Math 55b: Honors Real and Complex Analysis

Homework Assignment #8 (27 March 2017):
Trigonometric product formulas via integral manipulation;
a bit of multivariate differential calculus

> Fejér discovered his theorem[1] at the age of 19, Weierstrass published [his Polynomial Approximation Theorem] at 70. With time the reader may come to appreciate why many mathematicians regard the second circumstance as even more romantic and heart warming than the first.[2]

The first series of problems give a path to several classical product formulas and integrals that is more elementary than usual but requires finesse in several places. Be careful about justifying all steps!

1. [Wallis' integrals] Prove that $\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2} x\,dx$ for all $n \geq 2$. Deduce that
$$\int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{2}{3}\frac{4}{5}\frac{6}{7}\cdots\frac{n-1}{n}, & \text{if } n \text{ is odd;} \\ \frac{\pi}{2}\frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{n-1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

2. [Wallis' product (1655)] It follows that
$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\cdots\frac{2m}{2m-1}\,\frac{2m}{2m+1}\cdot\frac{\int_0^{\pi/2}\cos^{2m} x\,dx}{\int_0^{\pi/2}\cos^{2m+1} x\,dx}.$$
Show that
$$1 < \frac{\int_0^{\pi/2}\cos^{2m} x\,dx}{\int_0^{\pi/2}\cos^{2m+1} x\,dx} < \frac{\int_0^{\pi/2}\cos^{2m-1} x\,dx}{\int_0^{\pi/2}\cos^{2m+1} x\,dx} = 1 + \frac{1}{2m},$$
and therefore
$$\frac{\pi}{2} = \lim_{m\to\infty}\left(\frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\cdots\frac{2m}{2m-1}\,\frac{2m}{2m+1}\right).$$
[This is usually written as the "infinite product"
$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\cdots.]$$

3. Use the formulas of the previous problem to prove that
$$\lim_{n\to\infty}\int_0^{\sqrt{n}\frac{\pi}{2}} \cos^n\frac{x}{\sqrt{n}}\,dx = \sqrt{\pi/2}.$$

[1] That the uniform closure of trigonometric polynomials is the full space of continuous functions on $\mathbf{R}/2\pi\mathbf{Z}$; see Rudin, pages 199–200.

[2] Körner, *Fourier Analysis*, p.294 (end of Chapter 59: "Weierstrass's proof of Weierstrass's theorem").

Now show that $\lim_{n\to\infty} \cos^n(x/\sqrt{n}) = \exp(-x^2/2)$ for any $x \geq 0$, and use this to prove that[3]

$$\int_0^\infty e^{-x^2/2}\,dx = \sqrt{\pi/2}.$$

4. Define $I_n(\lambda)$ for $0 < \lambda < 1$ by

$$I_n(\lambda) = \int_0^{\pi/2} \cos^n x \cos(\lambda x)\,dx \qquad (n = 0, 1, 2, \ldots).$$

Integrate by parts twice to prove that $(n^2 - \lambda^2)I_n(\lambda) = (n^2 - n)I_{n-2}(\lambda)$ for $n \geq 2$. Then evaluate $I_0(\lambda)$ and $I_1(\lambda)$ to obtain a formula for $I_n(\lambda)$ for all n. Deduce a product formula for $\tan(\pi\lambda/2)$, and verify that Wallis' product can be recovered from your formula by taking the limit as $\lambda \to 0$. Can you obtain any further formulas by investigating the behavior of $I_n(\lambda)$ as $n \to \infty$?

Some problems introducing multivariate differential calculus and its interaction with mathematics that we developed earlier this year:

5. [Rudin, Problem 7 in Chapter 9] Suppose $E \subset \mathbf{R}^n$ is open and $f : E \to \mathbf{R}$ has partial derivatives $D_1 f(x), \ldots, D_n f(x)$ that are bounded as x varies over E. Prove that f is continuous.

6. [Rudin, Problem 10 in Chapter 9] Suppose the open set $E \subset \mathbf{R}^n$ is *convex*, i.e., E contains the line segment joining any two of its points (so $x, y \in E$ and $0 \leq t \leq 1 \Longrightarrow tx + (1-t)y \in E$). Prove that if $f : E \to \mathbf{R}$ has the property that $D_1 f(x)$ exists and is zero for all $x \in E$ then $f(x)$ depends only on $x_2, \ldots, x_n$. Show further that convexity can be replaced by a weaker hypothesis on the open set E, but that some condition is required; for instance if $n = 2$ and E is Ω-shaped the statement may be false.

7. The *Laplacian* Δf of a $\mathscr{C}^2$ function f on an open set $E \subseteq \mathbf{R}^n$ is defined by

$$\Delta f := \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = \sum_{j=1}^{n} D_{jj} f.$$

Let $A : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation. Show that $\Delta(f \circ A) = (\Delta f) \circ A$ for all $f \in \mathscr{C}^2$ if and only if A is orthogonal. In particular, the "harmonic functions" (those in the kernel of Δ, i.e. $\mathscr{C}^2$ functions f with $\Delta f = 0$) are preserved by orthogonal changes of variable.

8. [Cauchy-Riemann equations] Let E be a nonempty open subset of $\mathbf{C}$. The usual identification of $\mathbf{C}$ with $\mathbf{R}^2$ (identify the complex number $z = x + iy$ with the vector (x, y))

[3] As I may have noted some times ago in class, it is remarkable that this ubiquitous definite integral can be evaluated in closed form, considering that the indefinite integral $\int \exp(cx^2)\,dx$ cannot be simplified. We shall give another proof of $\int_0^\infty e^{-x^2/2}\,dx = \sqrt{\pi/2}$ when we come to the change of variable formula for multiple integrals.

lets us regard any map $w : E \to \mathbf{C}$ as a map from an open subset of $\mathbf{R}^2$ to $\mathbf{R}^2$, or equivalently as a pair of real-valued functions $u(x, y) = \operatorname{Re} f(x+iy)$, $v(x, y) = \operatorname{Im} f(x+iy)$ on that subset. Prove the following criterion for a $\mathscr{C}^1$ function w to be differentiable *as a map from a subset of* $\mathbf{C}$ *to* $\mathbf{C}$, i.e., for there to exist a function $w' : E \to \mathbf{C}$ such that $w(z + h) = w(z) + hw'(z) + o(|h|)$ as $h \to 0$: the functions u, v must satisfy the *Cauchy-Riemann equations*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Show that this is the case if w is any polynomial in z with complex coefficients, or the exponential function $w(z) = e^z = e^x(\cos y + i \sin y)$. Prove that such u, v, w are necessarily harmonic functions on $\mathbf{C}$. [We shall see that conversely every harmonic function on $\mathbf{C}$ is the real part of a differentiable function of a complex variable.]

This problem set is due Wednesday, 5 April at the beginning of class.