Math 250: Higher Algebra

Problem Set #1 (24 September 2004): Galois theory I

1. (Another construction of the trace and norm) Let K/F be a finite field extension with $[K:F]=n$. For each $a \in K$, we may consider the map $M_a : K \to K$, $x \mapsto ax$ as a linear operator on K *considered as a vector space over* F.

   i) Check that $a \mapsto M_a$ is a homomorphism from K to $\mathrm{End}_F(K)$, the algebra of F-linear operators on K.

   The trace and norm of a (relative to the extension K/F) are the trace and determinant of M_a. These are denoted $\mathrm{Tr}_{K/F}(a)$ and $\mathrm{N}_{K/F}(a)$, or simply $\mathrm{Tr}(a)$ and $\mathrm{N}(a)$ if K/F is understood.

   ii) Check that Tr is an F-linear map from K to F, and that the norm is multiplicative: $\mathrm{N}(ab) = \mathrm{N}(a)\mathrm{N}(b)$ for all $a, b \in K$. If $F = \mathbf{R}$ and $K = \mathbf{C}$, what are the trace and norm of $a = x + iy$? What are the eigenvalues of M_a?

   It is a fundamental result in linear algebra that a linear operator T on a finite-dimensional vector space satisfies $P(T) = 0$ where $P(\lambda) = \det(\lambda I - T)$ is the characteristic polynomial of T. This gives an explicit construction of a monic polynomial (NB not always the minimal such polynomial!) satisfied by an element of K.

   iii) Suppose $K = F(u)$ where u is a root of the irreducible polynomial $P(X) = X^n + \sum_{j=0}^{n-1} a_j X^j$ in $F[X]$. Determine the matrix of M_u relative to the F-basis $\{1, u, u^2, \ldots, u^{n-1}\}$ of K, and check directly that P is the characteristic polynomial of this matrix.

2. (An application of part (iii) of the last problem; cf. problem 2 of Jacobson 4.1) Let $F = \mathbf{Q}$, $K = \mathbf{Q}(\sqrt{2}, \sqrt{3})$, and $a = \sqrt{2} + \sqrt{3} \in K$. Determine $n = [K:F]$. Prove that $K = F(a)$. (Hint: what can $[K : F(a)]$ be?) Choose a basis for K as a F-vector space, and determine the matrix of M_a relative to this basis. Use this to compute the minimal polynomial of a over F. Check directly that this polynomial vanishes at a.

3. (Problem 7 of Jacobson 4.1) A field extension L/F is said to be algebraic if every element of L is algebraic over F. Suppose L/F is algebraic and $K \subseteq L$ is an *F-subalgebra*, i.e., a subring containing F (equivalently, an F-vector subspace containing F and closed under multiplication). Prove that K is a field.

4. (Problem 8 of Jacobson 4.1) Let L/F be the transcendental extension $F(u)$. Suppose that K is a subfield of L properly containing F. Prove that u is algebraic over K.

5. (Problem 2 of Jacobson 4.3) Construct a splitting field K of $x^5 - 2$ over $\mathbf{Q}$, and determine $[K:\mathbf{Q}]$. (You may assume the irreducibility of $x^5 - 2$ over $\mathbf{Q}$, and of the polynomial $(x^5-1)/(x-1)$ over $\mathbf{Q}(\sqrt[5]{2})$; we shall learn later how to prove such results.)

6. (Problem 4 of Jacobson 4.3) Let L/F be a splitting field over F of some polynomial $f(X)$, and let K be any subfield of L containing F. Suppose $\iota : K \to L$ is a homomorphism whose restriction to F is the identity. Prove that ι can be extended to an isomorphism of L.

7. (Problem 4 of Jacobson 4.4) Let F be a field of characteristic p that is not perfect. Thus there are elements of F not contained in F^p; let a be any such element. Prove that the polynomial $X^{p^e} - a$ is irreducible for every nonnegative integer e.

8*. (Curious behavior of an inseparable extension) Let k be a field, $K = k(X,Y)$ the field of rational functions in two variables, and $F = k(X^p, Y^p)$ for some prime p. Show that $[K:F] = p^2$. Now assume k is an infinite field of characteristic p. (For instance we may have $k = k_0(T)$ with $k_0 = \mathbf{Z}/p\mathbf{Z}$.) Prove that there are infinitely many intermediate fields E between K and F (necessarily with $[K:E] = [E:F] = p$).
We shall see that this cannot happen if K/F is a separable extension of finite degree.

Problem set is due in class Monday, October 4th.