois theory of $\mathbb{R}$ is rather simple. However, it is enlightening to rephrase the above results on $L$-packets for real groups in terms of the Weil group of $\mathbb{R}$. For non-archimedean fields $k$, the Weil group is essential for describing $L$-packets and branching laws.

In this chapter we define Weil groups for real and $p$-adic local fields, and give an introduction to some of the local number theory that is needed to construct $p$-adic $L$-packets. This is followed by some background on quasi-split groups, which is preliminary to the definition of the group ${}^LG$. With these ingredients in hand, we give a preliminary definition of a Langlands parameter, and then consider some refinements of this definition.

## 6.1 Weil groups

Let $k$ be a local field of characteristic zero. That is, $k$ is either $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. Fix an algebraic closure $\bar{k}$ of $k$. The Weil group $\mathcal{W}(k)$ of $k$ is a locally compact group, which comes with a group homomorphism

$$\mathcal{W}(k) \longrightarrow \mathrm{Gal}(\bar{k}/k)$$

with dense image, and an isomorphism

$$\mathcal{W}(k)^{ab} \simeq k^{\times}.$$

Here $\mathcal{W}(k)^{ab}$ is the quotient of $\mathcal{W}(k)$ by the closure of its commutator subgroup.

Weil groups can be described concretely. For archimedean $k$ they are given by:

$$\begin{aligned} \mathcal{W}(\mathbb{C}) &= \mathbb{C}^{\times}, \\ \mathcal{W}(\mathbb{R}) &= N_{\mathbb{H}^{\times}}(\mathbb{C}^{\times}). \end{aligned} \tag{22}$$

That is, $\mathcal{W}(\mathbb{R})$ is the normalizer of $\mathbb{C}^{\times}$ in $\mathbb{H}^{\times}$, the multiplicative group of Hamilton's quaternions. The group $\mathcal{W}(\mathbb{R})$ contains the normal subgroup $\mathbb{C}^{\times}$ with index two; the nontrivial coset $j\mathbb{C}^{\times}$ is represented by $j \in \mathbb{H}^{\times}$ where $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for all $z \in \mathbb{C}^{\times}$. Thus, we have an exact sequence

$$1 \longrightarrow \mathbb{C}^{\times} \longrightarrow \mathcal{W}(\mathbb{R}) \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1 \tag{23}$$

with $j$ mapping to complex conjugation. This sequence is not split; since $\mathbb{H}$ is a division algebra, the only involutions in $\mathbb{H}^{\times}$ are $\pm 1$.

If $k$ is non-archimedean, then the description of $\mathcal{W}(k)$ is a bit more involved. Roughly speaking, $\mathcal{W}(k)$ is the Galois group of $\bar{k}/k$ with the topology relaxed, so as to allow more continuous representations. More precisely, if $k$ has residue field $\mathfrak{f}$ of order $q$, then we have an exact sequence

$$1 \longrightarrow \mathcal{I} \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}) \longrightarrow 1,$$

whose kernel $\mathcal{I}$ is called the **inertia group** of $k$. Any element $F \in \mathrm{Gal}(\bar{k}/k)$ whose image in $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ is the automorphism $\alpha \mapsto \alpha^q$ of $\bar{\mathfrak{f}}$ is called a **Frobenius element**. We fix a choice of $F$ once and for all. The image of $F$ in $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ generates a dense subgroup, isomorphic to $\mathbb{Z}$.

The Weil group of $\mathcal{W}(k)$ is defined to be the subgroup of $\mathrm{Gal}(\bar{k}/k)$ generated by $F$ and the inertia subgroup $\mathcal{I}$. Thus,

$$\mathcal{W}(k) = \coprod_{n\in\mathbb{Z}} \mathcal{I}F^n = \mathcal{I} \rtimes \langle F\rangle,$$

and we have an exact sequence (analogous to (23))

$$1 \longrightarrow \mathcal{I} \longrightarrow \mathcal{W}(k) \longrightarrow \mathbb{Z} \longrightarrow 1. \tag{24}$$

A subset of $\mathcal{W}(k)$ is open iff its intersection with every coset of $\mathcal{I}$ is open, where $\mathcal{I}$ has the profinite topology inherited from $\mathrm{Gal}(\bar{k}/k)$. A homomorphism from $\mathcal{W}(k)$ to a discrete group is continuous iff the image of $\mathcal{I}$ is finite; such a homomorphism extends continuously to $\mathrm{Gal}(\bar{k}/k)$ iff the image of all of $\mathcal{W}(k)$ is finite.

To better understand the Weil group $\mathcal{W}(k)$ for non-archimedean $k$, we will now describe some subgroups of $\mathcal{W}(k)$ and the corresponding extension fields of $k$; these will be useful later on.

First of all, a Galois extension $L/k$ is **unramified** if a prime element in the integers of $k$ remains prime in the integers of $L$. Equivalently, $L/k$ is unramified if the inertia group $\mathcal{I}$ acts trivially on $L$. This implies that $\mathrm{Gal}(L/k)$ is generated by the image of $F$ under the canonical map $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(L/k)$. Since $\mathbb{Z}$ has a unique subgroup of every positive index, we see that for every $d \geq 1$, the field $k$ has a unique unramified extension $k_d \subset \bar{k}$ of degree $\deg(k_d/k) = d$. For example, the extension $k_2$ of $k = \mathbb{Q}_p$ was described in section 3.

The Weil group $\mathcal{W}(k_d)$ of $k_d$ is the subgroup of $W(k)$ generated by $\mathcal{I}$ and $F^d$. The composite field

$$K := \bigcup_{d\geq 1} k_d$$

is the **maximal unramified extension** of $k$ in $\bar{k}$; its Galois group is the inertia group of $k$:

$$\mathcal{I} = \mathrm{Gal}(\bar{k}/K).$$

The residue field of $K$ is $\bar{\mathfrak{f}}$, and the natural map $\mathrm{Gal}(K/k) \to \mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ is an isomorphism. This means that the unramified extension $k_d/k$ can be constructed as $k_d = k(a)$, where $a \in K$ is a lift of an element $\alpha \in \bar{\mathfrak{f}}$ which generates the degree $d$ extension $\mathfrak{f}_d/\mathfrak{f}$.

Thus, the theory of unramified extensions is more or less equivalent to the theory of extensions of finite fields. At the other extreme, a Galois extension $L/k$ of degree $e$ is **totally ramified** if a prime element in the ring of integers in $k$ is the

$e^{th}$ power of a prime in $L$. Equivalently, $L/k$ is totally ramified if the residue field of $L$ is equal to the residue field of $k$.

A general finite Galois extension $L/k$ is of the form $k \subset k_d \subset L$ for some $d$, where $L/k_d$ is totally ramified of degree $e = [L : k]/d$. We say that $L/k$ is **tame** if $e$ is not divisible by the residue characteristic $p$ of $k$. All of the $L$-packets constructed in this paper will arise from tame extensions.

Totally ramified Galois extensions correspond to normal subgroups $\Delta \trianglelefteq \mathcal{I}$ of finite index, and the extension is tame if $\Delta$ contains the maximal pro-$p$ subgroup $\mathcal{I}^+$ of $\mathcal{I}$. The quotient

$$\mathcal{I}_t := \mathcal{I}/\mathcal{I}^+$$

is called the **tame inertia group** ; it is the Galois group of the maximal tame extension $K_t$ of $K$. The field $K_t$ can be described very explicitly: it is obtained by adjoining to $K$ all the roots of the polynomials

$$X^{q^n-1} + \varpi, \qquad \text{for} \quad n \geq 1,$$

for any fixed choice of prime element $\varpi$ in the ring of integers of $k$. This description of $K_t$ implies that the tame inertia group can be viewed as an inverse limit

$$\mathcal{I}_t = \lim_{\leftarrow} \mathfrak{f}_n^\times, \tag{25}$$

where $\mathfrak{f}_n \subset \bar{\mathfrak{f}}$ is the extension of $\mathfrak{f}$ of degree $n$ and the transition maps are the norm homomorphisms $\mathfrak{f}_n^\times \to \mathfrak{f}_m^\times$, for $m \mid n$. Under the isomorphism (25), the action of $F$ by conjugation on $\mathcal{I}_t$ corresponds to the automorphism $\alpha \mapsto \alpha^q$ on each finite field $\mathfrak{f}_n^\times$.

## 6.2 Quasi-split groups and the L-group

More details for this section can be found, for example, in [36]. An excellent survey on the origins of the $L$-group can be found in [7].

In this section $G$ is a connected reductive algebraic group defined over the local field $k$. We also assume that the group $G$ is **quasi-split** over $k$. This means that the flag variety $\mathcal{B}$ of $G$ has a $k$-rational point, that is, $G$ has a Borel subgroup $B$ defined over $k$. (Note that $G$ is no longer a compact real Lie group, as it has been until now.) Let $T \subset B$ be a maximal torus defined over $k$, contained in $B$, and let

$$X = \mathrm{Hom}(GL_1, T), \qquad Y = \mathrm{Hom}(T, GL_1)$$

be the groups of algebraic co-characters (i.e., 1-parameter subgroups) and characters of $T$, respectively. Inside $Y$ have the roots $\Phi$ of $T$ in the Lie algebra of $G$, and the positive roots $\Phi^+$ of $T$ in the Lie algebra of $B$. Inside $X$ we have the corresponding co-roots $\check{\Phi}$ and positive co-roots $\check{\Phi}^+$. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $X$ and $Y$, preserving $\Phi^+$ and $\check{\Phi}^+$. Let $\mathrm{Gal}(\bar{k}/E)$ denote the kernel of this action; the field $E$ is a finite Galois extension of $k$, called the **splitting field** of $G$. We say $G$ is **split** if $E = k$.

Thus, $G$ determines a **based root datum** $(X, Y, \check{\Phi}^+, \Phi^+)$ with an action of $\mathrm{Gal}(E/k)$. There is a unique (up to isomorphism) complex reductive Lie group $\hat{G}$ whose based root datum $(Y, X, \Phi^+, \check{\Phi}^+)$ is dual to that of $G$. Moreover, there is a maximal torus and Borel subgroup $\hat{T} \subset \hat{B}$ in $\hat{G}$ such that

$$X = \mathrm{Hom}(\hat{T}, GL_1), \qquad Y = \mathrm{Hom}(GL_1, \hat{T}),$$

and $\check{\Phi}^+$ is the set of roots of $\hat{T}$ in the Lie algebra of $\hat{B}$. The action of $\mathrm{Gal}(E/k)$ on this root datum extends to an action of $\mathrm{Gal}(E/k)$ by automorphisms of $\hat{G}$, preserving $\hat{T}$ and $\hat{B}$.

The $L$-group of $G$ is defined as the semi-direct product

$${}^LG := \mathrm{Gal}(E/k) \ltimes \hat{G}.$$

Conversely, any continuous action of $\mathrm{Gal}(\bar{k}/k)$ on a based root datum arises from a unique quasi-split group over $k$, up to isomorphism. Thus, the group $G$ is determined by its $L$-group.

The center $Z({}^LG)$ of ${}^LG$ plays an important role, expecially in the $p$-adic case. One can show that

$$Z({}^LG) = Z(\hat{G})^{\mathrm{Gal}(E/k)}$$

is the fixed point group of $\mathrm{Gal}(E/k)$ in the center $Z(\hat{G})$ of $\hat{G}$. The group $Z({}^LG)$ is finite precisely when $G(k)$ has compact center. In particular, $Z({}^LG)$ is finite if $G$ is semisimple.

We now consider some examples. If $T$ is a torus over $k$, with character group $Y$, then $T$ is quasi-split, with dual torus

$$\hat{T} = Y \otimes \mathbb{C}^\times,$$

and the $L$-group

$${}^LT = \mathrm{Gal}(E/k) \ltimes \hat{T}$$

is given by the action of $\mathrm{Gal}(E/k)$ on $Y$.

At the opposite extreme from tori, we have semisimple quasi-split groups. Here the Galois action on the root datum is given by an automorphism of the Dynkin diagram of $G$. (For the passage from root data to Dynkin diagrams, see [3, VI.4].) Let us consider the possibilities of this action for orthogonal groups, assuming that $k \neq \mathbb{C}$.

If $G = SO_{2n+1}$ is a quasi-split odd orthogonal group, there are no nontrivial automorphisms of the Dynkin diagram, so the action of $\mathrm{Gal}(\bar{k}/k)$ is trivial. Hence $E = k$ and $G$ is split. This means there is only one quasi-split odd orthogonal group, up to $k$-isomorphism. It arises from the quadratic form $Q_n + x_{2n+1}^2$, where $Q_n = x_1x_{n+1} + \cdots + x_nx_{2n}$. In this case, we have ${}^LG = \hat{G} = Sp_{2n}(\mathbb{C})$.

If $G = SO_{2n}$ is an even orthogonal group with $n \geq 2$, then the Dynkin diagram has a symmetry of order two, so $G$ is either split, or $E$ is a quadratic field extension of $k$. These arise from the quadratic forms $Q_n,\ Q_{n-1} + N$, respectively, where $Q_n$ is as above, and $E$ is viewed as a two-dimensional $k$-vector space with quadratic form $N : E \to k$ given by the norm. We call these two quasi-split groups $SO_{2n}$ and $SO'_{2n}$, respectively. (We omit the dependence on $E$ since, in this paper, $E/k$ will always be $\mathbb{C}/\mathbb{R}$ or the unramified quadratic extension $k_2/k$.)

In the split case, we have ${}^LG = \hat{G} = SO_{2n}(\mathbb{C})$. In the non-split case, we have

$$ {}^LG = \mathrm{Gal}(E/k) \ltimes SO_{2n}(\mathbb{C}), $$

where the Galois action on $SO_{2n}(\mathbb{C})$ is given by conjugation by a reflection in $O_{2n}(\mathbb{C})$, and we have an isomorphism

$$ {}^LG \simeq O_{2n}(\mathbb{C}). $$

In summary, we have the following table of $L$-groups for quasi-split orthogonal groups.

| $G$ | ${}^LG$ |
|---|---|
| $SO_{2n+1}$ | $Sp_{2n}(\mathbb{C})$ |
| $SO_{2n}$ | $SO_{2n}(\mathbb{C})$ |
| $SO'_{2n}$ | $O_{2n}(\mathbb{C})$ |

In all three cases we have $Z({}^LG) = \{\pm I\}$ (except for $SO_2$, which is not semisimple).

When $k = \mathbb{R}$, these groups have real points

$$ \begin{aligned} SO_{2n+1}(\mathbb{R}) &= SO(n+1, n) \\ SO_{2n}(\mathbb{R}) &= SO(n, n) \\ SO'_{2n}(\mathbb{R}) &= SO(n+1, n-1), \end{aligned} \tag{26} $$

where $SO(p,q)$ is the special orthogonal group of a quadratic form on $\mathbb{R}^{p+q}$ of signature $(p,q)$.

## 6.3 Langlands parameters

Recall that $G$ is a connected quasi-split reductive group over the local field $k$, whose splitting field $E$ is a finite Galois extension of $k$. In the previous sections of this chapter we defined the Weil group $\mathcal{W}(k)$ of $k$, and the $L$-group ${}^LG = \mathrm{Gal}(E/k) \ltimes \hat{G}$.

In this paper, a Langlands parameter has two ingredients. The first is a continuous homomorphism

$$\varphi : \mathcal{W}(k) \longrightarrow {}^LG, \tag{27}$$

whose projection onto the first factor $\mathrm{Gal}(E/k)$ is the composition of canonical maps

$$\mathcal{W}(k) \hookrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Gal}(E/k), \tag{28}$$

and whose projection to the second factor $\hat{G}$ consists of semisimple elements.

For example, suppose $E = k$ is $p$-adic. If $\varphi$ is trivial on the inertia subgroup $\mathcal{I}$ of $\mathcal{W}(k)$, then $\varphi$ is completely determined by the semisimple element $\varphi(F) \in \hat{G}$. A more general Langlands parameter replaces the element $\varphi(F)$ by an arithmetic refinement: the homomorphism $\varphi : \mathcal{W}(k) \to \hat{G}$.

For another example, let $T$ be a torus over $k$ with character group $Y = \mathrm{Hom}(T, GL_1)$, with dual group $\hat{T} = \mathbb{C}^\times \otimes Y$, and $L$-group ${}^LT = \mathrm{Gal}(E/k) \ltimes \hat{T}$. A Langlands parameter

$$\varphi : \mathcal{W}(k) \longrightarrow {}^LT$$

amounts to a continuous one-cocycle on $\mathcal{W}(k)$ with values in $\hat{T}$, where $\mathcal{W}(k)$ acts on $\hat{G}$ via the natural map $\mathcal{W}(k) \to \mathrm{Gal}(E/k)$.

The second ingredient of a Langlands parameter is an irreducible representation of the finite group $A_\varphi = \pi_0\left(C_{\hat{G}}(\varphi)\right)$ of connected components of the centralizer in $\hat{G}$ of $\varphi$. From basic homotopy theory we have a surjective homomorphism

$$\pi_1(\hat{G} \cdot \varphi) \longrightarrow A_\varphi,$$

where $\hat{G} \cdot \varphi \subset \mathrm{Hom}(\mathcal{W}(k), {}^LG)$ is the $\hat{G}$-orbit of $\varphi$ under conjugation. So a representation of $A_\varphi$ gives rise to a representation of the fundamental group of the $\hat{G}$-conjugacy class of $\varphi$.

In this paper, the complete Langlands parameter is a pair $(\varphi, \rho)$, with $\varphi$ as in (27) and $\rho \in \mathrm{Irr}(A_\varphi)$ is an irreducible representation of $A_\varphi$. Two parameters are considered equivalent if they are conjugate under $\hat{G}$. Thus, the pair $(\varphi, \rho)$ may be thought of as a conjugacy-class in $\hat{G}$, with arithmetic and topological enhancements.

Note that the center $Z({}^LG)$ of ${}^LG$ is contained in $C_{\hat{G}}(\varphi)$. We say that a Langlands parameter is **discrete** (some say **elliptic**) if the group $C_{\hat{G}}(\varphi)$ has dimension as small as possible, that is, if $C_{\hat{G}}(\varphi)$ and $Z({}^LG)$ have the same identity component. For $G$ semisimple, the Langlands parameter $\varphi$ is discrete if and only if the group $C_{\hat{G}}(\varphi)$ is finite, in which case we have $A_\varphi = C_{\hat{G}}(\varphi)$.

A caveat: The Langlands parameters defined above are not sufficient to account for all representations in the $p$-adic case; one must also add a nilpotent part (see [37]). These more general parameters will play no role in this paper, but we will briefly mention them again in section 9, to give some perspective.

# 7 Real orthogonal groups again

In this section we explicitly describe Langlands parameters for real orthogonal groups, and show how they relate to the $L$-packets contructed in section 4.

Recall from (22) that the Weil group of $\mathbb{R}$ is $\mathcal{W}(\mathbb{R}) = N_{\mathbb{H}^\times}(\mathbb{C}^\times)$. In particular, $\mathcal{W}(\mathbb{R})$ contains $\mathbb{C}^\times$ as a normal subgroup of index two, with quotient $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. It has a family of two-dimensional complex representations $V(a)$, indexed by $a \in \frac{1}{2}\mathbb{Z}$, defined by

$$V(a) = \mathrm{Ind}_{\mathbb{C}^\times}^{\mathcal{W}(\mathbb{R})}(\chi_a),$$

where $\chi_a : \mathbb{C}^\times \to S^1$ is the unitary character

$$\chi_a(z) = \left(\frac{z}{\bar{z}}\right)^a = \frac{z^{2a}}{|z\bar{z}|^a}.$$

The representation $V(a)$ is isomorphic to $V(-a) = \mathrm{Ind}(\chi_a^{-1})$, and is self-dual. It is symplectic when $a \in \frac{1}{2} + \mathbb{Z}$ and orthogonal when $a \in \mathbb{Z}$. When $V(a)$ is orthogonal, its determinant is the sign character of $W(\mathbb{R})^{ab} = \mathbb{R}^\times$. Finally, $V(a)$ is irreducible unless $a = 0$, and $V(0)$ is the direct sum of the trivial representation and the sign character.

The Langlands parameters for $SO_{2n+1}$ over $\mathbb{R}$ are symplectic representations

$$\varphi : \mathcal{W}(\mathbb{R}) \longrightarrow {}^LG = Sp_{2n}(\mathbb{C}) = Sp(V).$$

Discrete series parameters for $SO_{2n+1}$ are symplectic representations $V$ of the form

$$V = \bigoplus_{i=1}^{n} V(\chi_i),$$

with

$$\chi_i \in \tfrac{1}{2} + \mathbb{Z}, \qquad \chi_1 > \chi_2 > \cdots > \chi_n > 0.$$

Thus, $\varphi$ corresponds to the positive character $\chi \in \rho + X(S_{\mathbb{C}})$ defined in section 5. We write $\Pi(\varphi) = \Pi(\chi)$.

The compact torus $S$ of section 4 is conjugate under $SO_{2n+1}(\mathbb{C})$ to the maximal compact subgroup of the complex torus $T$. Hence the involutions in $S$ correspond to involutions in $T$. The latter involutions may be identified, by duality, with characters of the group $\hat{T}_2$ of involutions in $\hat{T}$. In turn, $\hat{T}_2$ is the the center of the subgroup

$$Sp\left(V(\chi_1)\right) \times \cdots \times Sp\left(V(\chi_n)\right),$$

and since the $\chi_i$ are distinct, this center is exactly $A_\varphi = C_{\hat{G}}(\varphi)$. Thus, the $L$-packet $\Pi(\varphi)$ is in bijection with the set of $2^n$ characters of $A_\varphi$.

For even orthogonal groups over $\mathbb{R}$, recall from (26) that $SO_{2n}$ denotes the quasi-split orthogonal group with $SO_{2n}(\mathbb{R}) = SO(n,n)$, and $SO'_{2n}$ denotes the quasi-split orthogonal group with $SO'_{2n}(\mathbb{R}) = SO(n+1, n-1)$. A Langlands parameter for either of these groups is an orthogonal representation

$$\varphi : \mathcal{W}(\mathbb{R}) \longrightarrow {}^{L}G \subset O_{2n}(\mathbb{C}) = O(W)$$

whose image is contained in $SO_{2n}(\mathbb{C})$ when $G = SO_{2n}$ and is not contained in $SO_{2n}(\mathbb{C})$ when $G = SO'_{2n}$. Discrete series parameters for real even orthogonal groups are orthogonal representations $W$ of dimension $2n$, of the form

$$W = \bigoplus_{i=1}^{n} V(\eta_i),$$

with

$$\eta_i \in \mathbb{Z}, \qquad \eta_1 > \eta_2 > \cdots > |\eta_n|.$$

Since each $V(\eta_i)$ is orthogonal with nontrivial determinant equal to the sign character of $\mathcal{W}(\mathbb{R})^{ab} = \mathbb{R}^\times$, the resulting homomorphism $\varphi : \mathcal{W}(\mathbb{R}) \longrightarrow O(W)$ has image contained in $SO(W) = SO_{2n}(\mathbb{C})$ precisely when $n$ is even. Thus, $\varphi$ is a discrete series parameter for $SO_{2n}$ when $n$ is even and $SO'_{2n}$ when $n$ is odd.

In either case, $\varphi$ corresponds to the positive character $\eta \in \rho + X(S_{\mathbb{C}})$ defined in section 5. The involutions in $S$ again correspond to characters of the centralizer $A_\varphi$ of $\varphi$ in $SO(W)$, and $A_\varphi$ is the center of the subgroup

$$O(V(\eta_1)) \times \cdots \times O(V(\eta_n)).$$

The $2^n$ characters of $A_\varphi$ parametrize the representations in the $L$-packet $\Pi(\varphi) = \Pi(\eta)$ described in section 5.

The above calculations may seem more natural when viewed in greater generality, and this will also permit later comparison with $p$-adic Langlands parameters. For any quasi-split real group $G$, a Langlands parameter

$$\varphi : \mathcal{W}(\mathbb{R}) \longrightarrow {}^L G$$

maps the normal subgroup $\mathbb{C}^\times$ of $\mathcal{W}(\mathbb{R})$ to maximal torus in $\hat{G}$, which, after conjugation, we may arrange to be $\hat{T}$. To be discrete, $\varphi(\mathbb{C}^\times)$ must contain a regular element of $\hat{G}$. Then the centralizer of $\varphi(\mathbb{C}^\times)$ in $\hat{G}$ is exactly $\hat{T}$, and the full Weil group $\mathcal{W}(\mathbb{R})$ maps to the normalizer $N_{^L G}(\hat{T})$ of $\hat{T}$ in ${}^L G$. The element $j$, generating the quotient $\mathcal{W}(\mathbb{R})/\mathbb{C}^\times = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, maps to an element $n \in N_{^L G}(\hat{T})$, and conjugation by $n$ is an involution of $\hat{T}$. The centralizer $C_{\hat{G}}(\varphi)$ is the fixed points in $\hat{T}$ of this involution. The discrete condition forces this involution to be inversion, and the centralizer $A_\varphi$ of $\varphi$ is therefore always equal to the 2-torsion subgroup of $\hat{T}$. So the character group of $A_\varphi$ may be identified with the 2-torsion subgroup of a maximal compact torus in $G(\mathbb{R})$, as in section 4.

# 8 Pure inner forms of $p$-adic groups

The examples of $L$-packets for $p$-adic $SO_3$ in section 3 belong to a family of discrete series $L$-packets for $p$-adic groups whose Langlands parametrization (though not their construction) is similar to that for real groups. Later, we will discuss this family of $p$-adic $L$-packets in more detail.

First, recall that in those examples we had two versions of $SO_3$, one compact and the other non-compact. In this section we explain how the various versions of a $p$-adic group are controlled by Galois cohomology sets $H^1(k, G)$. The real case also involved Galois cohomology, but in disguise. There, it sufficed to consider involutions $s$, which were the concrete manifestations of real Galois cohomology classes. Indeed, if $G$ is a compact real group with maximal torus $S$ and Weyl group $W$, then the Galois cohomology is given by $H^1(\mathbb{R}, G) = S_2/W$ (see [33, III.4.5]).

In the $p$-adic case, we will see that $H^1(k, G)$ is nicely described by a theorem of Kottwitz. See [33] for an introduction to Galois cohomology, and [9] for more details in what follows.

*For the rest of this paper* , our local field $k$ is non-archimedean, of characteristic zero, with residue field $\mathfrak{f}$, and $p$ denotes the characteristic of $\mathfrak{f}$. Recall that $G$ is a connected quasi-split $k$-group.

For each continuous cocycle $c : \mathrm{Gal}(\bar{k}/k) \to G$, we define a new $\mathrm{Gal}(\bar{k}/k)$-action on $G$ by

$$\gamma \circ_c g = c(\gamma) \cdot g^\gamma \cdot c(\gamma)^{-1}.$$

This new $\mathrm{Gal}(\bar{k}/k)$-action is that of a new $k$-structure on $G$. Let $G_c$ be the group $G$ with this new $k$-structure, so that the $k$-rational points of $G_c$ are given by

$$G_c(k) = \{g \in G : \ \gamma \circ_c g = g \quad \text{for all} \quad \gamma \in \mathrm{Gal}(\bar{k}/k)\}.$$

The $k$-isomorphism class of $G_c$ is determined by the image of the class of $c$ in $H^1(k, G/Z)$, where $Z$ is the center of $G$.

For each class $\omega \in H^1(k, G)$, we choose a cocycle $c \in \omega$, and by abuse of notation we write $G_\omega = G_c$ (see [33, p.48]). We call $G_\omega$ a **pure inner form** of $G$. As in the real case, two such groups $G_\omega$, $G_{\omega'}$ may be $k$-isomorphic, even if $\omega \neq \omega'$. Usually, $G_\omega$ is no longer quasi-split. In fact $G_\omega$ is quasi-split if and only if $\omega$ has trivial image in $H^1(k, G/Z)$, and this is equivalent to having $G_\omega \simeq G$ over $k$.

The set $H^1(k, G)$ can be made completely explicit, thanks to a theorem of Kottwitz [23], who showed that there is a natural bijection between $H^1(k, G)$ and the set of irreducible characters of the group $\pi_0\left(Z({}^LG)\right)$ of connected components of the center $Z({}^LG)$ of the $L$-group ${}^LG$. Thus, for $p$-adic $k$, the Galois cohomology set $H^1(k, G)$ has a natural structure of a finite abelian group. Recall that if $G$ is semisimple, then $Z({}^LG)$ is finite, so Kottwitz' bijection takes the simpler form

$$H^1(k, G) \simeq \mathrm{Irr}\left(Z({}^LG)\right). \tag{29}$$

Let us use (29) to determine the pure inner forms of quasi-split orthogonal groups (see section 6.2). First, we have $H^1(k, SO_2) = 1$. In all other cases we have $Z({}^LG) = \{\pm 1\}$, so $G$ has a unique pure inner form $G_\omega$, corresponding to the nontrivial element $\omega \in H^1(k, G)$. These are tabulated as follows.

| $G$ | $G_\omega$ |
|---|---|
| $SO_{2n+1}$ | $SO^*_{2n+1}$ |
| $SO_{2n}$ | $SO^*_{2n}$ |
| $SO'_{2n}$ | $SO'_{2n}$ |

Here, $SO^*_{2n+1}$ is the special orthogonal group of the sum of $n-1$ hyperbolic planes and the three dimensional anisotropic quadratic space considered in section 3, and $SO^*_{2n}$ is the special orthogonal group of the sum of $n-2$ hyperbolic planes and the four dimensional anisotropic quadratic space arising from the unique non-split quaternion algebra over $k$. For $G = SO'_{2n}$, we have $G_\omega \simeq G$. Thus, we have a total of five families of orthogonal groups to consider.

The groups $SO'_2$, $SO^*_3$ and $SO^*_4$ have compact groups of $k$-rational points, and the remaining orthogonal groups have non-compact groups of $k$-rational points.

# 9 The $p$-adic Langlands conjecture

Unlike the real case, the discrete series representations of reductive $p$-adic groups have not yet been classified. However, there is a conjectural classification, due initially to Langlands and then refined by others. We now have most of the ingredients needed to state this conjectural classification for pure inner forms of a quasi-split group $G$. However, our Langlands parameters, as defined in section 6.3, while sufficing for the actual representations we will consider later, are not general enough to parametrize the whole discrete series. In this section only, we will consider more general parameters, which are expected to suffice, in order to give the reader an idea of how our representations should fit into the complete picture.

We assume $G$ is semisimple, just to make the statements cleaner. Given a Langlands parameter $\varphi$, the inclusion $Z({}^LG) \hookrightarrow C_{\hat{G}}(\varphi)$ will induce a homomorphism

$$i_\varphi : Z({}^LG) \longrightarrow A_\varphi$$

whose image is contained in the center of $A_\varphi$. Hence, by (29), for every $\rho \in \mathrm{Irr}\,(A_\varphi)$ there is $\omega_\rho \in H^1(k, G)$ such that $\rho \circ i_\varphi$ is the scalar character $\omega_\rho$ on $Z({}^LG)$.

The correspondence $\rho \mapsto \omega_\rho$ should be the means of distributing the different representations in the $L$-packet $\Pi(\varphi)$ among the different pure inner forms of $G$ (cf. [44]). This is part of the conjectural Langlands correspondence, as stated below.

**Conjecture 9.1** *For each class $\omega \in H^1(k, G)$, the discrete series of the pure inner form $G_\omega(k)$ is partitioned as*

$$\coprod_{\varphi} \Pi(\varphi, \omega),$$

*where*

$$\Pi(\varphi, \omega) = \{\pi(\varphi, \rho) : \ \rho \in \mathrm{Irr}\,(A_\varphi)\,, \quad \omega_\rho = \omega\},$$

*and $\varphi$ runs over the set of $\hat{G}$-conjugacy classes of Langlands parameters*

$$\varphi : \mathcal{W}(k) \times SL_2(\mathbb{C}) \longrightarrow {}^L G$$

*whose image has finite centralizer in $\hat{G}$.*

Note that the new ingredient here is the factor of $SL_2(\mathbb{C})$. This is the "nilpotent part" that we previously ignored. It is possible to have $\varphi$ being trivial on $SL_2(\mathbb{C})$; that is the case for the parameters as we originally defined them, to which we will confine ourselves after this section.

The last condition in 9.1 is the "discrete series" condition. It says that $A_\varphi = C_{\hat{G}}(\varphi)$, and is equivalent to there being no $\mathrm{Gal}(E/k)$-stable proper parabolic subgroup $\hat{P} \subset \hat{G}$ such that the image of $\varphi$ lies in $\mathrm{Gal}(E/k) \ltimes \hat{P}$.

Assuming Conjecture 9.1, the $L$-packet $\Pi(\varphi)$ would be the disjoint union

$$\Pi(\varphi) = \coprod_{\omega \in H^1(k,G)} \Pi(\varphi, \omega),$$

consisting of a certain number of representations $\pi(\varphi, \rho)$ on the various pure inner forms of $G$, determined by the restriction of representations of $A_\varphi$ to $Z({}^L G)$.

# 10 Parameters for some $p$-adic discrete series

We now consider some $p$-adic Langlands parameters $\varphi : \mathcal{W}(k) \longrightarrow {}^L G$, analogous to discrete series parameters for real groups (see section 7). In that case, $\mathcal{W}(\mathbb{R})$ had a normal subgroup $\mathbb{C}^\times$. In the $p$-adic case, the analogous normal subgroup is the inertia group $\mathcal{I} \subset \mathcal{W}(k)$. We will only consider the simplest nontrivial case, where $\varphi$ is trivial on the wild inertia subgroup $\mathcal{I}^+$. For similar parameters with higher ramification, see [29].

We continue to assume that $G$ is semisimple and quasi-split over $k$. We also now assume that the splitting field $E$ is unramified over $k$. Then $\mathrm{Gal}(E/k)$ is cyclic, generated by the image of Frobenius $F$. Hence the action of $\mathrm{Gal}(E/k)$ on the root datum of $G$ and on $\hat{G}$ is completely determined by an automorphism $\vartheta$ of order $\deg(E/k)$, given by the action of the $F$. We can then write

$$^{L}G = \langle\vartheta\rangle \ltimes \hat{G}$$

and

$$H^1(k, G) = \mathrm{Irr}(Z(\hat{G})^{\vartheta}).$$

## 10.1 Tame regular discrete parameters

Let $\mathcal{W} = \mathcal{W}(k)$ be the Weil group of $k$. We say that a homomorphism

$$\varphi : \mathcal{W} \longrightarrow {}^{L}G$$

as in (27) is **tame, regular and discrete** if the following three conditions hold.

1. $\varphi$ is trivial on the wild inertia group $\mathcal{I}^+$.
2. The centralizer in $\hat{G}$ of $\varphi(\mathcal{I})$ is the maximal torus $\hat{T}$ in $\hat{G}$.
3. The centralizer in $\hat{G}$ of $\varphi(\mathcal{W})$ is finite.

The first condition means that $\varphi$ factors through the tame inertia group

$$\mathcal{I}_t = \mathcal{I}/\mathcal{I}^+ \simeq \varprojlim \mathfrak{f}_m^{\times}. \tag{30}$$

This map induced on $\mathcal{I}_t$ by $\varphi$ must factor through $\mathfrak{f}_m^{\times}$ for some $m \geq 1$. Since $\mathfrak{f}_m^{\times}$ is cyclic, it follows that $\varphi(\mathcal{I}) = \langle s\rangle$ is cyclic, generated by an element $s \in \hat{T}$ of order prime to $p$.

The second condition means that this element $s \in \hat{T}$ is "regular" in the sense that its centralizer, namely $\hat{T}$, is as small as possible.

Outside of $\mathcal{I}$, the homomorphism $\varphi$ is determined by the single element $\varphi(F)$. Since $F$ normalizes $\mathcal{I}$, the element $n := \varphi(F) \in {}^{L}G$ must normalize the centralizer of $\varphi(\mathcal{I})$, which is $\hat{T}$, so $n$ lies in the normalizer $N_{^LG}(\hat{T})$. Let $w$ be the image of $n$ in the quotient group

$$^{L}W := N_{^LG}(\hat{T})/\hat{T} = \langle\vartheta\rangle \ltimes \hat{W},$$

where $\hat{W} = N_{\hat{G}}(\hat{T})/\hat{T}$ is the Weyl group of $\hat{T}$ in $\hat{G}$. Our conditions 1-3 force the element $w$ to have two significant properties.

First, recall that conjugation by $F$ on $\mathcal{I}_t$ corresponds, under (30), to the $q$-power automorphism of the groups $\mathfrak{f}_m^\times$. This implies the relation

$$s^w = n^{-1}sn = s^q.$$

The second property comes from condition 3: the centralizer in $\hat{G}$ of $\varphi(\mathcal{W})$ is $\hat{T}^w$, so $w$ must have the property that its fixed-point group $\hat{T}^w$ in $\hat{T}$ is finite, and we have

$$A_\varphi = \hat{T}^w.$$

The finiteness of $\hat{T}^w$ is equivalent to $w$ having no invariants (except zero) in the action of ${}^LW$ on the free abelian group $X = \mathrm{Hom}(\hat{T}, \mathbb{C}^\times)$ of characters of $\hat{T}$. In this case, the character group of $A_\varphi$ is isomorphic, by restriction of characters from $\hat{T}$ to $\hat{T}^w$, to the co-invariants of $w$ in $X$:

$$\mathrm{Irr}(A_\varphi) = X/(1-w)X.$$

In particular, $A_\varphi$ is an abelian group of order

$$|A_\varphi| = \det(1-w)|_X.$$

For example, if $w$ acts by inversion on $\hat{T}$, hence by $-1$ on $X$, we will have

$$\mathrm{Irr}(A_\varphi) = X/2X,$$

so that $|\,\mathrm{Irr}(A_\varphi)| = 2^n$, where $n = \dim T =$ rank $X$. This is the situation that most closely resembles the real case.

At the other extreme, suppose that $G$ is split with trivial center, and let $w$ be a Coxeter element of $\hat{W}$ (see [3, V.6]). Then $A_\varphi = \hat{T}^w = Z(\hat{G})$ is the center of $\hat{G}$. We will see that, for orthogonal groups, all examples of tame regular discrete parameters are built from copies of this Coxeter example.

## 10.2 Anisotropic tori

We now begin the construction of an $L$-packet of representations of our $p$-adic group $G$ and its pure inner forms, starting from a tame regular discrete Langlands parameter $\varphi : \mathcal{W} \longrightarrow {}^LG$, as defined in 10.1. The first step is to associate to $\varphi$ a pair $(T_w, \chi_\varphi)$, consisting of an anisotropic torus $T_w$ over $k$, determined by the element $w \equiv \varphi(F)$ in ${}^LW$ and a character $\chi_\varphi$ of $T_w(k)$.

A torus $S$ is **anisotropic** over $k$ if either of the following equivalent conditions holds:

1. No nontrivial element in the group $\mathrm{Hom}(GL_1, S)$ is fixed by $\mathrm{Gal}(\bar{k}/k)$.

2. The group $S(k)$ is compact.

Recall that we have chosen a maximal $k$-torus $T$ contained in a $k$-rational Borel subgroup $B$ in our quasi-split group $G$, and $X = \mathrm{Hom}(GL_1, T)$ denotes the lattice of 1-parameter subgroups of $T$. Since the splitting field $E$ is unramified over $k$, the Galois action on $X$ is trivial on $\mathcal{I}$, and the Frobenius $F$ acts by $\vartheta$. Let $w$ be the image of $\varphi(F)$ in ${}^LW$, and let $d$ be the order of $w$.

We define $T_w$ to have the same lattice $X = \mathrm{Hom}(GL_1, T_w)$, with Galois action on $X$ again trivial on $\mathcal{I}$, but now the Frobenius $F$ acts by $w$. Thus, we have

$$T_w(k_d) = X \otimes k_d^\times, \qquad T_w(k) = \left[X \otimes k_d^\times\right]^{w\otimes F}. \tag{31}$$

The discreteness condition 3 on $\varphi$ (see section 10.1) is equivalent to having $X^w = \{0\}$, which means that $T_w$ is anisotropic over $k$, and $T_w(k)$ is compact. In fact, we have

$$T_w(k) = \left[X \otimes \mathfrak{o}_d^\times\right]^{w\otimes F},$$

where $\mathfrak{o}_d$ is the ring of integers in $k_d$.

For example, if $w = -1$, the torus $T_w$ is a product of one-dimensional unitary groups, and

$$T_w(k) = X \otimes U_1,$$

where $U_1$ is the kernel of the norm homomorphism $k_2^\times \to k^\times$.

In general, the $L$-group of $T_w$ is

$${}^LT_w = \langle w\rangle \ltimes \hat{T}.$$

Note that ${}^LT_w$ need not be a subgroup of ${}^LG$, since $\varphi(F) = n$ can have order greater than that of $w$. Hence $\varphi$ does not, a priori, give a Langlands parameter $\varphi : \mathcal{W}(k) \to {}^LT_w$. However, a slight modification of $\varphi$ will give a parameter for $T_w$. Namely, we define

$$\varphi' : \mathcal{W}(k) \longrightarrow {}^LT_w = \langle w\rangle \ltimes \hat{T}$$

by making $\varphi'$ equal to $\varphi$ on $\mathcal{I}$, and defining

$$\varphi'(F) = w \ltimes 1 \in \langle w\rangle \ltimes \hat{T}.$$

If $\gamma \in \mathcal{I}$ and $t = \varphi(\gamma)$, we have

$$\varphi(F\gamma F^{-1}) = t^w = \varphi'(F\gamma F^{-1}),$$

so $\varphi'$ is a homomorphism.

By the Langlands correspondence for tori (which is essentially local class field theory, see [9] for an elementary treatment of the tame case), the parameter $\varphi'$ determines a character

$$\chi_\varphi : T_w(k) \longrightarrow \mathbb{C}^\times.$$

### 10.3 Summary

We summarize what has been shown in this chapter. We started with a Langlands parameter $\varphi : \mathcal{W}(k) \longrightarrow {}^L G$ whose restriction to $\mathcal{I}$ factors through the tame inertia group $\mathcal{I}_t$. We insisted that $\varphi$ satisfy the regularity condition

$$C_{\hat{G}}\left(\varphi(\mathcal{I})\right) = \hat{T},$$

as well as the discrete condition

$$A_\varphi = C_{\hat{G}}(\varphi) = \hat{T}^w \quad \text{is finite},$$

where $w \in \langle\vartheta\rangle \ltimes \hat{W}$ is the image of $\varphi(F)$. We call these $\varphi$ "tame regular discrete parameters".

Then we constructed an anisotropic torus $T_w$, splitting over the unramified extension of $k$ of degree equal to the order of $w$. Finally, using the abelian Langlands correspondence, we associated to $\varphi$ a character $\chi_\varphi$ of the compact group $T_w(k)$.

Note that all of this took place externally to $G$; the torus $T_w$ is not given as a subgroup of $G$ in any natural way. In the next chapter, we shall see that the various embeddings of $T_w$ into $G$ and its pure inner forms are controlled by the characters of the finite abelian group $A_\varphi = \hat{T}^w$.

## 11 Tame regular discrete series L-packets for p-adic groups: introduction

Let $\varphi : \mathcal{W}(k) \to {}^L G$ be a tame regular discrete Langlands parameter, as in section 10.3. We want to construct an $L$-packet

$$\Pi(\varphi) = \{\pi(\varphi, \rho) : \ \rho \in \mathrm{Irr}(A_\varphi)\}$$

parametrized by the irreducible representations $\rho$ of $A_\varphi$, in accordance with Conjecture 9.1.

We saw in Kottwitz' theorem (29) that $H^1(k, G) = \mathrm{Irr}\left(Z({}^L G)\right)$, and part of Conjecture 9.1 asserts that a representation $\pi(\varphi, \omega) \in \Pi(\varphi)$ should be a representation of the pure inner form $G_\omega(k)$ precisely when $\rho \in \mathrm{Irr}(A_\varphi)$ restricts to $\omega \in \mathrm{Irr}\left(Z({}^L G)\right)$.

In this chapter, we sketch the construction of such $L$-packets $\Pi(\varphi)$. A more detailed discussion of the construction requires more structure theory of $p$-adic groups, which we provide in the next chapter, along with illustrations for orthogonal groups.

As in the real case, maximal compact subgroups (or large subgroups of them) will play a key role in our construction. One difference here is that $p$-adic groups have several conjugacy-classes of maximal compact subgroups. Also, instead of the complex flag variety $\mathcal{B}$, the $p$-adic construction will use the Bruhat-Tits building.

We can outline the construction of $\Pi(\varphi)$, without reference to the Bruhat-Tits building, as follows. Rather than working with the quotient group $\mathrm{Irr}(A_\varphi) = X/(1-w)X$, we work in the lattice $X$ itself. Fix $\omega \in H^1(k, G) = \mathrm{Irr}\left(Z({}^L G)\right)$. For each $\lambda \in X$ whose restriction to $Z({}^L G)$ is $\omega$, we will construct a $k$-group $G_\lambda$ in the $k$-isomorphism class of $G_\omega$, along with a $k$-rational embedding

$$T_w \xrightarrow{\sim} T_\lambda \subset G_\lambda$$

of the anisotropic torus $T_w$ constructed in section 10.2 onto a maximal torus $T_\lambda$ in $G_\lambda$. Using this embedding, we transfer the character $\chi_\varphi$ of $T_w(k)$ from 10.2; we then have a character $\chi_\lambda$ of $T_\lambda(k)$.

As a general fact, the group of rational points of any anisotropic maximal torus in $G_\lambda$ is contained in a unique maximal compact subgroup of $G_\lambda(k)$. Let $K_\lambda$ be the maximal compact subgroup of $G_\lambda(k)$ containing $T_\lambda(k)$. Hence we have

$$T_\lambda(k) \subset K_\lambda \subset G_\lambda(k), \tag{32}$$

in analogy with the real case, where now $\lambda$ plays the role that $s$ did in (9). (In fact, the compact group $K_\lambda$ will be almost, but not quite, maximal. We ignore this difference while outlining the construction.)

From the data $(T_\lambda, \varphi_\lambda)$, a cohomological induction process, due to Deligne and Lusztig, gives us a finite dimensional representation $R_\lambda$ of $K_\lambda$. Via compact induction, we then have a representation

$$\pi_\lambda := \mathrm{Ind}_{K_\lambda}^{G_\lambda(k)} R_\lambda$$

of $G_\lambda(k)$, as in section 3. The regularity assumption on $\varphi$ ensures that $\pi_\lambda$ is irreducible. Since $\pi_\lambda$ has compactly-supported matrix coefficients, it is a discrete series representation of $G$.

Thus, starting from any $\lambda \in X$ which restricts to $\omega \in H^1(k, G)$, we will have constructed a group $G_\lambda$ in the $k$-isomorphism class of $G_\omega$ and a discrete series representation $\pi_\lambda$ of $G_\lambda(k)$. The isomorphism class of the pair $(G_\lambda, \pi_\lambda)$ depends only on the restriction of $\lambda$ to $\hat{T}^w = A_\varphi$. For $\rho \in \mathrm{Irr}(\hat{T}^w)$, we can then define $\pi(\varphi, \rho)$ to be the representation $\pi_\lambda$ of $G_\lambda(k)$, for any $\lambda \in X$ whose restriction to $\hat{T}^w$ is $\rho$.

This concludes our sketch of the construction of the $L$-packet $\Pi(\varphi)$. In fact, the actual construction of $\pi_\lambda$ finds $K_\lambda$ first, then $G_\lambda$, and the embedding of $T_\lambda(k)$ in $K_\lambda$ comes last.

# 12 Bruhat-Tits theory

To fill in details of the construction of $L$-packets just sketched, we need more structure theory for reductive $p$-adic groups. This was developed in great generality by Bruhat and Tits [5], but the working knowledge that we need is not easy to extract from the literature. We are going to cover the minimum amount of Bruhat-Tits theory sufficient to make our story coherent. We refer to [40] for a more thorough introduction to this theory.

## 12.1 The building

Recall that our quasi-split group $G$ splits over some unramified extension $E$ of $k$, and that for orthogonal groups we have $E = k$ or $E = k_2$. The field $E$ will now play the role that $\mathbb{C}$ did in the real case.

The main tool for us is the **Bruhat-Tits building** $\mathcal{B}(G)$ of the group $G(E)$ of $E$-rational points in $G$. This building $\mathcal{B}(G)$ is a simplicial complex with $G(E)$-action. In this paper, one can regard $\mathcal{B}(G)$ as a replacement for the complex flag variety $\mathcal{B}$ used in the real case.

As a set,

$$\mathcal{B}(G) = \bigcup_{g \in G(E)} g \cdot \mathcal{A}$$

is the union of $G(E)$-translates of an affine space

$$\mathcal{A} := \mathbb{R} \otimes X,$$

called an **apartment**, where we recall that $X = \mathrm{Hom}(GL_1, T)$ is the co-character group of $T$. The various translates $g \cdot \mathcal{A}$ are not disjoint in $\mathcal{B}(G)$; they are glued together in a way that will not concern us. For $G = SO_3 = PGL_2$, the building $\mathcal{B}(G)$ is an infinite homogeneous tree, and $\mathcal{A}$ is a path in $\mathcal{B}(G)$ without ends (cf. [35, chap II]).

A wealth of information about pure inner forms of $G$ and their maximal compact subgroups is contained in the simplicial structure on $\mathcal{B}(G)$. For our purposes, it suffices to study the simplicial structure on $\mathcal{A}$.

## 12.2 The geometry of an apartment and pure inner forms

A good reference for apartments and related affine Weyl groups is [3].

Recall that, by the definition of the dual group, we have

$$X = \mathrm{Hom}(\hat{T}, GL_1) = \mathrm{Hom}(GL_1, T)$$
$$Y = \mathrm{Hom}(GL_1, \hat{T}) = \mathrm{Hom}(T, GL_1)$$

and these groups are in duality via a the canonical pairing $\langle\ ,\ \rangle : Y \times X \to \mathbb{Z}$. Hence each element of $Y$ gives a linear functional on $\mathcal{A} = \mathbb{R} \otimes X$. Recall that $\Phi \subset Y$ is the set of roots of $T$ in $G$. The simplicial structure on $\mathcal{A}$ is given by the family of hyperplanes

$$H_{\alpha,n} := \{x \in \mathcal{A} :\ \langle \alpha, x\rangle = n\},$$

indexed by $\alpha \in \Phi$, $n \in \mathbb{Z}$. The apartment $\mathcal{A}$ is a disjoint union of **facets**. Two points $x, y \in \mathcal{A}$ lie in the same facet if, for every $\alpha \in \Phi$ and $n \in \mathbb{Z}$, either $x$ and $y$ both lie on $H_{\alpha,n}$ or are both strictly on the same side of $H_{\alpha,n}$. A **chamber** is a facet which is open in $\mathcal{A}$. Equivalently, a chamber is a connected component of the complement

$$\mathcal{A} - \bigcup_{\alpha,n} H_{\alpha,n}$$

of all hyperplanes in $\mathcal{A}$. The set $\Phi^+$ of positive roots determines one particular chamber

$$C := \{x \in \mathcal{A} :\ 0 < \langle \alpha, x\rangle < 1 \quad \text{for all } \alpha \in \Phi^+\}. \tag{33}$$

The Weyl group $W$ is generated, as a group of linear transformations on $\mathcal{A}$, by the reflections

$$s_\alpha \cdot x = x - \langle \alpha, x\rangle \check{\alpha}$$

(recall that each co-root $\check{\alpha}$ is an element of $X$). Moreover, each $\lambda \in X$ acts on $\mathcal{A}$ by the translation $t_\lambda \cdot x = x + \lambda$. These two actions generate an action of the **affine Weyl group**

$$W_{\text{aff}} = X \rtimes W \subset \text{Aff}(\mathcal{A}) \tag{34}$$

inside the group $\text{Aff}(\mathcal{A})$ of affine transformations on $\mathcal{A}$.

We let

$$\Omega = \{\omega \in W_{\text{aff}} \, : \, \omega \cdot C = C\}$$

be the stabilizer of $C$ in $W_{\text{aff}}$. This subgroup has a normal complement $W^\circ$ in $W_{\text{aff}}$, and $W^\circ$ acts simply-transitively on the set of chambers in $\mathcal{A}$. Hence we have a factorization

$$W_{\text{aff}} = W^\circ \rtimes \Omega. \tag{35}$$

Recall that $G$ is quasi-split over $k$, and the splitting field $E$ of $G$ is a finite unramified extension of $k$. We have seen that $\text{Gal}(E/k)$ acts on $X$ via an automorphism $\vartheta$ of order $\deg(E/k)$, which preserves the set $\Phi^+$ of positive roots. This means that the linear extension of $\vartheta$ to $\mathcal{A}$ preserves the chamber $C$ defined in (33). Hence $\vartheta$ acts (via conjugation in $\text{Aff}(\mathcal{A})$) on $W_{\text{aff}}$, preserving $\Omega$.

Note that we have used the same letter $\omega$ to denote an element of $\Omega$ as well as a class in $H^1(k, G)$. This was intentional; the projection $X \hookrightarrow W_{\text{aff}} \to \Omega$ induces an isomorphism

$$\Omega \simeq X/\mathbb{Z}\check{\Phi} \simeq \text{Irr}(Z(\hat{G})). \tag{36}$$

Let $\Omega/(1-\vartheta)\Omega$ denote the co-invariants of $\vartheta$ in $\Omega$. By restricting characters from $Z(\hat{G})$ to $Z(\hat{G})^\vartheta$, the isomorphism (36) induces another isomorphism

$$\Omega/(1-\vartheta)\Omega \simeq \text{Irr}\left(Z(\hat{G})^\vartheta\right) = \text{Irr}\left(Z(^LG)\right) \simeq H^1(k, G), \tag{37}$$

this last by Kottwitz' theorem (29). Thus, each element $\omega \in \Omega$ corresponds, via its image in $\Omega/(1-\vartheta)\Omega$, to a pure inner form $G_\omega$ of $G$.

To summarize, there are two automorphisms of $\mathcal{A}$ in play. Namely, $\vartheta$ gives the action of $F$ under the quasi-split action on $\mathcal{A}$, and $\omega \in \Omega$ measures the inner twisting of the quasi-split structure, via the isomorphism (37). Taken together, the product $\omega\vartheta \subset \Omega\vartheta$ tells us what group $G_\omega(k)$ we are looking at. For example, the Frobenius action for the $k$-structure on $G_\omega$ induces the operator $\omega\vartheta$ on $\mathcal{A}$. Thus, we see that there is a close relation between the geometry of the apartment $\mathcal{A}$ and the arithmetic of the group $G$ and its pure inner forms.

## 12.3 Parahoric subgroups

In this section we describe the appropriate analogues of maximal compact subgroups of real groups, using a minimum of structure theory. We will illustrate them for orthogonal groups in the next section.

Retain the notation of section 12.2. Recall that the Frobenius $F$ acts on $\mathcal{A}$ by the automorphism $\vartheta$. Take an element $\omega \in \Omega$, so that we have a pure inner form $G_\omega$ of $G$, via the isomorphism (37). Recall that the Frobenius for $G_\omega$ acts on $\mathcal{A}$ via the product $\omega\vartheta$.

Let $x \in \mathcal{A}^{\omega\vartheta}$ be a point fixed by $\omega\vartheta$. According to Bruhat-Tits theory, this point $x$ determines a certain compact and open subgroup

$$K_{\omega,x} \subset G_\omega(k),$$

called a **parahoric subgroup**. This group $K_{\omega,x}$ is profinite; it fits into an exact sequence

$$1 \longrightarrow K^+_{\omega,x} \longrightarrow K_{\omega,x} \longrightarrow \bar{K}_{\omega,x}(\mathfrak{f}) \longrightarrow 1,$$

where $K^+_{\omega,x}$ is an inverse limit of finite $p$-groups, and $\bar{K}_{\omega,x}(\mathfrak{f})$ is the group of $\mathfrak{f}$-rational points in a connected reductive group $\bar{K}_{\omega,x}$ over the residue field $\mathfrak{f}$.

In this paper, the only thing we need to know about $K_{\omega,x}$ is the structure of this $\mathfrak{f}$-group $\bar{K}_{\omega,x}$. As in section 6.2, the group $\bar{K}_{\omega,x}$ is determined by its based root datum with $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$-action. The root datum of $\bar{K}_{\omega,x}$ is that of $G$, except that $\Phi^+$ is replaced by the subset

$$\Phi^+_x := \{\alpha \in \Phi^+ : \langle \alpha, x\rangle \in \mathbb{Z}\}, \tag{38}$$

and $\check{\Phi}^+_x = \{\check{\alpha} : \alpha \in \Phi^+_x\}$. Moreover, the Frobenius automorphism in $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ acts on $X$ via $\omega\vartheta$.

For our discrete series representations, we will only be interested in certain parahoric subgroups. We call the $\omega\vartheta$-stable point $x$ and corresponding parahoric subgroup $K_{\omega,x}$ **good** if $x$ satisfies the two conditions

1. $x \in \bar{C}$, the closure of the chamber $C$ defined in (33).

2. The point $x$ is the unique $\omega\vartheta$-fixed point in the facet containing $x$.

If $\vartheta = \omega = 1$, the good points are just the vertices of the simplex $\bar{C}$. In general there are only finitely many good points $x$: they are the vertices in the fixed-point simplex $\bar{C}^{\omega\vartheta}$. Two good parahoric subgroups $K_{\omega,x}$ and $K_{\omega,y}$ are conjugate in $G_\omega(k)$ iff $x$ and $y$ are in the same same $\Omega^\vartheta$-orbit.

A good parahoric subgroup $K_{\omega,x}$ is almost, and is often equal to the full stabilizer $G_\omega(k)_x$ in $G_\omega(k)$ of the point $x \in \mathcal{B}(G)$. In fact, $K_{\omega,x}$ is a normal subgroup of $G_\omega(k)_x$ with finite quotient isomorphic to the stabilizer of $x$ in $\Omega^\vartheta$. For orthogonal groups, we have $|\Omega^\vartheta| \leq 2$. The groups $G_\omega(k)_x$ are maximal compact subgroups of $G_\omega(k)$. However, it is the good parahoric subgroups $K_{\omega,x}$ that play the role in the $p$-adic case that the maximal compact subgroups $K_s$ did in the real case.

# 13 Tame regular discrete series L-packets for p-adic groups: completion

Now we have all the tools to carry out the construction of $L$-packets as outlined in section 11, for a tame regular discrete parameter

$$\varphi : \mathcal{W}(k) \longrightarrow {}^L G.$$

The groups $W \rtimes \langle\vartheta\rangle$ and ${}^L W$ are canonically anti-isomorphic. Let $w \in W\vartheta$ correspond to the image of $\varphi(F)$ in ${}^L W$. Recall that our $L$-packet $\Pi(\varphi)$ should be parametrized by characters

$$\rho \in \mathrm{Irr}(A_\varphi) = X/(1-w)X,$$

where $X = \mathrm{Hom}(\hat{T}, GL_1)$.

For each $\lambda \in X$, we have a translation element $t_\lambda \in W_{\text{aff}}$. Also $w$ is a linear transformation on $\mathcal{A}$, so the product

$$t_\lambda w : x \mapsto \lambda + wx$$

is an element of $W_{\text{aff}}\,\vartheta$. The discrete condition 3 on $\varphi$ (see section 10.1) is equivalent to having $\mathcal{A}^w = 0$. It follows that $t_\lambda w$ has a unique fixed-point in $\mathcal{A}$, namely the point

$$x_\lambda := (1-w)^{-1} \cdot \lambda. \tag{39}$$

Recall that the pair $(\varphi, \rho)$ is only taken up to conjugacy in $\hat{G}$. Replacing the pair $(\varphi, \rho)$ by a $\hat{G}$-conjugate if necessary, we may arrange that $x_\lambda$ is contained in the closure of the chamber $C$ defined in (33).

We factor the element $t_\lambda w$ as in (35), to obtain

$$t_\lambda w = v_\lambda \omega_\lambda \vartheta, \tag{40}$$

with $v_\lambda \in W^\circ$ and $\omega_\lambda \in \Omega$.

The factorization (40) generates much of the inducing data of our Langlands correspondence. Indeed, (40) has the following properties:

1. The element $\omega_\lambda \in \Omega$ corresponds to the restriction of $\lambda$ to $Z(\hat{G})$ under the isomorphism (37). This implies that the pure inner form $G_\lambda := G_{\omega_\lambda}$ is the correct one, according to Conjecture 9.1.

2. The point $x_\lambda$ is fixed by $\omega_\lambda \vartheta$, and is good (see 12.3). Hence we have a good parahoric subgroup
$$K_\lambda := K_{\omega_\lambda, x_\lambda} \tag{41}$$
of $G_\lambda(k)$, as in section 12.3.

3. By the Lang-Steinberg theorem (see [6, p.32]), there is an element $p_\lambda \in G(K)$ which fixes $x_\lambda$, such that the conjugation map
$$\mathrm{Ad}(p_\lambda) : T_w \longrightarrow T_\lambda := p_\lambda T p_\lambda^{-1}$$
is a $k$-isomorphism. Since $p_\lambda \cdot x_\lambda = x_\lambda$, it follows that $T_\lambda(k) \subset K_\lambda$.

We now have our desired set-up:
$$T_\lambda(k) \subset K_\lambda \subset G_\lambda(k), \tag{42}$$
where $G_\lambda$ is the appropriate pure inner form of $G$, $K_\lambda$ is a good parahoric subgroup of $G_\lambda(k)$, and $T_\lambda$ is an anisotropic maximal $k$-torus of $G_\lambda$. Moreover, $K_\lambda$ is the unique parahoric subgroup of $G_\lambda(k)$ containing $T_\lambda(k)$.

To define the representation
$$\pi(\varphi, \rho) = \mathrm{Ind}_{K_\lambda}^{G_\lambda(k)} R_\lambda,$$
it remains to define the irreducible representation $R_\lambda$ of $K_\lambda$.

Recall from section 10.2 that $\varphi$ determines a character $\chi_\varphi$ of $T_w(k)$, via the abelian Langlands correspondence. Conjugating $\chi_\varphi$ via $p_\lambda$, we get a character
$$\chi_\lambda := \chi_\varphi \circ \mathrm{Ad}(p_\lambda)^{-1} \in \mathrm{Irr}\left(T_\lambda(k)\right).$$

Recall that $K_\lambda$ fits into the exact sequence
$$1 \longrightarrow K_\lambda^+ \longrightarrow K_\lambda \longrightarrow \bar{K}_\lambda(\mathfrak{f}) \longrightarrow 1. \tag{43}$$

Restricting this sequence to the subgroup $T_\lambda(k)$ of $K_\lambda$ gives an analogous sequence

$$1 \longrightarrow T_\lambda(k)^+ \longrightarrow T_\lambda(k) \longrightarrow \bar{T}_\lambda(\mathfrak{f}) \longrightarrow 1,$$

where $\bar{T}_\lambda(\mathfrak{f})$ is group of $\mathfrak{f}$-rational points in a maximal $\mathfrak{f}$-torus of the reductive group $\mathfrak{f}$-group $\bar{K}_\lambda$. Since our parameter $\varphi$ is tame, the character $\chi_\lambda$ of $T_\lambda(k)$ factors through the finite group $\bar{T}_\lambda(\mathfrak{f})$. Likewise, the representation $R_\lambda$ will factor through the finite group $\bar{K}_\lambda(\mathfrak{f})$.

These representations $R_\lambda$ are famous in the world of finite reductive groups. As mentioned in our example in section 3 for $SO_3(\mathfrak{f})$, the representation $R_\lambda$ has dimension $q-1$ and was known to Frobenius. In the late 1960s, Macdonald conjectured, based on the known character tables for $GL_n(\mathfrak{f})$ and $Sp_4(\mathfrak{f})$, that to every maximal torus $\bar{T}$ in a finite connected reductive group $\bar{K}$ and sufficiently regular character $\chi$ of $\bar{T}(k)$, there should correspond an irreducible character $R(\bar{T},\chi)$ of $\bar{K}(\mathfrak{f})$ whose dimension

$$\dim R(\bar{T},\chi) = [\bar{K}(\mathfrak{f}) : \bar{T}(\mathfrak{f})]_{p'}$$

is the maximal factor of the index $[\bar{K}(\mathfrak{f}) : \bar{T}(\mathfrak{f})]$ which is not divisible by the characteristic $p$ of $\mathfrak{f}$. Moreover, the restriction of $R(\bar{T},\chi)$ to sufficiently general elements of $\bar{T}(k)$ should agree up to sign with the sum of $\chi$ over the Weyl group of $\bar{T}$.

In the mid 1970s Macdonald's conjecture was proved by Deligne and Lusztig, building on the work of Drinfeld for $SL_2(\mathfrak{f})$. They found a cohomological construction of $R(\bar{T},\chi)$ that bears some resemblance to Schmid's construction of real discrete series. Besides the original paper [10], we refer the reader to any of the several good expositions of the Deligne-Lusztig construction, such as [6].

These Deligne-Lusztig representations are the final ingredient in our construction of $L$-packets: we define

$$R_\lambda := R(\bar{T}_\lambda, \chi_\lambda). \tag{44}$$

This is a representation of the finite group $\bar{K}_\lambda(\mathfrak{f})$, which we view as a representation of the good parahoric subgroup $K_\lambda$, via the exact sequence (43).

Putting everything together, we finally have our representation

$$\pi(\varphi,\rho) = \mathrm{Ind}_{K_\lambda}^{G_\lambda(k)} R_\lambda,$$

where $\lambda \in X$ projects to $\rho \in X/(1-w)X = \mathrm{Irr}(A_\varphi)$, and our $L$-packet $\Pi(\varphi)$ is then defined as

$$\Pi(\varphi) := \{\pi(\varphi,\rho) : \ \rho \in \mathrm{Irr}(A_\varphi)\}.$$

The simplest case of this construction is the following. Every semisimple group has a quasi-split form for which some $w \in {}^L W$ acts by inversion on $\hat{T}$, and this $w$ is unique. In this case $\hat{T}^w = \hat{T}_2$ is the 2-torsion subgroup of $\hat{T}$, and $A_\varphi = X/2X$. We have

$$x_\lambda = \tfrac{1}{2}\lambda,$$

and the root datum of $\bar{K}_\lambda$ is that of the centralizer of the involution $\lambda(-1) \in G$ (recall that characters of $\hat{T}$ are 1-parameter subgroups of $T$). The $L$-packet $\Pi(\varphi)$ has $2^n$ elements, where $n = \dim T$. These $p$-adic $L$-packets look quite similar to the real $L$-packets described in section 4. However, the Galois theory of a $p$-adic field $k$ is richer than that of $\mathbb{R}$. There are other tame regular discrete Langlands parameters, corresponding to other elements $w \in {}^L W$ for which $\hat{T}^w$ is finite. Thus the groups $A_\varphi$ will vary, and the cardinality $|A_\varphi|$ is not always a power of 2.

# 14 L-packets for $p$-adic orthogonal groups

We first describe the regular discrete parameters explicitly for orthogonal groups. As in the real case, we need a supply of some simple induced representations. For the moment, we make no assumption on tameness, since that plays no role in the following construction.

We fix a prime element $\varpi$ in the ring of integers of $k$. Recall that the Weil group $\mathcal{W}(k)$ is a semidirect product

$$\mathcal{W}(k) = \langle F \rangle \ltimes \mathcal{I}.$$

We change and simplify the notation slightly. Let

$$K := k_{2d}$$

be the unramified extension of $k$ of degree $2d$. The Weil group of $\mathcal{W}(K)$ is

$$\mathcal{W}(K) = \langle F^{2d} \rangle \ltimes \mathcal{I},$$

a normal subgroup of $\mathcal{W}(k)$, with cyclic quotient of order $2d$. The tautological embedding $\mathcal{W}(k) \hookrightarrow \mathrm{Gal}(\bar{k}/k)$ induces an isomorphism

$$\mathcal{W}(k)/\mathcal{W}(K) \simeq \mathrm{Gal}(K/k),$$

sending the coset of $F^d$ to the unique element $\tau$ of order two in the cyclic group $\mathrm{Gal}(K/k)$.

Let $\eta : \mathcal{W}(K) \to \mathbb{C}^\times$ be a character of finite order. Applying local class field theory to $K$, we have a homomorphism

$$\mathcal{W}(K) \longrightarrow \mathcal{W}(K)^{ab} \xrightarrow{\sim} K^\times, \tag{45}$$

sending $F^{2d}$ to $\varpi$, and sending $\mathcal{I}$ onto the unit group $\mathfrak{O}^\times$ in the ring of integers $\mathfrak{O}$ of $K$. We view $\eta$ as a character of $K^\times$, via (45). Note that $\eta(\varpi) = \eta(F^{2d})$ in these two viewpoints. We define

$$V(\eta) = \operatorname{Ind}_K^k \eta := \operatorname{Ind}_{\mathcal{W}(K)}^{\mathcal{W}(k)} \eta. \tag{46}$$

This gives a representation

$$\varphi : \mathcal{W}(k) \longrightarrow GL\left(V(\eta)\right) \tag{47}$$

of dimension $2d$.

We assume further that the conjugates $\eta^\sigma$ of $\eta$ by the elements of $\mathrm{Gal}(K/k)$ are pairwise distinct, and that $\eta^\tau = \eta^{-1}$. Since $\eta$ has finite order, we have $\eta^{-1} = \bar{\eta}$. The first hypothesis implies, by Mackey's theorem (cf. [34, II.7]) that $V(\eta)$ is irreducible and $V(\eta) \simeq V(\eta^\sigma)$ for any $\sigma \in \mathrm{Gal}(K/k)$. Since the dual of $V(\eta)$ is $V(\bar{\eta})$, the second hypothesis implies that $V(\eta)$ is self-dual.

Hence there is a nondegenerate bilinear form $B$ on $V(\eta)$ which is invariant under $\mathcal{W}(k)$, and unique up to scaling. Since $V(\eta)$ is irreducible, the form $B$ either symplectic or orthogonal. We have an orthogonal decomposition into two dimensional subspaces

$$V(\eta) = \bigoplus_{i=0}^{d-1} V(\eta, i),$$

where $\mathcal{I}$ acts on $V(\eta, i)$ as $\eta^{q^i} \oplus \bar{\eta}^{q^i}$, and $\varphi(F^d)$ interchanges these two summands. The form $B_i$ on $V(\eta, i)$ is either symplectic for all $i$, or orthogonal for all $i$. Let $i = 0$. Since $\eta^\tau = \bar{\eta}$, we can choose eigenvectors for $\eta$ and $\bar{\eta}$ so that the matrix of $\varphi(F^d)$ on $V(\eta, 0)$ has the form $\begin{pmatrix} 0 & \eta(F^{2d}) \\ 1 & 0 \end{pmatrix}$. This matrix preserves $B_0$. It follows that $\eta(F^{2d}) = \pm 1$, with $\eta(F^{2d}) = +1$ if $B_0$ is orthogonal, $\eta(F^{2d}) = -1$ if $B_0$ is symplectic. Recalling that $F^{2d}$ corresponds to $\varpi$ under (45), we conclude that

$$V(\eta) \quad \text{is} \quad \begin{cases} \text{orthogonal} & \text{if} \quad \eta(\varpi) = +1, \\ \text{symplectic} & \text{if} \quad \eta(\varpi) = -1. \end{cases} \tag{48}$$

Moreover, in the orthogonal case, we have $\det \varphi(F) = -1$, so $\det \varphi$ is the unique quadratic character of $\mathcal{W}(k)$ which is trivial on $\mathcal{I}$.

## 14.1 L-packets for odd p-adic orthogonal groups

Here $G = SO_{2n+1}$ and we have ${}^L G = Sp_{2n}(\mathbb{C})$, and a regular discrete Langlands parameter is a homomorphism

$$\varphi : \mathcal{W}(k) \longrightarrow Sp(V),$$

with

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s, \qquad \text{and} \quad V_i = V(\eta_i) = \mathrm{Ind}_{K_i}^k \eta_i,$$

as in (46), where the $K_i$ are unramified extensions of $k$ of even degree $2d_i \geq 2$, such that $\sum d_i = n$. By (48), the characters $\eta_i$ of $K_i^\times$ satisfy $\eta_i(\varpi) = -1$. For $\varphi$ to be regular and discrete (see 10.1), the representations $V_i$ must be pairwise non-isomorphic. Hence if $K_i = K_j$, we must assume that $\eta_j$ is not equal to any conjugate $\eta_i^\sigma$ of $\eta_i$. In order to apply the construction of $L$-packets as described in section 13, we now assume that each $\eta_j$ factors through the tame inertia group $\mathcal{I}_t$ (see 6.1), so that $\varphi$ will be tame.

We now describe explicitly the pair $(T_w, \chi_\varphi)$ associated to $\varphi$ in section 10.2. The building blocks of $T_w$ are **Coxeter tori**, defined as follows. The Weyl group $W_d$ of $Sp_{2d}(\mathbb{C})$ is the group of the hypercube in $\mathbb{R}^d$, and acts on $\mathbb{R}^d$ by permuting and changing the signs of the standard basis $\{e_1, \ldots, e_d\}$. A **Coxeter element** in $W_d$ is an element $v \in W_d$ of order $2d$ which acts by a single orbit on the set $\{\pm e_i : 1 \leq i \leq d\}$. Coxeter elements form a single conjugacy class in $W_d$. A Coxeter torus $T_{\text{cox}}$ over the $p$-adic field $k$ has character group $Y = \sum_{i=1}^d \mathbb{Z} e_i$, splitting over $k_{2d}$, for which $F$ acts on $Y$ via $v$. The torus $T_{\text{cox}}$ has $k$-rational points

$$T_{\text{cox}}(k) \simeq \{a \in K^\times : \ \tau(a) = a^{-1}\},$$

where, as above, $K = k_{2d}$ and $\tau$ is the nontrivial element of $\mathrm{Gal}(K/k_d)$.

The element $w = \varphi(F)$ is the product

$$w = w_1 \times w_2 \times \cdots \times w_s$$

of Coxeter elements $w_i \in W_{d_i}$, of order $2d_i$, in the Weyl group $W_{d_i}$ of $Sp(V_i)$, where $d_1 + \cdots + d_s = n$. The torus $T_w$ is the product

$$T_w = T_1 \times T_2 \times \cdots \times T_s$$

of corresponding Coxeter tori. The splitting field $L$ of $T_w$ is the composite of the unramified extensions $K_i$, so $L$ is unramified over $k$ of degree twice the least common multiple of $\{d_1, \ldots, d_s\}$. Each $T_i$ has $k$-rational points

$$T_i(k) := \{a \in K_i^\times : \tau_i(a) = a^{-1}\},$$

where $\tau_i$ is the unique element of order two in $\mathrm{Gal}(K_i/k)$. The character $\chi_\varphi$ is given by

$$\chi_\varphi = \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_s.$$

The group $A_\varphi \simeq \{\pm 1\}^s$ is the center of $Sp(V_1) \times \cdots \times Sp(V_s)$, and the diagonal subgroup of $A_\varphi$ is the center $\{\pm I_{2n}\}$ of $\hat{G} = Sp_{2n}(\mathbb{C})$. Recall that

$$H^1(k, SO_{2n+1}) = \mathrm{Irr}\left(\{\pm I_{2n}\}\right) = \{1, \omega\},$$

where $\omega$ gives the pure inner form $SO^*_{2n+1}$ (see section 8).

The $L$-packet $\Pi(\varphi)$ contains $2^s$ representations, parametrized by characters $\rho = \rho_1 \otimes \cdots \otimes \rho_s \in \mathrm{Irr}(A_\varphi)$. The representation $\pi(\varphi, \rho)$ lives on the quasi-split group $G(k) = SO_{2n+1}(k)$ if an even number of $\rho_i$ are nontrivial, and on the pure inner form $G_\omega(k) = SO^*_{2n+1}(k)$ otherwise .

We now determine the inducing subgroup $K_\lambda$ of $\pi(\varphi, \rho)$ (see (41)). Replacing $(\varphi, \rho)$ by a conjugate in $Sp_{2n}(\mathbb{C})$ if necessary, we can assume that for some $0 \leq \ell \leq s$, the first $\ell$ components $\rho_1, \ldots, \rho_\ell$ of $\rho$ are non-trivial and the remaining components $\rho_{\ell+1}, \ldots, \rho_s$ are trivial. We write $X = \mathrm{Hom}(\hat{T}, GL_1)$ and $\mathcal{A} = \mathbb{R} \otimes X$ as

$$X = \bigoplus_{i=1}^n \mathbb{Z} x_i, \qquad \mathcal{A} = \bigoplus_{i=1}^n \mathbb{R} x_i,$$

where $x_i$ is the $i^{th}$ coordinate function of $\hat{T} = (\mathbb{C}^\times)^n$. As lift $\lambda \in X$ of $\rho$, we take the sum of one $x_i$ from each block $V_1, \ldots, V_\ell$, as long as $\ell > 0$. If $\ell = 0$ we take $\lambda = 0$.

As in (15), the positive roots are

$$\Phi^+ = \{e_i \pm e_j : \ 1 \leq i < j \leq n\} \cup \{e_k : \ 1 \leq k \leq n\},$$

where $\{e_i\}$ is the dual basis of $\{x_i\}$. The corresponding chamber $C$ is the interior of the hypertetrahedron in $\mathcal{A}$, defined by the inequalities

$$1 - e_2 > e_1 > e_2 > \cdots > e_n > 0. \tag{49}$$

We have $\Omega = \{1, \omega\}$, where, in this viewpoint, $\omega$ acts on $\mathcal{A}$ as a reflection in the first coordinate:

$$\omega \cdot (a_1, a_2, \ldots, a_n) = (1 - a_1, a_2, \ldots, a_n). \tag{50}$$

Using the formula (39), we find that the unique fixed point of $t_\lambda w$ in $\mathcal{A}$ is

$$x_\lambda = \tfrac{1}{2}(x_1 + x_2 + \cdots + x_m) \in \bar{C},$$

where $m = d_1 + \cdots + d_\ell$, and $x_\lambda$ is read as $0 \in \mathcal{A}$ if $\ell = 0$. Note that $x_\lambda$ is good, in the sense of section 12.3. According to (38), the root datum of $\bar{K}_\lambda$ is determined by the roots in $\Phi^+$ which take integer values at $x_\lambda$.

In order to describe the groups $\bar{K}_\lambda$, we need some notation for orthogonal groups over the finite field $\mathfrak{f}$. All such groups are quasi-split, and are classified as in section 6.2. Up to $\mathfrak{f}$-isomorphism, there are three families, which we denote by

$$\mathsf{SO}_{2n+1}, \quad \mathsf{SO}_{2n}, \quad \mathsf{SO}'_{2n}, \tag{51}$$

corresponding to the quadratic forms

$$\bar{Q}_n + x_{2n+1}^2, \quad \bar{Q}_n, \quad \bar{Q}_{n-1} + \bar{N},$$

where $\bar{Q}_n = x_1 x_{n+1} + \cdots + x_n x_{2n}$ and $\bar{N}$ is the norm form on the quadratic extension $\mathfrak{f}_2$, viewed as a two-dimensional space over $\mathfrak{f}$.

Using (38), one finds that the reductive $\mathfrak{f}$-group $\bar{K}_\lambda$ is given by

$$\bar{K}_\lambda = \begin{cases} \mathsf{SO}_{2m} \times \mathsf{SO}_{2(n-m)+1} & \text{for } \ell \text{ even} \\ \mathsf{SO}'_{2m} \times \mathsf{SO}_{2(n-m)+1} & \text{for } \ell \text{ odd}, \end{cases} \tag{52}$$

using the notation (51) for finite orthogonal groups.

Finally, recall that $\pi(\varphi, \rho)$ is obtained by inducing from $K_\lambda$ the representation $R_\lambda$ in (44), whose dimension is

$$\dim R_\lambda = [\bar{K}_\lambda(\mathfrak{f}) : T_w(\mathfrak{f})]_{p'}.$$

For example, if $\rho$ is the trivial character, we have $\ell = 0$, and we get

$$\dim R_0 = \frac{(q^2-1)(q^4-1)\cdots(q^{2n}-1)}{(q^{d_1}+1)(q^{d_2}+1)\cdots(q^{d_s}+1)}.$$

## 14.2 L-packets for even p-adic orthogonal groups

The description of tame, regular, elliptic Langlands parameters for even orthogonal groups is similar, with just a few added twists, so we shall be brief. Recall that

$${}^L G = \begin{cases} SO_{2n}(\mathbb{C}) & \text{if} \quad G = SO_{2n} \\ O_{2n}(\mathbb{C}) & \text{if} \quad G = SO'_{2n}. \end{cases}$$

A tame regular discrete Langlands parameter for either $SO_{2n}$ or $SO'_{2n}$ is a homomorphism

$$\varphi : \mathcal{W}(k) \longrightarrow O(W),$$

with

$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_r, \qquad \text{and} \quad W_j = V(\mu_j) = \operatorname{Ind}_{L_j}^k \mu_j,$$

as in (46), where the $L_j$ are unramified extensions of $k$ of even degree $2d'_j \geq 2$ such that $\sum d'_j = n$. Note that the image of $\varphi$ lies in $SO(W)$ precisely if $r$ is even, in which case $\varphi$ is a parameter for $SO_{2n}$. If $r$ is odd then $\varphi$ is a parameter for $SO'_{2n}$.

The characters $\mu_j$ are orthogonal, so $\mu_j(\varpi) = +1$. Again, for $\varphi$ to be regular and discrete, the orthogonal representations $W_j$ must be pairwise non-isomorphic. Hence if $L_i = L_j$, the character $\mu_j$ is not equal to any Galois conjugate $\mu_i$.

The class of $w = \varphi(F)$, the torus $T_w = \prod T_j$, the character $\chi_\varphi = \prod \mu_j$, and the component group $A_\varphi \simeq \{\pm 1\}^r$ are as described in the odd orthogonal case. However, there is a minor relaxation of the irreducibility of $W_j = V(\mu_j)$ in the two-dimensional case. Namely, if $L_j$ is the unramified quadratic extension of $k$, we can take $\mu_j = 1$ or $\mu_j =$ the nontrivial quadratic character of $\mathfrak{f}_2^\times/\mathfrak{f}^\times$, but not both. We allow this for at most one $j$. Then $W_j = V(\mu_j)$ is reducible, but $\varphi$ remains regular.

Again we have $H^1(k, G) = \{1, \omega\}$. For $r$ even, half of the $2^r$ representations in $\Pi_\varphi$ live on $G(k) = SO_{2n}(k)$ and the other half live on $G_\omega(k) = SO^*_{2n}(k)$. For $r$ odd, each representation of $SO'_{2n}(k)$ in $\Pi(\varphi)$ will appear twice: once for $G(k)$ and once for $G_\omega(k)$. The descriptions of the inducing data $K_\lambda,\ R_\lambda$ are similar to the odd orthogonal case. We will describe $\bar{K}_\lambda$ more precisely when it is needed, in (62) below.

# 15 Symplectic root numbers

Our eventual aim is to show how the restriction of representations from pure inner forms of $SO_{2n+1}$ to pure inner forms of either $SO_{2n}$ or $SO'_{2n}$ (see section 8) is determined by symplectic root numbers. These are signs $\pm 1$ attached to the Langlands parameters of the representations in question, using ideas from number theory. In this section we give a short introduction to symplectic root numbers.

Fix a non-trivial additive character $\psi : k^+ \longrightarrow S^1$ whose kernel is the ring of integers $\mathfrak{o}$ of $k$, and let $dx$ be the Haar measure on $k^+$ giving unit volume to $\mathfrak{o}$.

Let $\varphi : \mathcal{W}(k) \longrightarrow GL(V)$ be a continuous representation of $\mathcal{W}(k)$ on a finite-dimensional $\mathbb{C}$-vector space $V$. In the notation of [37, 3.6], the **root number** of $\varphi$, with respect to $\psi$, is the number

$$\varepsilon(\varphi) := \varepsilon_D(\varphi \otimes |\cdot|^{1/2}, \psi, dx).$$

This is a non-zero complex number, which was introduced and studied by Tate, in his thesis (see [8]), for one-dimensional representations $\varphi$ of $W(k)^{ab} = k^\times$. These root numbers give a product decomposition of the constants which appear in the functional equations of Artin $L$-series (cf. [12]).

**Proposition 15.1** *The root number $\varepsilon(\varphi)$ has the following properties:*

1. $\varepsilon(\varphi \oplus \varphi') = \varepsilon(\varphi) \cdot \varepsilon(\varphi')$.
2. *If $\varphi$ is is trivial on $\mathcal{I}$, then $\varepsilon(\varphi) = 1$.*
3. *Let $K/k$ be a finite unramified extension and let $\psi_K = \psi \circ \mathrm{tr}_{K/k}$. Let $\varphi$ be a representation of $\mathcal{W}(K)$, and use $\psi_K$ to define $\varepsilon(\varphi)$ as above. Then*

$$\varepsilon(\mathrm{Ind}_{\mathcal{W}(K)}^{\mathcal{W}(k)} \varphi) = \varepsilon(\varphi).$$

4. *Let $\chi : \mathcal{W}(k) \to \mathbb{C}^\times$ be a tame character which is nontrivial on $\mathcal{I}_t$. Then we have the formula*

$$\varepsilon(\chi) = \frac{\chi(\varpi)}{\sqrt{|\mathfrak{f}|}} \sum_{t \in \mathfrak{f}^\times} \bar{\chi}(t)\psi(\frac{t}{\varpi}).$$

   *Here $\chi$ is viewed as a character of $\langle\varpi\rangle \times \mathfrak{f}^\times$ via class field theory; see* (45).

5. *If $\varphi$ is self-dual and* $\det\varphi = 1$ *then $\varepsilon(\varphi) = \pm 1$.*

See [37] for proofs of these properties.

If $V$ is a symplectic space and $\varphi : \mathcal{W}(k) \longrightarrow Sp(V)$, then the conditions of item 5 hold, and we have a **symplectic root number**

$$\varepsilon(\varphi) = \pm 1.$$

Symplectic root numbers play a role in many important questions in modern number theory. For example, if $A$ is an abelian variety over a global field $F$, then at each completion of $F$ the Tate module of $A$ defines a symplectic representation of the local Weil group (or more generally, representations of the type discussed in section 9.1). Almost all of these local representations are unramified, and the product of their symplectic root numbers is the sign in the conjectural functional equation of the $L$-series of $A$ over $F$. Since this sign determines the parity of the order of vanishing of the $L$-series in the center of the critical strip, it should determine the parity of the rank of the Mordell-Weil group, by the conjecture of Birch and Swinnerton-Dyer (cf. [13]).

Symplectic root numbers are also defined for $k = \mathbb{R}$ and $\mathbb{C}$. The branching laws for restrictions of discrete series of real orthogonal groups may be expressed in terms of real symplectic root numbers [14, section 12]. This raises the hope that $p$-adic root numbers may also determine branching laws in the $p$-adic case. In the next section we recall the Gross-Prasad conjecture on this question.

# 16 Restriction and symplectic root numbers

Fix two parameters which satisfy the conditions of being tame, regular, and discrete, as in section 14:

$$\varphi_1 : \mathcal{W}(k) \longrightarrow Sp(V), \qquad V = \bigoplus_{i=1}^{s} V(\eta_i),$$
$$\varphi_2 : \mathcal{W}(k) \longrightarrow O(W), \qquad W = \bigoplus_{j=1}^{r} V(\mu_j).$$

Then $\varphi_1$ is (part of) a Langlands parameter for an odd orthogonal group and $\varphi_2$ is (part of) a Langlands parameter for an even orthogonal group. To get complete Langlands parameters, we need characters $\rho_i$ of the respective centralizers $A_{\varphi_i}$. We define these characters $\rho_i$ using symplectic root numbers, as follows.

The tensor product $V \otimes W$ is a symplectic representation of $\mathcal{W}(k)$, centralized by $A_{\varphi_1} \times A_{\varphi_2}$. Hence for each pair $(a, b) \in A_{\varphi_1} \times A_{\varphi_2}$ the eigenspace

$$(V \otimes W)^{a \otimes b = -1}$$

is again a symplectic representation of $\mathcal{W}(k)$. We define the **branching character**

$$\rho : A_{\varphi_1} \times A_{\varphi_2} \longrightarrow \{\pm 1\}, \tag{53}$$

by the formula

$$\rho(a,b) = \varepsilon\left((V \otimes W)^{a\otimes b=-1}\right), \tag{54}$$

where the right side of (54) is the symplectic root number of $(V \otimes W)^{a\otimes b=-1}$, as in section 15 (this is simpler than the general definition of $\rho$ in [14], because each $\det V(\mu_j)$ is an unramified character in the present case). It is shown in [14] that $\rho$ is actually a character of $A_{\varphi_1} \times A_{\varphi_2}$.

Writing the branching character as $\rho = \rho_1 \otimes \rho_2$, we now have complete Langlands parameters $(\varphi_1, \rho_1)$ and $(\varphi_2, \rho_2)$ for odd and even orthogonal groups $G$ and $H$, respectively, whose $k$-isomorphism classes are determined by the values of $\rho_1$ and $\rho_2$ on the centers of $Sp(V)$ and $O(W)$, via Kottwitz' theorem (see sections 14.1 and 14.2). Using our notation for orthogonal groups in section 8, this works out as follows.

$$G \simeq \begin{cases} SO_{2n+1} & \text{if} \quad \rho_1(-I_V) = +1 \\ SO^*_{2n+1} & \text{if} \quad \rho_1(-I_V) = -1. \end{cases} \tag{55}$$

$$H \simeq \begin{cases} SO_{2n} & \text{for } r \text{ even, } \rho_2(-I_W) = +1 \\ SO^*_{2n} & \text{for } r \text{ even, } \rho_2(-I_W) = -1 \\ SO'_{2n} & \text{for } r \text{ odd.} \end{cases} \tag{56}$$

Note that previously, $G$ always denoted a quasi-split group; we have made this slight change in order to ease the notation later on. For the rest of the paper, $G$ and $H$ will be as defined in (55) and (56). Since $\rho(-I_V, -I_W) = 1$, it follows from [14, 10.9] that $H$ is always a subgroup of $G$.

Using our $L$-packets from section 11, the Langlands parameters $(\varphi_i, \rho_i)$ define irreducible representations

$$\begin{aligned} \pi = \pi(\varphi_1, \rho_1) \quad &\text{of} \quad G(k), \\ \sigma = \pi(\varphi_2, \rho_2) \quad &\text{of} \quad H(k). \end{aligned}$$

The conjectures of [14] predict that $\sigma$ occurs in the restriction of $\pi$ to $H(k)$ with multiplicity one. More precisely, it is proposed that

$$\langle \pi, \sigma \rangle := \dim \operatorname{Hom}_{H(k)}(\pi, \sigma) = 1. \tag{57}$$

Unpublished work of Bernstein and Rallis (independently) shows that $\langle \pi, \sigma \rangle \leq 1$. The rest of this paper will be devoted to proving that $\langle \pi, \sigma \rangle \geq 1$.

The conjectures of [14] also predict that no other representation in the $L$-packet $\Pi(\varphi_2)$ can appear in the restriction of a representation in $\Pi(\varphi_1)$, but at present we do not know how to prove this.

## 17 Calculation of the branching character

We begin with the explicit determination of the branching character $\rho$, defined in (53) by symplectic root numbers. The following lemma is crucial; it can be proved using the properties 15.1, see also [12].

**Lemma 17.1** *Let*

$$V = \mathrm{Ind}_K^k \eta, \qquad W = \mathrm{Ind}_L^k \mu$$

*be irreducible symplectic and orthogonal representations, respectively, of $\mathcal{W}(k)$, induced from tamely ramified characters of the unramified extensions $K$ and $L$ of $k$. Then $\varepsilon(V \otimes W) = +1$, unless $V$ and $W$ are isomorphic representations of the inertia subgroup $\mathcal{I} \subset \mathcal{W}(k)$, in which case $\varepsilon(V \otimes W) = -1$.*

We now use Lemma 17.1 to determine the character $\rho$ of the elementary abelian 2-group $A_{\varphi_1} \times A_{\varphi_2}$. Recall that the tame, regular discrete parameters for odd and even orthogonal groups, respectively, have the form

$$V = \bigoplus_{i=1}^{s} V_i, \qquad V_i = \mathrm{Ind}_{K_i}^k \eta_i,$$
$$W = \bigoplus_{j=1}^{r} W_j, \qquad W_j = \mathrm{Ind}_{L_j}^k \mu_j.$$

The group $A_{\varphi_1}$ has $\mathbb{Z}/2\mathbb{Z}$-basis elements $a_i$, the non-trivial central elements in $Sp(V_i)$, and the group $A_{\varphi_2}$ has $\mathbb{Z}/2\mathbb{Z}$-basis elements $b_j$, the non-trivial central elements in $O(W_j)$. Using Lemma 17.1, we find that

$$\rho(a_i, 1) = \varepsilon(V_i \otimes W) = \prod_{j=1}^{r} \varepsilon(V_i \otimes W_j)$$
$$= \begin{cases} -1 & \text{if} \quad \mathrm{Res}_{\mathcal{I}}(V_i) \simeq \mathrm{Res}_{\mathcal{I}}(W_j) \quad \text{for some } j, \\ +1 & \text{otherwise.} \end{cases}$$

Note that if $W_j$ exists then it is unique, since $W_1, \ldots, W_r$ are pairwise non-isomorphic. Similarly, we find that

$$\rho(1, b_j) = \varepsilon(V \otimes W_j) = \prod_{i=1}^{s} \varepsilon(V_i \otimes W_j)$$
$$= \begin{cases} -1 & \text{if} \quad \mathrm{Res}_{\mathcal{I}}(W_j) \simeq \mathrm{Res}_{\mathcal{I}}(V_i) \quad \text{for some } i, \\ +1 & \text{otherwise.} \end{cases}$$

Again, $V_i$ is unique, if it exists.

# 18 The representations $\pi$ and $\sigma$ of $SO_{2n+1}$ and $SO_{2n}$

Recall that $G$ and $H$ are the orthogonal groups determined by the Langlands parameters $(\varphi_1, \rho_1)$ and $(\varphi_2, \rho_2)$ (see (55) and (56)). We now use our formula for $\rho = \rho_1 \otimes \rho_2$ in the previous section to determine the representations

$$\pi = \pi(\varphi_1, \rho_1) \in \mathrm{Irr}\,(G(k))\,, \quad \text{and} \quad \sigma = \pi_2(\varphi_2, \rho_2) \in \mathrm{Irr}\,(H(k))\,.$$

Then we will show that $\sigma$ occurs in the restriction of $\pi$ from $G(k)$ to $H(k)$, in the sense of (57).

As in section 17, we have

$$V = \bigoplus_{i=0}^{s} V_i, \qquad W = \bigoplus_{j=0}^{r} W_j.$$

Suppose that exactly $\ell$ of these factors give isomorphic representations of the inertia subgroup $\mathcal{I}$ of $\mathcal{W}(k)$. Reorder the factors so that

$$V_1 \simeq W_1, \quad V_2 \simeq W_2, \quad \ldots, \quad V_\ell \simeq W_\ell$$

as representations of $\mathcal{I}$ and set $2m = \sum_{i=1}^{\ell} \dim V_i = \sum_{j=1}^{\ell} \dim W_j$.

Hence $\rho_1$ is nontrivial precisely on the basis vectors $a_1, \ldots, a_\ell$ of $A_{\varphi_1}$, so that $\rho_1(-I_V) = (-1)^\ell$, and (55) takes the more explicit form

$$G = \begin{cases} SO_{2n+1} & \text{for} \quad \ell \quad \text{even} \\ SO^*_{2n+1} & \text{for} \quad \ell \quad \text{odd.} \end{cases} \tag{58}$$

Recall that $X = \mathbb{Z}^n$, with basis $\{e_i\}$. As lift $\lambda_1 \in X$ of $\rho_1$, we may take the sum of one $e_i$ from each block $V_1, \ldots, V_\ell$. Our good parahoric subgroup $K_{\lambda_1} \subset G$ has reductive quotient given, in the notation of (51), by

$$\bar{K}_{\lambda_1} = \bar{K}'_{\lambda_1} \times \bar{K}''_{\lambda_1} = \begin{cases} \mathsf{SO}_{2m} \times \mathsf{SO}_{2(n-m)+1} & \text{for} \quad \ell \quad \text{even} \\ \mathsf{SO}'_{2m} \times \mathsf{SO}_{2(n-m)+1} & \text{for} \quad \ell \quad \text{odd.} \end{cases} \tag{59}$$

The representation $\pi$ is then given explicitly by

$$\pi = \mathrm{Ind}_{K_{\lambda_1}}^{G(k)} R_{\lambda_1}, \tag{60}$$

where $R_{\lambda_1}$ is the Deligne-Lusztig representation defined in (44).

Likewise $\rho_2$ is nontrivial precisely on the basis vectors $b_1, \ldots, b_\ell$ of $A_{\varphi_2}$, so that $\rho_2(-I_W) = (-1)^\ell$, and (56) takes the more explicit form

$$H \simeq \begin{cases} SO_{2n} & \text{for } r, \ell \text{ both even} \\ SO_{2n}^* & \text{for } r \text{ even, } \ell \text{ odd} \\ SO_{2n}' & \text{for } r \text{ odd.} \end{cases} \tag{61}$$

For $\rho_2$, we may choose the lift $\lambda_2 \in X$ so that $x_{\lambda_2} = x_{\lambda_1}$. Using the method of section 14.1, we find that our good parahoric subgroup $K_{\lambda_2} \subset H$ has reductive quotient given by

$$\bar{K}_{\lambda_2} = \bar{K}'_{\lambda_2} \times \bar{K}''_{\lambda_2}, \tag{62}$$

where

$$\bar{K}'_{\lambda_2} = \bar{K}'_{\lambda_1} \tag{63}$$

and

$$\bar{K}''_{\lambda_2} = \begin{cases} \mathsf{SO}_{2(n-m)} & \text{for} \quad r-\ell \quad \text{even} \\ \mathsf{SO}'_{2(n-m)} & \text{for} \quad r-\ell \quad \text{odd.} \end{cases}$$

Finally, the representation $\sigma$ is given explicitly by

$$\sigma = \mathrm{Ind}_{K_{\lambda_2}}^{H(k)} R_{\lambda_2}, \tag{64}$$

where $R_{\lambda_2}$ is the Deligne-Lusztig representation defined in (44).

# 19 The multiplicity of $\sigma$ in the restriction of $\pi$

In the next two sections we prove that $\langle \pi, \sigma \rangle \geq 1$, in the sense of (57). Elementary properties of induced representations show that

$$\mathrm{Hom}_{\bar{K}_{\lambda_2}(\mathfrak{f})}\left(R_{\lambda_1}, R_{\lambda_2}\right) \subset \mathrm{Hom}_{H(k)}\left(\pi, \sigma\right).$$

We will show that $\mathrm{Hom}_{\bar{K}_{\lambda_2}(\mathfrak{f})}\left(R_{\lambda_1}, R_{\lambda_2}\right) \neq 0$.

Our inducing representations $R_{\lambda_i}$ factor as products of Deligne-Lusztig representations

$$R_{\lambda_i} = R'_{\lambda_i} \otimes R''_{\lambda_i},$$

according to the factorizations (59) and (62).

Recall from section 14 that for $\varphi_1$ our torus and character are products

$$\prod_i T_i, \qquad \prod_i \eta_i, \tag{65}$$

and the characters $\eta_i$ factor through the reductions $T_i(k) \longrightarrow \bar{T}_i(\mathfrak{f})$. Likewise, for $\varphi_2$ our torus and character are products

$$\prod_j S_j, \qquad \prod_j \mu_j, \tag{66}$$

and the characters $\mu_j$ factor through the reductions $S_j(k) \longrightarrow \bar{S}_j(\mathfrak{f})$. By our assumption that $V_1 \simeq W_1, \ \ldots, \ V_\ell \simeq W_\ell$, we have

$$\prod_{i=1}^{\ell} \bar{T}_i \simeq \prod_{j=1}^{\ell} \bar{S}_j, \tag{67}$$

and the characters

$$\prod_{i=1}^{\ell} \bar{\eta}_i, \qquad \prod_{j=1}^{\ell} \bar{\mu}_j,$$

are equal on the torus (67). Recall from (63) that $\bar{K}'_{\lambda_2} = \bar{K}'_{\lambda_1}$. Hence we have $R'_{\lambda_1} \simeq R'_{\lambda_2}$.

Setting $\nu = n - m$, it remains only to show that $R''_{\lambda_2}$ appears in the restriction of $R''_{\lambda_1}$ from $\mathsf{SO}_{2\nu+1}(\mathfrak{f})$ to either $\mathsf{SO}_{2\nu}(\mathfrak{f})$ or $\mathsf{SO}'_{2\nu}(\mathfrak{f})$ according as $r - \ell$ is even or odd. This is a consequence of the result which is stated in the next section, and proved in [28].

# 20 Restriction of Deligne-Lusztig representations

This last section concerns only representations of groups over the finite field $\mathfrak{f}$, so we will simplify our notation.

Let $T$ be a maximal $\mathfrak{f}$-torus in the group $G = \mathsf{SO}_{2n+1}$ over $\mathfrak{f}$, and let $W(T)$ be the $\mathfrak{f}$-rational points in the Weyl group of $T$ in $G$. We say a character of $T(\mathfrak{f})$ is **regular** if it has trivial stabilizer in $W(T)$.

Assume that $T$ is anisotropic over $\mathfrak{f}$. Then $T = \prod_{i=1}^{s} T_i$ is a product of Coxeter tori in odd orthogonal groups. Let $\eta = \prod_i^s \eta_i$ be a regular character of $T(\mathfrak{f})$.

Let $S = \prod_{j=1}^{r} S_j$ be another such product of Coxeter tori, with $\sum \dim S_j = n$. Then $S$ is an anisotropic maximal torus in the $\mathfrak{f}$-group $H$ which is $\mathsf{SO}_{2n}$ or $\mathsf{SO}'_{2n}$ according as $r$ is even or odd. Let $\mu = \prod_{j=1}^{r} \mu_j$ be a regular character of $S(\mathfrak{f})$.

We then have irreducible Deligne-Lusztig representations $R(T, \eta)$ on $G(\mathfrak{f})$ and $R(S, \mu)$ on $H(\mathfrak{f})$. The following result is proved in [28].

**Theorem 20.1** *The $H(\mathfrak{f})$-invariants in $R(T, \eta) \otimes R(S, \mu)$ have dimension*

$$\dim\left[R(T, \eta) \otimes R(S, \mu)\right]^{H(\mathfrak{f})} = 0 \quad \textit{or} \quad 1.$$

*This dimension is 0 if any factor matches, in the sense that there are $i, j$ such that $T_i \simeq S_j$ and $\eta_i$ is a Galois conjugate of $\mu_j$. If there is no matching, the dimension is 1.*

In the case of $R''_{\lambda_1}$ and $R''_{\lambda_2}$ of the previous section, we have no matching, by the definition of $\ell$. This concludes the proof that $\sigma$ occurs in $\mathrm{Res}_{H(k)}(\pi)$.

It is a nice exercise to verify 20.1 for $n = 1$, given the character table of $\mathsf{SO}_3(\mathfrak{f}) = PGL_2(\mathfrak{f})$ (cf. [11, section 9]). One finds that $R(T, \eta)$ contains all characters of $\mathsf{SO}_2(\mathfrak{f})$, and all characters of $\mathsf{SO}'_2(\mathfrak{f})$ except $\eta$ and $\eta^{-1}$. We used this in section 3.

The main idea of the proof of 20.1 for general $n$ is to show that the sum over $H(\mathfrak{f})$ of the the character of $R(T, \eta) \otimes R(S, \mu)$ is a rational function in $q = |\mathfrak{f}|$ of degree $\leq 0$ whose leading term can be computed explicitly. This idea goes back to Thoma [38] for the pair $GL_n \supset GL_{n-1}$, and was used by Hagedorn [16] in his 1994 Harvard PhD thesis to give some abstract formulas for restriction for other classical groups. Pushing this method further in [28] one obtains closed formulas as in 20.1, as well as qualitative Deligne-Lusztig restriction formulas for a general pair of reductive $\mathfrak{f}$-groups $G \supset H$.

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