Math 229:[1] Introduction to Analytic Number Theory

pseudo-syllabus

0. Introduction: What is analytic number theory?

1. Distribution of primes before complex analysis: classical techniques (Euclid, Euler); primes in arithmetic progressions via Dirichlet characters and L-series; Čebyšev's estimates on $\pi(x)$.

2. Distribution of primes using complex analysis: $\zeta(s)$ and $L(s,\chi)$ as functions of a complex variable, and the proof of the Prime Number Theorem and its extension to Dirichlet; blurb for Čebotarev density; functional equations; the Riemann hypothesis, extensions, generalizations and consequences.

3. Selberg's quadratic sieve and applications.

4. Analytic estimates on exponential sums (van der Corput etc.); prototypical applications: Weyl equidistribution, upper bounds on $|\zeta(s)|$ and $|L(s,\chi)|$ on vertical lines, lattice point sums.

5. Lower bounds on discriminants, conductors, etc. from functional equations; geometric analogue: how many points can a curve of genus $g \to \infty$ have over a given finite field?

6. Analytic bounds on coefficients of modular forms and functions; applications to counting representations of integers as sums of squares, etc.

Prerequisites. While Math 229 will proceed at a pace appropriate for a graduate-level course, its prerequisites are perhaps surprisingly few: complex analysis at the level of Math 113, and linear algebra and basic number theory (up to say arithmetic in the field $\mathbf{Z}/p\mathbf{Z}$ and Quadratic Reciprocity). Some considerably deeper results (such as estimates on Kloosterman sums) will be cited but may be regarded as black boxes for our purposes. If you know about algebraic number fields or modular forms or curves over finite fields, you'll get more from the course at specific points, but these points will be in the nature of scenic detours that are not required for the main journey.

Texts. Lecture notes will be handed out periodically, and can also be found on the course webpage `http://www.math.harvard.edu/~elkies/M229.26` . There is no textbook: this class is an introduction to several different flavors of analytic methods in number theory, and I know of no one work that covers all this material. Thus I intend to expand and edit the lecture notes to put together a textbook, which may finally become available by the next time I teach the class… Supplementary readings such as Serre's *A Course in Arithmetic* and Titchmarsh's *The Theory of the Riemann Zeta-Function* will be suggested as we approach their respective territories.

Office Hours will be set up soon; for now — and also once regular office hours begin — you can set up a meeting, or ask questions directly, by e-mailing me at `elkies@math.harvard.edu` .

September calendar. Monday, September 7, is a University holiday. Wednesday, September 23, I'll be at a conference out of town, and will not teach the class; I'll make it up during Reading Period. Therefore I will <u>not</u> cancel class for Yom Kippur, which this year falls on Monday, September 21. Instead, on that day I'll cover material that shows some of our techniques in action, but not results or methods that we'll need later in the course.

[1] The course formerly known as Math 259.

Grading. This section and the next are relevant to you if and only if you are taking Math 229 for a grade (i.e. are officially enrolled and are not a post-Qual math graduate student exercising your EXC option).

Part of your grade will be based on homework, assigned regularly ($\simeq$weekly, excepting the week of September 21) and drawn from exercises in the lecture notes. As usual in our department, you are allowed — indeed encouraged — to collaborate on solving homework problems, but must write up your own solutions.

In the past, the homework counted for most of your grade. That wouldn't make sense now that we have entered the era of Claude and its peers. (See also "AI policy" below.) All math classes, even 200-level ones, are now expected to minimize the grade component that depends on purely take-home tasks, including homework.[2] Therefore there will be two in-class midterm exams, necessarily on questions that test basic concepts and techniques.

The homework and each exam will count for about 1/4 of your grade. The remaining quarter will come from a final project, which (depending on class size and composition) will be an expository paper and/or an in-class presentation on some aspect of analytic number theory related to but just beyond what we cover in class. If there are too many final projects for in-class presentations, I will meet individually with each student who writes a final project.

I may put together some Extended Exercises that can be used as final topics; the supplementary references will be another good source for paper/presentation topics.

There is still no cap on A grades. Anybody who does A-level work will earn an A grade. Grades of A-minus and below are still available too...

AI matters. Fall 2026 is the first time that computers can solve graduate-level exercises almost as routinely as they can factor 8675309 and 7184981043. So nobody has experience teaching graduate-level mathematics in such an environment. I figure that it's somewhat analogous to studying an elementary text, or reading a puzzle book, that has hints, answers, or full solutions in the back of the book. Consulting the answer key can be a legitimate part of the learning process. But if we resort to the answer key too often or too quickly, we don't learn as much or as deeply. In particular, we miss out on the process of learning the territory by exploring interesting paths that do not go to our immediate destination.

Therefore, I recommend that you not ask your preferred AI agent(s) to solve homework problems until you have tried to solve them on your own, and you solve *at least* one problem yourself on each problem set (and indicate in your solutions which problem(s) you solved unaided). Once you're satisfied that you have solved it, you're welcome to ask an AI how to do it; even when you're right, you may learn yet more by seeing its alternative solution(s).

Some problems are partly computational; you're welcome to use arbitrary AI assistance coding up calculations and computer searches.

In any case, your write-ups <u>must</u> be your own work in your own words; As with naturally intelligent classmates, you may ask an artificial-intelligence system to check or proofread your problem sets, but not to write them for you. Also, as a special case of Harvard's policy on working with sources, when and if you do use AI you must cite it and specify what you did and how.

[2] Yes, Harvard has an honor code, and one could declare that using AI is cheating. But an honor code also places implicit responsibility on faculty to not place students in a position where they may feel they need to cheat. That aside, strong mathematical AI can be a valuable tool in the practice of mathematics, including learning and problem solving (see the AI section of this pseudo-syllabus); so forbidding the use of AI has its own costs, even if everybody complies.