Math 229: Introduction to Analytic Number Theory

Functions of finite order: product formula and logarithmic derivative

This chapter is another review of standard material in complex analysis. See for instance Chapter 11 of [Davenport 1967], keeping in mind that Davenport uses "integral function" for what we call an "entire function"; Davenport treats only the case of order (at most) 1, which is all that we need, but it is scarcely harder to deal with any finite order as we do here.

The *order* of an entire function $f(\cdot)$ is the smallest $\alpha \in [0, +\infty]$ such that $f(z) \ll_\epsilon \exp|z|^{\alpha+\epsilon}$ for all $\epsilon > 0$. Hadamard showed that entire functions of finite order are given by nice product formulas. We have seen already the cases of $\sin z$ and $1/\Gamma(z)$, both of order 1. As we shall see, $(s^2 - s)\xi(s)$ also has order 1 (as do analogous functions that we'll obtain from Dirichlet L-series). From the product formula for $\xi(s)$ we shall obtain a partial-fraction decomposition of $\zeta'(s)/\zeta(s)$, and then use it to manipulate the contour-integral formula for $\psi(x)$.

Hadamard's product formula for a general entire function of finite order is given by the following result.

Theorem. *Let f be an entire function of order $\alpha < \infty$. Assume that f does not vanish identically on $\mathbf{C}$. Then f has a product formula*

$$f(z) = e^{g(z)} z^r \prod_{k=1}^{\infty} \left((1 - \frac{z}{z_k}) \exp\left[\sum_{m=1}^{a} \frac{1}{m} \left(\frac{z}{z_k} \right)^m \right] \right), \tag{1}$$

where $a = \lfloor \alpha \rfloor$, the integer r is the order of vanishing of f at $z = 0$, the z_k are the other zeros of f listed with multiplicity, g is a polynomial of degree at most a, and the product converges uniformly in bounded subsets of $\mathbf{C}$. Moreover, for $R > 1$ we have

$$\#\{k : |z_k| < R\} \ll_\epsilon R^{\alpha+\epsilon}. \tag{2}$$

Conversely, suppose r is any nonnegative integer, g is a polynomial of degree at most $a = \lfloor \alpha \rfloor$, and z_k are nonzero complex numbers such that $|z_k| < R$ for at most $O_\epsilon(R^{\alpha+\epsilon})$ choices of k. Then the right-hand side of (1) defines an entire function of order at most α.

To prove this, we first show:

Lemma. *A function f has finite order and no zeros if and only if $f = e^g$ for some polynomial g.*

Proof: Clearly e^g satisfies the hypotheses if g is a polynomial. Conversely, f is an entire function with no zeros if and only if $f = e^g$ for some entire function g; we shall show that if also $|f| \ll_\epsilon \exp|z|^{\alpha+\epsilon}$ then g is a polynomial. Indeed the real part of g is $< O(|z|^{\alpha+\epsilon})$ for large z. But then the same is true of $|g(z)|$, as the following argument shows. Let $h = g - g(0)$, so $h(0) = 0$; and let $M(R) = \sup_{|z| \le 2R} \operatorname{Re} h(z)$. By assumption $M(R) \ll R^{\alpha+\epsilon}$ for large R. Then

$h_1 := h/(2M(R) - h)$ is analytic in the closed disc $D := \{z \in \mathbf{C} : |z| \leq 2R\}$, with $h_1(0) = 0$ and $|h_1(z)| \leq 1$ in D. Consider now the analytic function $\phi(z) := 2Rh_1(z)/z$ on D. On the boundary of that disc, $|\phi(z)| \leq 1$. Thus by the maximum principle the same is true for all $z \in D$. In particular, if $|z| \leq R$ then $|h_1(z)| \leq 1/2$. But then $|h(z)| \leq 2M(R)$. Hence $|g(z)| \leq 2M(R) + g(0) \ll |z|^{\alpha+\epsilon}$ for large $|z|$, and g is a polynomial in z as claimed. Moreover, the degree of that polynomial is just the order of f. $\square$

We shall reduce the Theorem to this Lemma by dividing a given function f of finite order by a product $P(z)$ whose zeros match those of f. To show that this product converges, we first need to obtain the bound (2) on the number of zeros of f in a disc. We shall deduce this bound from *Jensen's inequality* for the function $f_0 = f/z^r$. This inequality states: if f_0 is an analytic function on the disc $|z| \leq R$ then

$$|f_0(0)| \leq \prod_{\zeta} \frac{|\zeta|}{R} \cdot \sup_{|z|=R} |f_0(z)|, \tag{3}$$

where the product ranges over the zeros ζ of f_0 in the disc, counted with multiplicity.

We recall the proof of (3). If $f_0(0) = 0$ we are done. Else, let $z_1, z_2, \ldots$ be the zeros of f_0, listed with the correct multiplicity in non-decreasing order of $|z_k|$:

$$0 < |z_1| \leq |z_2| \leq |z_3| \leq \cdots.$$

For $R > 0$, let $n(R)$ be the left-hand side of (2), which is the number of k such that $|z_k| < R$. Thus $n(R) = k$ if and only if $|z_k| < R \leq |z_{k+1}|$. We first prove (3) for $R = 1$. Let $\phi(z)$ be the *Blaschke product* $\prod_{k=1}^{n(1)} (z - z_k)/(1 - \bar{z}_k z)$. This is a rational function designed to have the same zeros as f_0 in the unit disc but with $|\phi(z)| = 1$ on $|z| = 1$. Then $f_1 := f_0/\phi$ is analytic on $|z| \leq 1$, and $|f(z)| = |f_0(z)| = |f_1(z)|$ on the boundary $|z| = 1$. Therefore by the maximum principle $|f_1(0)| \leq \max_{|z|=1} |f(z)|$, so

$$|f_0(0)| = |\phi(0) f_1(0)| = \prod_{k=1}^{n(1)} |z_k| \cdot |f_1(0)| \leq \prod_{k=1}^{n(1)} |z_k| \cdot \max_{|z|=1} |f(z)|.$$

Applying this to the function $f_0(Rz)$, whose zeros in the unit disc are z_k/R for $k \leq n(R)$, we obtain Jensen's inequality (3).

Taking logarithms, we find

$$\begin{aligned}
\log \max_{|z|=R} |f(z)| &\geq r \log R + \log |f_0(0)| + \sum_{k=1}^{n(R)} \log \frac{R}{|z_k|} \\
&= r \log R + \log |f_0(0)| + \int_0^R n(r) \frac{dr}{r}.
\end{aligned}$$

If f has order at most $\alpha < \infty$ then $\log \max_{|z|=R} |f(z)| \ll_\epsilon R^{\alpha+\epsilon}$, and we conclude that

$$n(R) = \int_R^{eR} n(R) \frac{dr}{r} \leq \int_0^{eR} n(r) \frac{dr}{r} \ll_\epsilon R^{\alpha+\epsilon}.$$

We have thus proved (2). It follows that $\sum_{k=1}^\infty |z_k|^{-\beta}$ converges if $\beta > \alpha$, since the sum is

$$\int_0^\infty r^{-\beta} dn(r) = \beta \int_{|z_1|}^\infty r^{-\beta-1} n(r)\, dr \ll \int_{|z_1|}^\infty r^{\alpha+\epsilon-\beta-1}\, dr < \infty$$

for any positive $\epsilon < \beta - \alpha$. Therefore the product

$$P(z) := z^r \prod_{k=1}^\infty \left((1 - \frac{z}{z_k}) \exp \left[\sum_{m=1}^a \frac{1}{m} \left(\frac{z}{z_k} \right)^m \right] \right) \tag{4}$$

converges for all $z \in \mathbf{C}$, and is not affected by any permutation of the zeros z_k. Moreover, the convergence is uniform in bounded subsets of $\mathbf{C}$, because on $|z| \leq R$ we have

$$\log(1 - (z/z_k)) + \sum_{m=1}^a (z/z_k)^m/m \ll |z/z_k|^{a+1} \ll |z_k|^{-a-1} \tag{5}$$

uniformly once $k > n(2R)$. Therefore $P(z)$ is an entire function, with the same zeros and multiplicities as f.

It follows that f/P is an entire function without zeros. We claim that it too has order at most α, and is thus $\exp g(z)$ for some polynomial g of degree at most a. This would be clear if it were true that

$$\frac{1}{P(z)} \ll_\epsilon \exp |z|^{\alpha+\epsilon},$$

but such an inequality cannot hold for all z due to the zeros of P. But it is enough to show that for each $R > 0$ a bound

$$\frac{1}{P(z)} \ll_\epsilon \exp R^{\alpha+\epsilon} \tag{6}$$

holds on the circle $|z| = r$ for some $r \in (R, 2R)$, because then we would have $|f(z)/P(z)| \ll_\epsilon \exp R^{\alpha+\epsilon}$ for all z on that circle, and thus also on $|z| = R$ by the maximum principle. We do this next.

Write $P = z^r P_1 P_2$, with P_1, P_2 being the product in (4) over $k \leq n(4R)$ and $k > n(4R)$ respectively. We may ignore the factor z^r, whose norm exceeds 1 once $R > 1$. The k-th factor of $P_2(z)$ is $\exp O(|z/z_k|^{a+1})$ by (5), so

$$\log |P_2(z)| \ll R^{a+1} \sum_{k>n(4R)} |z_k|^{-a-1} \ll R^{a+1} \int_{4R}^\infty r^{-a-1}\, dn(r) \ll_\epsilon R^{\alpha+\epsilon},$$

using integration by parts and $n(r) \ll_\epsilon r^{\alpha+\epsilon}$ in the last step (check this!). As to P_1, it is a finite product, which we write as $e^{h(z)}P_3(z)$ where $P_3(z) := \prod_{k\le n(4R)}(1-(z/z_k))$ and $h(z)$ is the polynomial

$$h(z) = \sum_{k=1}^{n(4R)} \sum_{m=1}^{a} \frac{1}{m}\left(\frac{z}{z_k}\right)^m$$

of degree at most a. Thus $h(z) \ll R^a \sum_{k\le n(4R)} |z_k|^{-a}$, which readily yields $h(z) \ll R^{\alpha+\epsilon}$. (Again you should check this by carrying out the required partial summation and estimates; note too that the upper bounds on the *absolute value* of $\log|P_2(z)|$ and $h(z)$ yield lower as well as upper bounds on $|P_2(z)|$ and $|\exp h(z)|$.) So far, our lower bounds on the factors of $P(z)$ hold for all z in the annulus $R < |z| < 2R$, but we cannot expect the same for $P_3(z)$, since it may vanish at some points of the annulus. However, we can prove that *some* r works by estimating the average[1]

$$-\frac{1}{R}\int_R^{2R} \min_{|z|=r} \log|P_3(z)|\,dr \le -\sum_{k=1}^{n(4R)} \frac{1}{R}\int_R^{2R} \log\left|1-\frac{r}{|z_k|}\right| dr.$$

The integral is elementary, if not pretty, and at the end we conclude that the average is again $\ll R^{\alpha+\epsilon}$. This shows that for some $r \in (R, 2R)$ the desired lower bound holds, and we have finally proved the product formula (1).

To complete the proof of our Theorem we need only show the converse: (1) converges to an entire function of order at most α under the stated hypotheses on r, g, z_k. The convergence was proved already, and the upper bound on $|f(z)|$ follows readily from (5). □□

Taking logarithmic derivatives in (1), we deduce

$$\frac{f'}{f}(z) = g'(z) + \frac{P'}{P}(z) = g'(z) + \frac{r}{z} + \sum_{k=1}^{\infty}\left[\frac{1}{z-z_k} + \sum_{m=1}^{a}\frac{z^{m-1}}{z_k^m}\right]$$

$$= g'(z) + \frac{r}{z} + \sum_{k=1}^{\infty} \frac{(z/z_k)^a}{z-z_k}.$$

We note too that if $\alpha > 0$ and $\sum_k |z_k|^{-\alpha} < \infty$ then there exists a constant C such that $f(z) \ll \exp C|z|^\alpha$. This follows from the existence of a constant C_α such that

$$\left|(1-w)\exp\sum_{m=1}^{a} w^m/m\right| \ll \exp C_\alpha |w|^\alpha$$

for all $w \in \mathbf{C}$. Contrapositively, if $f(z)$ is a function of order α that grows faster than $\exp C|z|^\alpha$ for all C then $\sum_k |z_k|^{-\alpha}$ diverges. For instance this happens for

[1] This averaging trick is a useful technique that we'll encounter again several times; it is closely related to the "probabilistic method" in combinatorics, in which an object with some property is proved to exist by showing that the property holds with positive probability.

$f(s) = 1/\Gamma(s)$. [This approach may appear circular because it is proved from the product formula for $\Gamma(s)$, but it need not be; see Exercise 6 below.] As we shall see, the same is true for $f(s) = (s^2 - s)\xi(s)$; it will follow that ξ, and thus ζ, has infinitely many zeros ρ with real part in $[0, 1]$, and in fact that $\sum_\rho |\rho|^{-1}$ diverges.

Exercises

1. The bound $f(z) \ll \exp C|z|^\alpha$ for functions satisfying $\sum_k |z_k|^{-\alpha} < \infty$ was proved under the hypothesis $\alpha > 0$. Is this hypothesis necessary?

2. Find an entire function $f(z)$ of order 1 such that $|f(z)| \ll \exp O(|z|)$ but $\sum_{k=1}^\infty |z_k^{-1}| = \infty$. [Hint: you don't have to look very far.]

3. Supply the missing steps in our proof of (1).

4. Suppose z_k $(k = 1, 2, 3, \ldots)$ are distinct complex numbers with $0 < |z_k| < 1$, and m_k are some positive integers. Prove that $\prod_k |z_k|^{m_k} > 0$ if and only if there exists a *bounded* nonzero analytic function $f \not\equiv 0$ on the open disc $|z| < 1$ with a root at each z_k of multiplicity m_k.

5. Prove Jensen's formula: if f is an analytic function on $|z| \leq R$ such that $f(0) \neq 0$ then $(2\pi)^{-1} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta = \log |f(0)| + \sum_k \log(R/|z_k|)$, where the z_k are the zeros of f in $|z| \leq R$ with the correct multiplicities. What is $(2\pi)^{-1} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta$ if $f(0) = 0$ but f does not vanish identically?

6. Show that $1/\Gamma(s)$ is an entire function of order 1, using only the following tools available to Euler: the integral formulas for $\Gamma(s)$ and $\mathrm{B}(s, s')$, and the identities $B(s, s') = \Gamma(s)\Gamma(s')/\Gamma(s+s')$ and $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$. [The hard part is getting an upper bound for $1/|\Gamma(s)|$ on a vertical strip; remember how we showed that $\Gamma(s) \neq 0$, and use the formula for $|\Gamma(1/2 + it)|^2$ to get a better lower bound on $|\Gamma(s)|$.] Use this to recover the product formula for $\Gamma(s)$, up to a factor e^{A+Bs} which may be determined from the behavior of $\Gamma(s)$ at $s = 0, 1$.

7. Prove that if $f(z)$ is an entire function of order $\alpha > 0$ then

$$\iint_{|z|<r} |f'(z)/f(z)| \, dx \, dy \ll r^{\alpha+1+\epsilon} \qquad (z = x + iy)$$

as $r \to \infty$. [Note that the integral is improper (except in the trivial case that f has no zeros) but still converges: if ϕ is a meromorphic function on a region $U \subset \mathbf{C}$ with simple but no higher-order poles then $|\phi|$ is integrable on compact subsets $K \subset U$, even K that contain poles of ϕ.]

Reference

[Davenport 1967] Davenport, H.: *Multiplicative Number Theory.* Chicago: Markham, 1967; New York: Springer-Verlag, 1980 (GTM 74). [9.67.6 & 9.80.6 / QA 241.D32]