On the values of Artin L-functions

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0. Information

I wrote this paper in 1979, as an attempt to extend the results of Borel [2] on zeta functions at negative integers to Artin L-functions. The conceptual framework was provided by Tate's formulation [10] of Stark's conjectures. What I needed was a workable definition of the regulator homomorphism in complex K-theory. I discussed this with Borel at the Institute, first over lunch and then in his office. It is an honor to dedicate this paper to his memory.

Using results of Bloch and Thurston, I was able to treat the special case of Dirichlet L-series at $s = -1$. I had the hope of treating Dirichlet L-series at all negative integers, where the order of vanishing is either zero or one, but was unable to construct the required "cyclotomic classes" in K-theory. This was done by Beilinson [11], who also found the generalization of my conjecture, and the conjectures of Deligne [6] on special values, to all motivic L-functionss.

I didn't publish this paper, but it has circulated as a preprint for 25 years. For reasons of historical interest, I decided to publish it in its original form here. I have updated the references, and added some comments on the recent literature at the end of the paper.

1. Higher logarithms

If A is a commutative ring with unit, we let K_iA, $i > 0$, be the abelian groups defined by Quillen [9]. Then

$$K_iA = \pi_i(BGL(A)^+) \tag{1.1}$$

and one has a canonical isomorphism:

$$K_iA \otimes \mathbb{Q} \simeq \text{Prim } H_i(GL(A), \mathbb{Q}). \tag{1.2}$$

In this section we will define canonical homomorphisms

$$e_{2m+1} : K_{2m+1}\mathbb{C} \longrightarrow \mathbb{R}. \tag{1.3}$$

If we identify $K_1\mathbb{C} \simeq \mathbb{C}^*$ then e_1 becomes the homomorphism

$$\mathbb{C}^* \to \mathbb{R}$$

$$z \mapsto \log|z| = \text{ Re } (\log z).$$

Similarly, e_3 may be identified with the imaginary part of the dilogarithm. In general, if $\alpha \mapsto \overline{\alpha}$ is the involution on $K_{2m+1}\mathbb{C}$ induced by complex conjugation, we have

$$e_{2m+1}(\overline{\alpha}) = (-1)^m e_{2m+1}(\alpha). \tag{1.4}$$

To define e_{2m+1} we will use Quillen's isomorphism (1.2) and construct an (indecomposible) class

$$c_{2m+1} \in H^{2m+1}(GL(\mathbb{C}), \mathbb{R}). \tag{1.5}$$

This class will be in the image of the map

$$H_c^*(GL(\mathbb{C}), \mathbb{R}) \xrightarrow[f]{} H^*(GL(\mathbb{C}), \mathbb{R}), \tag{1.6}$$

where H_c^* denotes the continuous group cohomology of of $GL(\mathbb{C})$.

For $n > m$ let X_n be the symmetric space

$$X_n = U_n \backslash U_n \times U_n, \tag{1.7}$$

where U_n is the unitary group of degree n, embedded in the product along the diagonal. Since X_n is the "compact dual" of $GL_n(\mathbb{C})$, one has a canonical isomorphism [3]

$$H^*_{\text{top}}(X_n, \mathbb{R}) \xrightarrow[g]{\sim} H_c^*(GL_n(\mathbb{C}), \mathbb{R}). \tag{1.8}$$

Complex conjugation acts as an involution on both exterior algebras; if u is a topological class of pure degree

$$\overline{g(u)} = (-1)^{\deg u} \cdot g(\overline{u}). \tag{1.9}$$

The integral cohomology of X_n is a free exterior algebra, generated by the classes of odd spheres [4]:

$$H^*_{\text{top}}(X_n, \mathbb{Z}) \simeq \Lambda^*_{\mathbb{Z}}(u_1, u_3, \ldots, u_{2n-1}). \tag{1.10}$$

We have

$$\overline{u}_{2m-1} = (-1)^m \cdot u_{2m-1} \tag{1.11}$$

as complex conjugation on $\mathbb{C}^m$ induces an automorphism of S^{2m-1} of degree $(-1)^m$. Setting

$$v_{2m+1} = g(u_{2m+1}) \tag{1.12}$$

and passing to the limit over n [3] gives an isomorphism

$$H_c^*(GL(\mathbb{C}), \mathbb{R}) \simeq \Lambda_{\mathbb{R}}^*(v_1, v_3, v_5, \ldots). \tag{1.13}$$

Furthermore, by combining (1.9) and (1.11) we obtain

$$\overline{v}_{2m+1} = (-1)^m \cdot v_{2m+1}. \tag{1.14}$$

Finally, let

$$c_{2m+1} = 2\pi \cdot f(v_{2m+1}) \tag{1.15}$$

where f is the map in (1.6). This is the canonical class which corresponds to the honomorphisms e_{2m+1}. An explicit cocycle for c_{2m+1} can be constructed geometrically, using the methods of Dupont [7].

2. The theorems of Dirichlet and Borel

Let F be a number field of finite degree n over $\mathbb{Q}$. Let A be the ring of integers in F. Let

$$Y = \mathrm{Hom}(R_{F/Q}\mathbb{G}_m, \mathbb{G}_{m/\mathbb{C}}) \simeq \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})}. \tag{2.1}$$

We have an exact sequence

$$0 \to Y_0 \to Y \xrightarrow{\deg} \mathbb{Z} \to 0 \tag{2.2}$$

$$\sum a_\phi \cdot \phi \mapsto a_\phi.$$

The groups $\mathrm{Aut}(F)$ and $\mathrm{Aut}_{\mathbb{R}}\mathbb{C} \simeq \langle c \rangle$ act on Y and stabilize Y_0; they both act trivially on the quotient $Y/Y_0 \simeq \mathbb{Z}$.

Let m be a nonnegative integer. Any complex embedding $\phi : F \to \mathbb{C}$ induces a map $\phi_* : K_{2m+1}A \to K_{2m+1}\mathbb{C}$ by functorality. Define the homomorphism

$$\begin{aligned} K_{2m+1}A \; &\underset{\lambda_{2m+1}}{\longrightarrow} Y \otimes \mathbb{R} \\ \alpha \; &\mapsto \sum_\phi e_{2m+1}(\phi_*(\alpha)) \cdot \phi, \end{aligned} \tag{2.3}$$

where e_{2m+1} is the homomorphism described in Section 1. By (1.4) the image of λ_{2m+1} lies in the subspace $(Y \otimes \mathbb{R})^{(-1)^m}$, where $\pm$ denotes the eigenspaces of c. If $m = 0$ then $K_1A \simeq A^*$ and $e_1(\phi_*(\alpha)) = \log|\phi_*(\alpha)|$. Consequently the image of λ_1 lies in the subspace $(Y_0 \otimes \mathbb{R})^+$, by the product formula.

Theorem 2.4 (Dirichlet, cf. [5]). *The map* λ_1 *induces an isomorphism*

$$K_1 A \otimes \mathbb{R} \xrightarrow[\lambda_1]{\sim} (Y_0 \otimes \mathbb{R})^+.$$

Theorem 2.5 (Borel, [2], [3]). *For* $m > 0$ *the map* λ_{2m+1} *induces an isomorphism*

$$K_{2m+1} A \otimes \mathbb{R} \xrightarrow[\lambda_{2m+1}]{\sim} (Y \otimes \mathbb{R})^{(-1)^m}.$$

The above isomorphisms are equivariant for $\mathrm{Aut}(F)$.

3. Artin L-series

Let k be a number field, and let F be a finite Galois extension of k. Let $G = \mathrm{Gal}(F/k)$; if w is a place of k let G_w be a decomposition group for w in G.

Let E be a finite extension field of $\mathbb{Q}$. Let V be a vector space of finite dimension over E which admits a linear action of G:

$$\rho : G \to \mathrm{Aut}_E(V). \tag{3.1}$$

The class of this representation corresponds to an Artin motive over k with coefficients in E [6].

For any $s \in \mathrm{Hom}(E, \mathbb{C})$ we obtain a complex representation

$$\rho^\sigma : G \to \mathrm{Aut}_{\mathbb{C}}(V^\sigma) \tag{3.2}$$

by change of base. Let $L(V^\sigma, s)$ be its Artin L-series, and let

$$L(V, s) = (\dots, L(V^\sigma, s), \dots)_{\mathrm{Hom}(E,\mathbb{C})}. \tag{3.3}$$

This is a meromorphic function on $\mathbb{C}$ with values in $\mathbb{C} \otimes E \simeq \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$.

For $m \leq 0$ the order of $L(V^\sigma, s)$ at $s = -m$ is independent of σ. Let $d_m(V)$ be this order; we have the formulas

$$d_0(V) = \sum_{w|\infty} \dim V^{G_w} - \dim V^G. \tag{3.4}$$

$$d_m(V) = \sum_{\substack{\text{complex} \\ w|\infty}} \dim V^{G_w} + \sum_{\substack{\text{real} \\ w|\infty}} \dim V^{G_w} \qquad \begin{array}{l} m \ \text{even} \\ m > 0 \end{array} \tag{3.5}$$

$$d_m(V) = \sum_{\substack{\text{complex} \\ w|\infty}} \dim V^{G_w} + \sum_{\substack{\text{real} \\ w|\infty}} \dim \left(V/V^{G_w}\right) \qquad m \ \text{odd}. \tag{3.6}$$

We define

$$L(V,-m)^* = \lim_{s\to -m} \frac{L(V,s)}{(s+m)^{d_m(V)}} \tag{3.7}$$

in $(\mathbb{C}\otimes E)^*$, and wish to conjecture its value modulo E^*.

The theorems of Dirichlet and Borel, applied to the field F, give us canonical G-isomorphisms:

$$K_1 A\otimes\mathbb{R} \xrightarrow[\lambda_1]{\sim} (Y_0\otimes\mathbb{R})^+$$

$$K_{2m+1}A\otimes\mathbb{R} \xrightarrow[\lambda_{2m+1}]{\sim} (Y\otimes\mathbb{R})^{(-1)^m} \qquad m>0.$$

We may therefore *choose* G-isomorphisms of the underlying $\mathbb{Q}$-vector spaces:

$$K_1 A\otimes\mathbb{Q} \xleftarrow[\phi_1]{\sim} (Y_0\otimes\mathbb{Q})^+$$

$$K_{2m+1}A\otimes\mathbb{Q} \xleftarrow[\phi_{2m+1}]{\sim} (Y\otimes\mathbb{Q})^{(-1)^m} \qquad m>0.$$

Let

$$W_{2m+1} = \begin{cases} (Y_0\otimes\mathbb{C})^+ & \text{if } m=0,\\ (Y\otimes\mathbb{C})^{(-1)^m} & \text{if } m>0.\end{cases} \tag{3.8}$$

Then $W_{2m+1}\otimes V$ is a free $(\mathbb{C}\otimes E)$-module. The group G acts linearly and one has

$$\operatorname{rank}_{\mathbb{C}\otimes E}(W_{2m+1}\otimes V)^G = d_m(V) \tag{3.9}$$

by Frobenius reciprocity.

The autmorphism $(\lambda_{2m+1}\circ\phi_{2m+1})\otimes 1$ is $\mathbb{C}\otimes E$-linear and commutes with the action of G. Define

$$c(V, 2m+1) = \det((\lambda_{2m+1} \circ \phi_{2m+1}) \otimes 1 | (W_{2m+1} \otimes V)^G). \tag{3.10}$$

Modulo E^* this is independent of the choice of ϕ_{2m+1}.

Conjecture 3.11. *For $m \geq 0$ the value $L(V, -m)^*$ is equal to the product of $c(V, 2m+1)$ by an element of E^*.*

4. Remarks

For $m = 0$, Conjecture 3.11 is Stark's conjecture, as formulated by Tate [10]. It is known to be true in the following cases:

V has a rational character	(Dirichlet, Ono, Lichtenbaum, Tate)
V is abelian and $k = \mathbb{Q}$	(Dirichlet, Lerch).
V is abelian and $k = \mathbb{Q}(\sqrt{-D})$	(Kronecker).
$d_0(V) = 0$	(Siegel).

When $d_m(V) = 0$, Conjecture 3.11 is a special case of Deligne's conjecture on the critical values of motivic L-series [6]. This case is known to be true by results of Siegel.

When $m > 0$ and $d_m(V) > 0$ less is known. For the *trivial* representation one has the much sharper conjectures of Lichetenbaum [8], and conjecture 3.11 was proved by Borel [2]. One deduces that (3.11) is true whenever V has the form $\oplus m_i \mathrm{Ind}_{H_i}^{G} l_{H_i}$ with $m_i \in \mathbb{Z}$.

It would be very interesting to prove (3.11) either for all V with rational character, or for all abelian representations V of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

We will show how the methods of Bloch [1] can be used to settle the latter case in the affirmative when $m = 1$.

5. Character sums

For $k > 0$ and $|z| \leq 1$ define

$$D_k(z) = \sum_{n \geq 1} \frac{z^n}{n^{k+1}}.$$

Theorem 5.1. *Let $k \geq 1$ be an integer. If χ is any Dirichlet character* $(\mathrm{mod}\ f)$ *then*

$$\sum_{(\mathbb{Z}/f)^*} \chi(a) D_k(e^{2\pi i a/f}) \neq 0.$$

Corollary 5.2. *Let* $\mathrm{dilog}(z) = \mathrm{Im}\ D_1(z)$. *If χ is any odd Dirichlet character* $(\mathrm{mod}\ f)$ *then*

$$\sum_{(\mathbb{Z}/f)^*} \chi(a) \mathrm{dilog}(e^{2\pi i a/f}) \neq 0.$$

Proof of Corollary. Since $D_k(\overline{z}) = \overline{D_k(z)}$, the real part of $D_1(z)$ is an even function. Since χ is assumed odd:

$$\sum \chi(a) D_1(e^{2\pi i a/f}) = i \cdot \sum \chi(a) \mathrm{dilog}(e^{2\pi i a/f}).$$

Thus the Corollary follows from the theorem.

Translation. $\mathrm{dilog}(e^{2\pi i a/f}) = 2\mathfrak{h}(\pi a/f)$ when $\mathfrak{h}$ is Lobachevsky's function.

Proof of Theorem. First assume χ is a primitive character (mod f), and let $L(s,\chi) = \sum_{n\geq 1} \chi(n) n^{-s}$ be its Dirichlet L-series ($s > 1$).

Define the Fourier transform of χ on $\mathbb{Z}/f$ by

$$\hat{\chi}(a) = \frac{1}{f} \sum_{(\mathbb{Z}/f)^*} \chi(b) e^{2\pi i ab/f}.$$

Since χ is primitive, $\hat{\chi}(a) = 0$ unless $(a, f) = 1$. Hence

$$\chi(n) = \sum_{(\mathbb{Z}/f)^*} \hat{\chi}(a) e^{-2\pi i an/f}.$$

by Fourier inversion. Putting this formula into the L-series, we obtain

$$L(s,\chi) = \sum_{(\mathbb{Z}/f)^*} \hat{\chi}(a) D_{s-1}(e^{-2\pi i a/f}).$$

Define the Gauss sum

$$g(\chi) = \frac{1}{f} \sum_{(\mathbb{Z}/f)^*} \chi(a) e^{2\pi i a/f}.$$

Then making the change of variables $a \mapsto ab$, we see that $g(\chi) = \chi(a)\hat{\chi}(a)$ for all $(a, f) = 1$. Consequently if $k \geq 1$:

$$L(1+k,\chi) = g(\chi) \sum_{(\mathbb{Z}/f)^*} \chi^{-1}(a) D_k(e^{-2\pi i a/f}). \tag{5.1}$$

But $L(s,\chi)$ admits an Euler product which converges for $s > 1$:

$$L(s,\chi) = \prod_p (1 - \chi(p) p^{-s})^{-1}.$$

Consequently, $L(1+k,\chi) \neq 0$ and by (5.1):

$$\sum_{(\mathbb{Z}/f)^*} \chi^{-1}(a) D_k(e^{-2\pi i a/f}) \neq 0.$$

Now start with the primitive character χ^{-1} to obtain

$$\sum_{(\mathbb{Z}/f)^*} \chi(a) D_k(e^{2\pi i a/f}) = \chi(-1) \sum_{(\mathbb{Z}/f)^*} \chi(a) D_k(e^{-2\pi i a/f}) \neq 0.$$

This completes the proof when χ is primitive (mod f).

Now assume $g|f$ and χ comes from a primitive character (mod g). Since the function $D_k : S^1 \to \mathbb{C}$ is a distribution of weight k, we have the formula:

$$(f/g)^k \sum_{(\mathbb{Z}/f)^*} \chi(a) D_k(e^{2\pi i a/f}) = \prod_{\substack{p|f \\ p\nmid g}} (1 - \chi(p) p^k) \cdot \sum_{(\mathbb{Z}/g)^*} \chi(b) D_k(e^{2\pi i b/g}).$$

The sum on the right-hand side is nonzero, by our previous argument. So is the product, as $\chi(p)$ is a root of unity and $k \geq 1$. This completes the proof.

Notes:

1) The result of Theorem 5.1 also holds when $k = 0$, if we assume that χ is *primitive* and $\chi \neq 1$.

2) The function $L(s, \chi)$ satisfies a functional equation, which relates the value at $s = k + 1$ to the leading term in the Taylor expansion at $s = -k$. For example, if χ is *primitive* and $\chi(-1) = (-1)^k$ then

$$L(-k,\chi)=0$$

$$L'(-k,\chi)=-\frac{1}{2}\frac{f^k k!}{(2\pi i)^k}\sum_{(\mathbb{Z}/f)^*}\chi(a)D_k(e^{2\pi i a/f}).$$

6. K_3 OF ABELIAN FIELDS

Let $F = \mathbb{Q}(\mu_f)$ and let $A = \mathbb{Z}[\mu_f]$. The Artin map gives a canonical isomorphism:

$$\begin{aligned} (\mathbb{Z}/f)^* &\simeq \mathrm{Gal}(F/\mathbb{Q}) \\ b &\to \mathrm{Frob}(b) \qquad\qquad \text{where} \qquad \zeta^{\mathrm{Frob}(b)} = \zeta^b. \end{aligned}$$

Fix an embedding $\phi : F \to \mathbb{C}$; this induces a map $\phi_* : K_3A \to K_3\mathbb{C}$ by functorality. Recall the map $e_3 : K_3\mathbb{C} \to \mathbb{R}$ defined in §1.

Theorem 6.1 (Bloch, Thurston). *For $a \in (\mathbb{Z}/f)^*$ there exist elements $C_f(a)$ in $K_3A \otimes \mathbb{Q}$ which satisfy*

1) $C_f(a)^{\mathrm{Frob}(b)} = C_f(ab)$,
2) $C_f(-a) = -C_f(a)$,
3) $e_3(\phi_* C_f(a)) = \frac{1}{2}\mathrm{dilog}(e^{2\pi i a/f})$.

These properties (and the choice of ϕ) completely determine the elements $C_f(a)$.

We will combine this result with Corollary 5.2 to prove the following result.

Theorem 6.2. *1) For any character χ $(\mathrm{mod}\ f)$ the element $C_\chi = \sum_{(\mathbb{Z}/f)^*} \chi(a) \otimes C_f(a)$ spans the χ^{-1}-eigenspace of $\mathbb{C} \otimes K_3A$.*

2) The elements $\{C_f(a) : a \in (\mathbb{Z}/f)^*\}$ span the vector space $K_3A \otimes \mathbb{Q}$. Any relation between them is a consequence of the relation 2) in Theorem 6.1.

Combining these results with the explicit formulas for $L'(-1,\chi)$ given at the end of §5, one obtains a proof of conjecture 3.11 for all abelian representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ when $m = 1$.

Proof of 6.1. To exhibit elements in $K_3A \otimes \mathbb{R}$ we will use Borel's isomorphism (2.5): $K_3A \otimes \mathbb{R} \underset{\lambda_3}{\xrightarrow{\sim}} (Y \otimes \mathbb{R})^-$. The elements $\{\phi^{-\mathrm{Frob}(a)} - \phi^{\mathrm{Frob}(a)} : a \in (\mathbb{Z}/f)^*/\pm 1\}$ give a basis for the space $(Y \otimes \mathbb{R})^-$. Associating to each basis element a copy of hyperbolic 3-space

$$H = \{(x + iy, z) \in \mathbb{C} \times \mathbb{R}_+^*\}$$

we may specify an element $\alpha \in K_3A \otimes \mathbb{R}$ by giving a collection of 3-simplices (or, more generally, 3-chains) in H:

$$\Delta(\alpha) = (\ldots, \Delta_a(\alpha), \ldots)_{a \in (\mathbb{Z}/f)^*}/\pm 1.$$

Then $\lambda_3(\Delta_a(\alpha)) = (\ldots, \mathrm{vol}\,\Delta_a(\alpha), \ldots)$, where the volume is computed with respect to the invariant 3-form $(dxdydz)/\pi z^3$. It is sufficient, for the purpose of representation, to use ideal simplices $\Delta_a(\alpha)$ with four vertices on the boundary.

What condition will insure that $\Delta(\alpha)$ lies in the rational vector space $K_3A \otimes \mathbb{Q}$? We might hope this would hold when the four vertices of $\Delta(\alpha)$ lie in the *rational* boundary $\mathbb{P}^1(F) \hookrightarrow (\ldots, \mathbb{P}^1(\mathbb{C})_a, \ldots)$. At least this condition makes it easy to compute the Galois action: Since Borel's isomorphism commutes with $G = \mathrm{Gal}(F/\mathbb{Q})$, the Galois group acts on these cycles via its action on the vertices in $\mathbb{P}^1(F)$.

However, the rational boundary assumption does *not* suffice to give a rational cycle: There is a further geometric invariant which must vanish. If we normalize the vertices of $\Delta(\alpha)$ to be at the points $\{\infty, 0, 1, z_\alpha\}$ with $z_\alpha \in F - \{0, 1\}$, Thurston has shown that α is a rational cycle iff $(z_\alpha) \wedge (1 - z_\alpha) = 0$ in the rational vector space $\Lambda^2(F^* \otimes \mathbb{Q})$. More generally, the chain $\sum_i a_i \Delta(\alpha_i)$ with $a_i \in \mathbb{Q}$ gives a rational cycle iff

$$\sum_i a_i \cdot (z_{\alpha_i}) \wedge (1 - z_{\alpha_i}) = 0 \qquad \text{in} \quad \Lambda^2(F^* \otimes \mathbb{Q}).$$

The condition is *certainly* met when z_α is a root of unity! Moreover, the volume of an ideal simplex with vertices $\{\infty, 0, 1, e^{2\pi i a/f}\}$ is $\frac{1}{\pi}\mathrm{dilog}(e^{2\pi i a/f})$. Taking $C_f(a)$ to be the element in $K_3 A \otimes \mathbb{Q}$ represented by the ideal simplex with vertices $\{\infty, 0, 1, \phi^{-1}(e^{2\pi i a/f})\}$ in $\mathbb{P}^1(F)$, we get the theorem.

Proof of 6.2. Since Borel's isomorphism $K_3 A \otimes \mathbb{C} \xrightarrow[\lambda_3]{\sim} (Y \otimes \mathbb{C})^-$ is G-equivariant the χ^{-1}-eigenspace of $K_3 A \otimes \mathbb{C}$ is zero when χ is even and one-dimensional when χ is odd. By 6.1 we see that the element C_χ always lies in the χ^{-1}-eigenspace; it suffices to show $C_\chi \neq 0$ when χ is odd.

But the ϕ-coordinate of $\lambda_3(C_\chi)$ is equal to

$$\frac{1}{\pi} \sum_{(\mathbb{Z}/f)^*} \chi(a) \mathrm{dilog}(e^{2\pi i a/f})$$

by 6.1. This character sum is nonzero by 5.2. Hence $C_\chi \neq 0$.

Now observe that the elements $C_f(a)$ with $(a, f) = 1$ span $K_3 A \otimes \mathbb{C}$, as they span all the nontrivial eigenspaces for G. Hence they span

$K_3A \otimes \mathbb{Q}$; since this space has dimension $\phi(f)/2$, there can be no additional relations beyond those in 6.1. This completes the Proof of 6.2.

Finally, we will discuss the norms of elements in $K_3A \otimes \mathbb{Q}$ to abelian subfields E of F. Let B be the ring of integers in E and $H = \mathrm{Gal}(F/E)$. The inclusion $B \underset{i}{\hookrightarrow} A$ induces a G-homomorphism $i_* : K_3B \to K_3A$. Using Borel's isomorphism, one can show

Lemma 6.3. *The map* $i_* \otimes 1 : K_3B \otimes \mathbb{Q} \to K_3A \otimes \mathbb{Q}$ *is injective and has image equal to the subspace* $(K_3A \otimes \mathbb{Q})^H$.

Now for $\sigma \in G/H$ let a_σ be any lifting of σ to G. Define the element

$$C_f(\sigma) = \sum_{a \in H} C_f(aa_\sigma) = \sum_{a \in H} C_f(a_\sigma)^{\mathrm{Frob}(a)}.$$

By Lemma 6.3, this gives an element of $K_3B \otimes \mathbb{C}$ which is independent of the lifting chosen.

Corollary 6.4.

1) For $\tau \in G/H = \mathrm{Gal}(E/\mathbb{Q}) : C_f(\sigma)^\tau = C_f(\sigma\tau)$,

2) $C_f(\overline{\sigma}) = -C_f(\sigma)$,

3) The elements $\{C_f(\sigma) : \sigma \in \mathrm{Gal}(E/\mathbb{Q})\}$ *span the vector space* $K_3B \otimes \mathbb{Q}$. *All relations between them are a consequence of 2).*

This follows immediately from the definition of $C_f(\sigma)$ and Theorem 6.2. For a given abelian extension $E/\mathbb{Q}$, we may take f to be the non-Archimedian part of the conductor of E.

7. **Bibliography**

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