Math 229: Introduction to Analytic Number Theory

More about the Gamma function

We collect some more facts about $\Gamma(s)$ as a function of a complex variable that will figure in our treatment of $\zeta(s)$ and $L(s,\chi)$. All of these, and most of the Exercises, are standard textbook fare; one basic reference is Ch. XII (pp. 235–264) of [WW 1940]. One reason for not just citing Whittaker & Watson is that some of the results concerning Euler's integrals B and Γ have close analogues in the Gauss and Jacobi sums associated to Dirichlet characters, and we shall need these analogues before long.

The product formula for $\Gamma(s)$. Recall that $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \ldots$ and no zeros. We readily concoct a product that has the same behavior: let

$$g(s) := \frac{1}{s} \prod_{k=1}^{\infty} e^{s/k} \Big/ \Big(1 + \frac{s}{k}\Big),$$

the product converging uniformly in compact subsets of $\mathbf{C} - \{0, -1, -2, \ldots\}$ because $e^x/(1+x) = 1 + O(x^2)$ for small x. Then Γ/g is an entire function with neither poles nor zeros, so it can be written as $\exp \alpha(s)$ for some entire function α. We show that $\alpha(s) = -\boldsymbol{\gamma} s$, where $\boldsymbol{\gamma} = 0.57721566490\ldots$ is *Euler's constant*:

$$\boldsymbol{\gamma} := \lim_{N\to\infty} \Big(-\log N + \sum_{k=1}^{N} \frac{1}{k}\Big).$$

That is, we show:

Lemma. *The Gamma function has the product formulas*

$$\Gamma(s) = e^{-\boldsymbol{\gamma} s} g(s) = \frac{e^{-\boldsymbol{\gamma} s}}{s} \prod_{k=1}^{\infty} e^{s/k} \Big/ \Big(1 + \frac{s}{k}\Big) = \frac{1}{s} \lim_{N\to\infty} \left(N^s \prod_{k=1}^{N} \frac{k}{s+k} \right). \quad (1)$$

Proof: For $s \neq 0, -1, -2, \ldots$, the quotient $g(s+1)/g(s)$ is the limit as $N\to\infty$ of

$$\frac{s}{s+1} \prod_{k=1}^{N} e^{1/k} \frac{1 + \frac{s}{k}}{1 + \frac{s+1}{k}} = \frac{s}{s+1} \left(\exp \sum_{k=1}^{N} \frac{1}{k} \right) \prod_{k=1}^{N} \frac{k+s}{k+s+1}$$
$$= s \cdot \frac{N}{N+s+1} \cdot \exp\left(-\log N + \sum_{k=1}^{N} \frac{1}{k}\right).$$

Now the factor $N/(N+s+1)$ approaches 1, while $-\log N + \sum_{k=1}^{N} \frac{1}{k} \to \boldsymbol{\gamma}$. Thus $g(s+1) = s e^{\boldsymbol{\gamma}} g(s)$, and if we define $\Gamma^?(s) := e^{-\boldsymbol{\gamma} s} g(s)$ then $\Gamma^?$ satisfies the same functional equation $\Gamma^?(s+1) = s\Gamma^?(s)$ satisfied by Γ. We are claiming that in fact $\Gamma^? = \Gamma$.

Consider $q := \Gamma/\Gamma^?$, an entire function of period 1. Thus it is an analytic function of $e^{2\pi i s} \in \mathbf{C}^*$. We wish to show that $q = 1$ identically. By the definition of g we have $\lim_{s\to 0} sg(s) = 1$; hence

$$\lim_{s\to 0} s\Gamma^?(s) = \lim_{s\to 0} sg(s) = 1 = \lim_{s\to 0} s\Gamma(s),$$

and $q(0) = 1$. We claim that there exists a constant C such that

$$|q(\sigma + it)| \leq Ce^{\pi|t|/2} \tag{2}$$

for all real σ, t; since the coefficient $\pi/2$ in the exponent is less than 2π, it will follow that q is constant, and thus that $\Gamma^? = \Gamma$ as claimed.

Since q is periodic, we need only prove (2) for $s = \sigma + it$ with $\sigma \in [1, 2]$. For such s, we have $|\Gamma(\sigma + it)| \leq \Gamma(\sigma)$ by the integral formula and

$$\left|\frac{\Gamma^?(\sigma + it)}{\Gamma^?(\sigma)}\right| = \prod_{k=0}^{\infty} \frac{\sigma + k}{|\sigma + k + it|} = \exp\left(-\frac{1}{2}\sum_{k=0}^{\infty} \log\left(1 + \frac{t^2}{(\sigma + k)^2}\right)\right).$$

The summand is a decreasing function of k, so the sum is

$$\leq \int_0^\infty \log\bigl(1 + (t/x)^2\bigr)\, dx = |t| \int_0^\infty \log\bigl(1 + (1/x)^2\bigr)\, dx,$$

which on integration by parts becomes $2|t| \int_0^\infty dx/(x^2 + 1) = \pi|t|$. This proves (2) with $C = \sup_{1\leq\sigma\leq 2} q(\sigma)$, and completes the proof of (1). □

Consequences of the product formula. Our most important application of the product formula for $\Gamma(s)$ is the *Stirling approximation*[1] to $\log \Gamma(s)$. Fix $\epsilon > 0$ and let R_ϵ be the region

$$\{s \in \mathbf{C}^* : |\mathrm{Im}(\log s)| < \pi - \epsilon\}.$$

Then R_ϵ is a simply-connected region containing none of the poles of Γ, so there is an analytic function $\log \Gamma$ on R_ϵ, real on $R_\epsilon \cap \mathbf{R}$, and given by the above product formula:

$$\log \Gamma(s) = \lim_{N\to\infty} \Bigl(s \log N + \log N! - \sum_{k=0}^{N} \log(s + k)\Bigr). \tag{3}$$

We prove:

Lemma. *The approximation*

$$\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log(2\pi) + O_\epsilon(|s|^{-1}) \tag{4}$$

holds for all s in R_ϵ.

[1] Originally only for $n! = \Gamma(n + 1)$, but we need it for complex s as well.

Proof: The estimate holds for small s, say $|s| < 1$, because $O_\epsilon(|s|^{-1})$ well exceeds all the other terms. We thus assume $|s| \geq 1$, and estimate the sum in (3) as we did for $\log x!$ in obtaining the original form of Stirling's approximation. The sum differs from

$$\int_{-\frac{1}{2}}^{N+\frac{1}{2}} \log(s+x)\,dx = (N+\frac{1}{2}+s)\log(N+\frac{1}{2}+s) - (s-\frac{1}{2})\log(s-\frac{1}{2}) - N - 1$$

$$= (N+\frac{1}{2}+s)\log N + (N+\frac{1}{2}+s)\log(1+\frac{1}{N}(s+\frac{1}{2})) - (s-\frac{1}{2})\log(s-\frac{1}{2}) - N - 1$$

by

$$\frac{1}{2}\int_{-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\|x+\frac{1}{2}\|^2}{(s+x)^2}\,dx \ll_\epsilon |s|^{-1}.$$

We already know that $\log N! = (N+1/2)\log N - N + A + O(N^{-1})$ for some constant A. The estimate (4) follows upon taking $N \to \infty$, except for the value $\frac{1}{2}\log(2\pi)$ of the constant term. This constant can be obtained by letting $s \to \infty$ in the duplication formula $\Gamma(2s) = \pi^{-1/2}2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2})$. $\Box$

One can go on to expand the $O_\epsilon(|s|^{-1})$ error in an asymptotic series in inverse powers of s (see the Exercises), but (4) is already more than sufficient for our purposes, in that we do not need the identification of the constant term with $\frac{1}{2}\log 2\pi$.

The logarithmic derivative of our product formula for $\Gamma(s)$ is

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{s+k}\right) = \lim_{N\to\infty}\left[\log N - \sum_{k=0}^{N}\frac{1}{s+k}\right].$$

Either by differentiating[2] (4) or by applying the same Euler-Maclaurin step to $\sum_0^N 1/(s+k)$ we find that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O_\epsilon(|s|^{-2}). \tag{5}$$

Remark

The product formula for $\Gamma(s)$ can also be obtained for real s by elementary means, starting from the characterization of Γ as the unique logarithmically convex function on $(0,\infty)$ satisfying the recursion $\Gamma(s+1) = s\Gamma(s)$ and normalized by $\Gamma(1) = 1$ (the Bohr-Mollerup theorem, see for instance [Rudin 1976, p.193]). The theorem for complex s can then be obtained by analytic continuation. The method used here, though less elegant, generalizes to a construction

[2] While real asymptotic series cannot in general be differentiated (why?), complex ones can, thanks to Cauchy's integral formula for the derivative. The logarithmic derivative of $\Gamma(s)$ is often called $\psi(s)$ in the literature, but alas we cannot use this notation because it conflicts with $\psi(x) = \sum_{n<x}\Lambda(n)$...

of product formulas for a much more general class of functions, as we shall see next.

Exercises

On the product formula:

1. Verify that the duplication formula for $\Gamma(2s)$ yields the correct constant term in (4). Apply Euler-Maclaurin to the sum in (3) to show that the $O_\epsilon(|s|^{-1})$ error can be expanded in an asymptotic series in inverse powers of s.

2. Use (1) to obtain a product formula for $\Gamma(s)\Gamma(-s)$, and deduce that

$$\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s. \tag{6}$$

(This can also be obtained from $\Gamma(s)\Gamma(1-s) = \mathrm{B}(s, 1-s)$ by using the change of variable $x = y/(y-1)$ in the Beta integral and evaluating the resulting expression by contour integration.) Use this together with the duplication formula and Riemann's formula for $\zeta(1-s)$ to obtain the equivalent asymmetrical form

$$\zeta(1-s) = \pi^{-s} 2^{1-s} \Gamma(s) \cos \frac{\pi s}{2} \zeta(s)$$

of the functional equation for $\zeta(s)$. Note that the duplication formula, and its generalization

$$\Gamma(ns) = (2\pi)^{\frac{1-n}{2}} n^{ns-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(s + \frac{k}{n}\right),$$

can also be obtained from (1).

3. Show that $\log \Gamma(s)$ has the Taylor expansion

$$\log \Gamma(s) = -\boldsymbol{\gamma}(s-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n)(s-1)^n$$

about $s = 1$. Recover from this the Laurent expansion

$$\Gamma(s) = \frac{1}{s} - \boldsymbol{\gamma} + \big(\boldsymbol{\gamma}^2 + \frac{\pi^2}{6}\big)\frac{s}{2} + O(s^2)$$

of $\Gamma(s)$ about $s = 0$.

Behavior of $\Gamma(s)$ on vertical lines:

4. Deduce from (4) that for fixed $\sigma \in \mathbf{R}$

$$\mathrm{Re}\big(\log \Gamma(\sigma + it)\big) = (\sigma - \frac{1}{2}) \log |t| - \frac{\pi}{2}|t| + C_\sigma + O_\sigma(|t|^{-1})$$

as $|t| \to \infty$. Check that for $\sigma = 0, 1/2$ this agrees with the exact formulas

$$|\Gamma(it)|^2 = \frac{\pi}{t \sinh \pi t}, \quad |\Gamma(1/2 + it)|^2 = \frac{\pi}{\cosh \pi t}$$

obtained from (6).

5. For $a, b, c > 0$, determine the Fourier transform of $f(x) = \exp(ax - be^{cx})$, and check your answer by using contour integration to calculate the Fourier transform of $\hat{f}$. Now apply Poisson summation, let $a \to 0$ and $C = e^c > 1$, and describe the behavior of $\sum_{n=0}^{\infty} z^{C^n}$ as $z \to 1$ from below. What does

$$\sum_{n=0}^{\infty} (-1)^n z^{2^n} = z - z^2 + z^4 - z^8 + z^{16} - + \cdots$$

do as $z \to 1$? Use this to prove that $\mathbf{Z} \cap \bigcup_{m=0}^{\infty} \big[2^{2^{2m}}, 2^{2^{2m+1}}\big)$ is an explicit example of a set of integers that does not have a logarithmic density.

An alternative proof of the functional equation for $\zeta(s)$ (also in Riemann's fundamental paper of 1859):

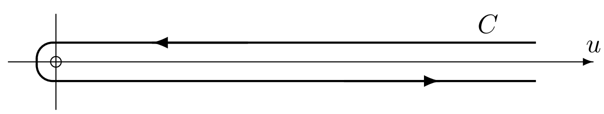
6. Prove that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1} \frac{du}{e^u - 1}$$

for $\sigma > 1$, and that when s is not a positive integer an equivalent formula is

$$\zeta(s) = -\frac{e^{-\pi i s}}{2\pi i} \Gamma(1-s) \int_C u^{s-1} \frac{du}{e^u - 1}$$

where C is a contour coming from $+\infty$, going counterclockwise around $u = 0$, and returning to $+\infty$:



Show that this gives the analytic continuation of ζ to a meromorphic function on $\mathbf{C}$; shift the line of integration to the left to obtain the functional equation relating $\zeta(s)$ to $\zeta(1-s)$ for $\sigma < 0$, and thus for all s by analytic continuation.

References

[WW 1940] Whittaker, E.T., Watson, G.N.: *A Course of Modern Analysis...*[3] (fourth edition). Cambridge University Press, 1940 (reprinted 1963). [HA 9.40 / QA295.W38]

[Rudin 1976] Rudin, W.: *Principles of Mathematical Analysis* (3rd edition). New York: McGraw-Hill, 1976.

[3] The full title is 26 words long, which was not out of line when the book first appeared in 1902. You can find the title in Hollis.