Math 229: Introduction to Analytic Number Theory

How many points can a curve of genus g have over $\mathbf{F}_q$?
Motivation for the Drinfeld-Vlăduţ bound

Suppose c_n ($n \in \mathbf{Z}$) are real numbers such that:
i) $c_n \geq 0$ for all n;
ii) $c_{-n} = c_n$ for all n,
iii) $\sum_{n\in\mathbf{Z}} q^{|n|/2} c_n < \infty$, and
iv) $C(\mu) := \sum_{n\in\mathbf{Z}} c_n \mu^n$ satisfies $C(\mu) \geq 0$ for all $\mu \in \mathbf{C}$ such that $|\mu| = 1$.

[Note that the sum defining $C(\mu)$ converges absolutely by hypothesis (iii), and is real by hypothesis (ii). Absolute convergence would already be guaranteed by the ℓ^1 condition $\sum_n c_n < \infty$, but the stronger condition $\sum_n q^{|n|/2} c_n < \infty$ is needed for our application.]

Write $\lambda_j = q^{1/2}\mu_j$ for $1 \leq j \leq 2g$, so that $|\mu_j| = 1$ for each j and $\mu_{g+j} = \mu_j^{-1}$ for $1 \leq j \leq g$. Then applying (iv) to each μ_j ($1 \leq j \leq g$), summing over j, and using $N_n \geq N_1$ yields

$$\begin{aligned} 0 &\leq c_0 g + \sum_{n=1}^{\infty} c_n \Big(\sum_{j=1}^{g} (\mu_j^n + \mu_j^{-n}) \Big) \\ &= c_0 g + \sum_{n=1}^{\infty} c_n q^{-n/2} (q^n + 1 - N_n) \\ &\leq c_0 g + \sum_{n=1}^{\infty} c_n q^{-n/2} (q^n + 1 - N_1), \end{aligned}$$

whence

$$\Big(\sum_{n=1}^{\infty} c_n q^{-n/2} \Big) N_1 \leq c_0 g + \sum_{n=1}^{\infty} c_n (q^{n/2} + q^{-n/2}).$$

Finding the best choice of c_n is then an infinite-dimensional linear-programming problem: conditions (i), (ii), (iv) define a convex subset of $\ell^1(\mathbf{R})$, which for our q we can intersect with a hyperplane $\sum_{n=1}^{\infty} c_n q^{-n/2} = A$ and seek to minimize for each g the linear functional $c_0 g + \sum_{n=1}^{\infty} c_n (q^{n/2} + q^{-n/2})$, or equivalently intersect with a hyperplane $c_0 g + \sum_{n=1}^{\infty} c_n (q^{n/2} + q^{-n/2}) = B$ and seek to maximize the linear functional $\sum_{n=1}^{\infty} c_n q^{-n/2}$. (This is why we had to require $\sum_n q^{|n/2|} c_n < \infty$, else the "upper bound" on N_1 would be ∞.)

In general such problems require nontrivial computation. (For fixed q and g the best $\{c_n\}$ will have finite support, but (iv) still imposes infinitely many linear inequalities in a finite-dimensional space so numerical solution requires some subtlety.) But for the asymptotic problem where we fix q and let $g \to \infty$ there is a simple answer. Let us first ignore the contribution of $\sum_{n=1}^{\infty} c_n (q^{n/2} + q^{-n/2})$, and simply ask to maximize $\sum_{n=1}^{\infty} c_n q^{-n/2}$ subject to $c_0 = 1$.

Now (iv) implies $c_n \leq c_0$ for each n, because

$$c_n = \int_0^1 C(x) \cos 2\pi n x \, dx \leq \int_0^1 C(x)\, dx = c_0.$$

Thus if $c_0 = 1$ then $\sum_{n=1}^{\infty} c_n q^{-n/2} \le \sum_{n=1}^{\infty} q^{-n/2} = 1/(\sqrt{q}-1)$, so the best upper bound we can hope for is $N_1 \le (\sqrt{q}-1)g$, and that only when $c_n = 1$ for all n. We cannot actually make each $c_n = 1$, because then $\sum_n q^{|n/2|} c_n$ and even $\sum_n c_n$ would diverge; but we might still expect that we can come arbitrarily close, because "morally speaking" taking $c_n = 1$ for all n should make C the Dirac delta function, which is indeed everywhere nonnegative.

To get a rigorous bound $N_1 \le (\sqrt{q} - 1 + o(1))\, g$ out of this, we need to find approximations to "$c_n = 1$ for all n", with $c_n \to 1$ for each n (albeit not uniformly), that still make $C(\mu) \ge 0$ for all μ on the unit circle but with $\sum_{n=1}^{\infty} q^{n/2} c_n < \infty$. There are many ways to do this,[1] but the simplest is to use positivity of the Fejér kernel, whose Fourier coefficients are finitely supported and thus satisfy $\sum_{n=1}^{\infty} q^{n/2} c_n < \infty$ automatically. Hence we fix some large integer M, and take $c_n = \max(0, 1 - M^{-1}|n|)$, and proceed as in the `many_pts` handout.

[1]For example, let $c_n = \exp(-\epsilon n^2)$, for which we can verify $C > 0$ using Poisson summation or properties of the heat equation.