Math 229: Introduction to Analytic Number Theory

Formulas for $L(1,\chi)$

Let χ be a primitive character mod $q > 1$. We shall obtain a finite closed form for $L(1,\chi)$. As with several of our other formulas involving $L(s,\chi)$, this one will have one shape if χ is even ($\chi(-1) = +1$), another if the character is odd ($\chi(-1) = -1$).

Recall our formula

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a \bmod q} \overline{\chi}(a) e^{2\pi i n a/q}.$$

This yields

$$L(1,\chi) = \frac{1}{\tau(\overline{\chi})} \sum_{a \bmod q} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{1}{n} e^{2\pi i a n/q}, \tag{1}$$

the implied interchange of sums being justified if the inner sum converges for each a mod q coprime with q. But this convergence follows by partial summation from the boundedness of the partial sum $\sum_{n=1}^{M} e^{2\pi i a n/q}$ for all nonzero a mod q. In fact we recognize it as the Taylor series for

$$-\log(1 - e^{2\pi i a/q}) = -\log\left(2 \sin \frac{a\pi}{q}\right) + \frac{i\pi}{2}\left(1 - \frac{2a}{q}\right)$$

(in which we choose the representative of $a \bmod q$ with $0 < a < q$). Either the real or the imaginary part will disappear depending on whether χ is odd or even.

Assume first that χ is even. Then the terms $\overline{\chi}(a)(i\pi/2)(1 - (2a/q))$ cancel in pairs $\{a, q-a\}$. Moreover, the terms $\overline{\chi}(a)\log 2$ sum to zero. Therefore

$$L(1,\chi) = -\frac{1}{\tau(\overline{\chi})} \sum_{a \bmod q} \overline{\chi}(a) \log \sin \frac{a\pi}{q}. \tag{2}$$

For example, if χ is a real character then $\tau(\overline{\chi}) = \tau(\chi) = +q^{1/2}$, so

$$q^{1/2} L(1,\chi) = 2 \log \epsilon$$

where

$$\epsilon = \prod_{a=1}^{\lfloor q/2 \rfloor} \sin^{-\chi(a)} \frac{a\pi}{q}$$

is a *cyclotomic unit* of $\mathbf{Q}(\sqrt{q}\,)$. The Dirichlet class number formula then asserts in effect that $\epsilon = \epsilon_0^h$ where ϵ_0 is the fundamental unit of that real quadratic field and h is its class number.

If on the other hand χ is odd then it is the logarithm terms that cancel in symmetrical pairs. Using again the fact that $\sum_{a \bmod q} \overline{\chi}(a) = 0$, we simplify (1) to

$$L(1,\chi) = -\frac{i\pi}{q\tau(\overline{\chi})} \sum_{a=1}^{q-1} a\overline{\chi}(a) \tag{3}$$

In particular if χ is real then (again using the sign of $\tau(\chi)$ for real characters)

$$L(1,\chi) = -\pi q^{-3/2} \sum_{a=1}^{q-1} a\chi(a).$$

Thus $\sum_{a=1}^{q-1} a\chi(a)$ is negative, and by Dirichlet equals $-q$ times the class number of the imaginary quadratic field $\mathbf{Q}(\sqrt{-q})$, except for $q = 3, 4$ when that field has extra roots of unity.

Let us concentrate on the case of real characters to prime modulus $q \equiv -1 \bmod 4$. The inequality $\sum_{a=1}^{q-1} a\chi(a) < 0$ suggests that the quadratic residues mod q tend to be more numerous in the interval $(0, q/2)$ than in $(q/2, q)$. We prove this by evaluating the sum

$$S_\chi(N) := \sum_{n=1}^{N} \chi(n)$$

at $N = q/2$. We noted already that for any nontrivial character $\chi \bmod q$ we have $S_\chi(N) = 0$ for all $N \equiv 0 \bmod q$, and thus $|S_\chi(N)| < q$ for all N. In fact, using the Gauss-sum formula for $\chi(n)$ we have

$$S_\chi(N) = \frac{1}{\tau(\overline{\chi})} \sum_{a \bmod q} \overline{\chi}(a) \sum_{n=1}^{N} e^{2\pi i n a/q} = \frac{-1}{\tau(\overline{\chi})} \sum_{a \bmod q} \overline{\chi}(a) \frac{1 - e^{2\pi i N a/q}}{1 - e^{-2\pi i a/q}}. \tag{4}$$

We note in passing that this formula quickly yields:

Lemma ([Pólya 1918], [Vinogradov 1918])**.** *There exists an absolute constant A such that*

$$|S_\chi(N)| < A q^{1/2} \log q$$

for all primitive Dirichlet characters $\chi \bmod q$ $(q > 1)$ and all $N \in \mathbf{Z}$.

Proof: We have $(1-e^{2\pi i N a/q})/(1-e^{-2\pi i a/q}) \ll \max(q/a, q/(q-a))$. We already saw that $|\tau(\chi)| = q^{1/2}$. Therefore (4) yields

$$S_\chi(N) \ll q^{1/2} \sum_{a=1}^{\lfloor q/2 \rfloor} \frac{1}{a} \ll q^{1/2} \log q. \quad \square$$

For the special case $N = (q-1)/2$ we can also analyze $S_\chi(N)$ as follows. If χ is a nontrivial even character then $S_\chi((q-1)/2)$ vanishes (why?). We thus assume χ is odd, and write

$$S_\chi((q-1)/2) = \sum_{n=1}^{q-1} \chi(n)\phi(n/q),$$

where $\phi(x)$ is the periodic function defined by

$$\phi(x) = \begin{cases} 0, & \text{if } 2x \in \mathbf{Z}; \\ +1/2, & \text{if } 0 < x - \lfloor x \rfloor < 1/2; \\ -1/2, & \text{otherwise} \end{cases}$$

("square wave"). This has the Fourier series

$$\phi(x) = \frac{2}{\pi}\Big(\sin(2\pi x) + \frac{1}{3}\sin(6\pi x) + \frac{1}{5}\sin(10\pi x) + \frac{1}{7}\sin(14\pi x) + \cdots\Big).$$

We thus have

$$S_\chi((q-1)/2) = \frac{1}{i\pi}\sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{1}{m}\sum_{a=1}^{q-1}\chi(a)(e^{2\pi ima/q} - e^{-2\pi ima/q}).$$

The inner sum is

$$\tau(\chi)(\overline{\chi}(m) - \overline{\chi}(-m)) = 2\tau(\chi)\overline{\chi}(m).$$

Therefore

$$S_\chi((q-1)/2) = \frac{2\tau(\chi)}{i\pi}\sum_{m \text{ odd}}\frac{\overline{\chi}(m)}{m} = \frac{(2-\overline{\chi}(2))\tau(\chi)}{i\pi}L(1,\overline{\chi}).$$

In particular if χ is real then $\tau(\chi) = +iq^{1/2}$ so

$$S_\chi((q-1)/2) = \frac{(2-\chi(2))q^{1/2}}{\pi}L(1,\chi).$$

It follows, as claimed, that there are more quadratic residues than nonresidues in $[1, q/2]$; in fact, once $q > 3$ the difference between the counts is either h or $3h$ according as $\chi(2) = 1$ or -1, that is, according as q is 7 or 3 mod 8. Even the positivity of $S_\chi((q-1)/2)$ has yet to be proved without resort to such analytic methods!

Exercises

1. Show directly that if χ is a primitive, odd, real character mod $q > 4$ then $\sum_{a=1}^{q-1} a\chi(a)$ is a multiple of q, at least when q is prime.

2. Suppose χ is a primitive character mod q, and n is a positive integer such that $(-1)^n = \chi(-1)$. Prove that $q^{1/2}\pi^{-n}L(n,\chi)$ is a rational number by finding a closed form that generalizes our formula for $n = 1$.

For instance, if $\chi = \chi_4$ we have $\pi^{-n}L(n,\chi) = (-1)^n E_{n-1}/(2^{n+1}(2n-1)!)$, where the integer E_{n-1} is the $(n-1)$-st Euler number.

3. Using the functional equation, conclude that $L(n,\chi) \in \mathbf{Q}$ for all real Dirichlet characters χ (possibly trivial and/or non-primitive) and integers $n \leq 0$.

4. What can you say of $S_\chi(\lfloor q/4 \rfloor)$? What about the sums $\sum_{a=1}^{q-1} a^m\chi(a)$ for $m = 2, 3, \ldots$? (See [ACW 1967], [TW 1999].)

References

[ACW 1967] Ayoub, R., Chowla, S., Walum, H.: On sums involving quadratic characters, *J. London Math. Soc.* **42** (1967), 152–154.

[Pólya 1918] Pólya, G.: Über die Verteilung der quadratische Reste und Nichtreste, *Königl. Ges. Wiss. Göttingen Nachr.* (1918), 21–29.

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[TW 1999] Teske, E., Williams, H.C.: A Problem Concerning a Character Sum, *Experimental Math.* **8** (1999) #1, 63–72.