Math 229: Introduction to Analytic Number Theory

Elementary approaches II: the Euler product

Euler [Euler 1737] achieved the first major advance beyond Euclid's proof by combining his method of generating functions with another highlight of ancient Greek number theory, unique factorization into primes.

Theorem [Euler product for the zeta function]**.** *The identity*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}. \tag{1}$$

holds for all s such that the left-hand side converges absolutely.

Proof: Here and henceforth we adopt the convention:

> *The notation $\prod_p$ or $\sum_p$ means a product or sum over* prime p*.*

Every positive integer n may be written uniquely as $\prod_p p^{c_p}$, with each c_p a nonnegative integer that vanishes for all but finitely many p. Thus the formal expansion of the infinite product

$$\prod_{p \text{ prime}} \Bigl(\sum_{c_p=0}^{\infty} p^{-c_p s} \Bigr) \tag{2}$$

contains each term

$$n^{-s} = \Bigl(\prod_p p^{c_p} \Bigr)^{-s} = \prod_p p^{-c_p s}$$

exactly once. If the sum of the n^{-s} converges absolutely, we may rearrange the sum arbitrarily and conclude that it equals the product (2). On the other hand, each factor in this product is a geometric series whose sum equals $1/(1-p^{-s})$. This establishes the identity (2). □

The sum on the left-hand side of (1) is nowadays called the *zeta function*

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s} \; ;$$

the formula (2) is called the *Euler product* for $\zeta(s)$. Euler did not actually impose the convergence condition: the rigorous treatment of limits and convergence was not yet available, and Euler either handled such issues intuitively or ignored them. If s is a real number — the only case that concerned Euler — then it is well known that $\sum_{n=1}^{\infty} n^{-s}$ converges if and only if $s > 1$, by comparison with $\int_1^{\infty} x^{-s}\,dx$ (that is, by the "Integral Test" of elementary calculus). We shall use complex s as well, but the criterion for absolute convergence is still easy: if s has real part σ then

$$|n^{-s}| = |\exp(-s\log n)| = \exp(\mathrm{Re}(-s\log n)) = \exp(-\sigma\log n) = n^{-\sigma},$$

so the Euler product holds in the half-plane $\sigma > 1$.

Euler's next step was to set $s = 1$ in (2). This equates $\prod_p 1/(1-p^{-1})$ with the sum $\sum_{n=1}^{\infty} 1/n$ of the harmonic series. Since the sum diverges to $+\infty$, whereas each factor $\prod_p 1/(1-p^{-1})$ is finite, there are infinitely many factors. Therefore, there are infinitely many primes. This proof does not meet modern standards of rigor, but it is easy enough to fix: instead of setting s equal 1, let s approach 1 from above. The next result is an easy estimate on the behavior of $\zeta(s)$ for s near 1.

Lemma. *The inequalities*

$$\frac{1}{s-1} < \zeta(s) < \frac{1}{s-1} + 1 \tag{3}$$

hold for all $s > 1$.

(More accurate estimates are available using the Euler-Maclaurin formula, but we do not yet need them.)

Proof: For all $n > 0$ we have

$$\int_n^{n+1} x^{-s}\,dx = \frac{1}{s-1}\bigl(n^{1-s} - (n+1)^{1-s}\bigr),$$

whence

$$(n+1)^{-s} < \frac{n^{1-s} - (n+1)^{1-s}}{s-1} < n^{-s}.$$

Now sum over $n = 1, 2, 3, \ldots$. The sum of $(n^{1-s} - (n+1)^{1-s})/(s-1)$ telescopes to $1/(s-1)$. This sum is bounded above by $\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$, and below by $\sum_{n=1}^{\infty}(n+1)^{-s} = \zeta(s) - 1$. This proves the inequalities (3). $\Box$

The lower bound in (3) shows that $\zeta(s)\to\infty$ as $s\to 1$ from above (an equivalent notation is "as $s\to 1+$"). Since each factor $(1-p^{-s})^{-1}$ in the Euler product remains bounded, we have vindicated Euler's argument for the infinitude of primes.

The divergence of $\prod_p p/(p-1)$ and the behavior of $\prod_p 1/(1-p^{-s})$ as $s\to 1+$ give us much more specific information on the distribution of primes than we could hope to extract from Euclid's argument. For instance, we cannot have constants C, θ with $\theta < 1$ such that $\pi(x) < Cx^{\theta}$ for all x, because then the Euler product would converge for $s > \theta$. To go further along these lines it is convenient to use the logarithm of the Euler product:

$$\log\zeta(s) = \sum_p -\log(1-p^{-s}). \tag{4}$$

Euler again took $s = 1$ and concluded that $\sum_p 1/p$ diverges. Again we justify his conclusion by letting s approach 1 from above:

Theorem. *For any $s_0 > 1$ there exists M such that*

$$\Big|\sum_p p^{-s} - \log\frac{1}{s-1}\Big| < M \tag{5}$$

for all $s \in (1, s_0]$. In particular, $\sum_p 1/p$ diverges.

Proof: By our Lemma, $\log\zeta(s)$ is between $\log 1/(s-1)$ and $\log s/(s-1)$. Since $0 < \log s < s-1$, we conclude that $\log\zeta(s)$ differs from $\log 1/(s-1)$ by less than $s-1 < s_0-1$. In the right-hand side of (4), we approximate each summand $-\log(1-p^{-s})$ by p^{-s}. The error is at most p^{-2s}, so

$$\Big|\sum_p p^{-s} - \sum_p(-\log(1-p^{-s}))\Big| < \sum_p p^{-2s} < \zeta(2).$$

Hence (5) holds with $M = s_0 - 1 + \zeta(2)$. Letting $s \to 1$ we obtain the divergence of $\sum_p 1/p$. $\Box$

Interlude on the "Big Oh" notation $O(\cdot)$. The point of (5) is that $\sum_p p^{-s}$ equals $\log\frac{1}{s-1}$ within a bounded error, not the specific upper bound M on this error — which is why we were content with a bound $s_0 - 1 + \zeta(2)$ weaker than what the method can give. Usually in such approximate formulas we shall be interested only in the existence of constants such as M, not in their exact values. To avoid distractions such as "$s_0 - 1 + \zeta(2)$", we henceforth use "big Oh" notation. In this notation, (5) appears as

$$\sum_p p^{-s} = \log\frac{1}{s-1} + O(1). \tag{6}$$

In general, $f = O(g)$ means that f, g are functions on some space S with g nonnegative, and there exists a constant M such that $|f(z)| \leq Mg(z)$ for all $z \in S$. In particular, $O(1)$ is a bounded function, so (6) is indeed the same as (5). An equivalent notation, more convenient in some circumstances, is $f \ll g$ (or $g \gg f$). For instance, a linear map T between Banach spaces is continuous iff $Tv = O(|v|)$ iff $|v| \gg |Tv|$. Each instance of $O(\cdot)$ or $\ll$ or $\gg$ is presumed to carry its own implicit constant M. If the constant depends on some parameter(s), we may use the parameter(s) as a subscript to the "O" or "$\ll$". For instance, we may write $O_{s_0}(1)$ instead of $O(1)$ in (6); for any $\epsilon > 0$, we have $\log x \ll_\epsilon x^\epsilon$ on $x \in [1, \infty)$. For basic properties of $O(\cdot)$ and $\ll$ see the Exercises at the end of this section.

Back to $\pi(x)$. The estimate (6) for $\sum_p p^{-s}$ does not explicitly involve $\pi(x)$. We thus rearrange this sum as follows. Write p^{-s} as an integral $s\int_p^\infty y^{-1-s}\,dy$, and sum over p. Then y occurs in the interval of integration $[p, \infty)$ iff $p < y$, that is, with multiplicity $\pi(y)$. Therefore

$$\sum_p p^{-s} = s\int_1^\infty \pi(y) y^{-1-s}\,dy, \tag{7}$$

and (6) becomes an estimate for an integral involving $\pi(\cdot)$.

This transformation from the sum in (6) to the integral (7) is an example of a method we shall use often, known either as partial summation or integration by parts. To explain the latter name, consider that the sum may be regarded as the Stieltjes integral $\int_1^\infty y^{-s}\,d\pi(y)$, which integrated by parts yields (7); that is how we shall write this transformation from now on.

Our eventual aim is the Prime Number Theorem (PNT), which asserts that $\pi(x)$ is asymptotic to $x/\log x$ as $x \to \infty$. Our estimate (6) on the integral (7) does not suffice to prove the PNT, but does provide support for it: the estimate holds if we replace $\pi(x)$ with $x/\log x$. That is,[1]

$$\int_2^\infty \frac{y^{-s}}{\log y}\,dy = \log\frac{1}{s-1} + O(1) \qquad (1 < s \le 2).$$

To prove this, let $I(s) = \int_2^\infty \frac{y^{-s}}{\log y}\,dy$, and differentiate under the integral sign to obtain $I'(s) = -\int_2^\infty y^{-s}\,dy = 2^{1-s}/(1-s) = 1/(1-s) + O(1)$. Thus for $1 < s \le 2$ we have

$$I(s) = I(2) - \int_s^2 I'(\sigma)\,d\sigma = +\int_s^2 \frac{d\sigma}{\sigma - 1} + O(1) = \log\frac{1}{s-1} + O(1)$$

as claimed. While this does not prove the Prime Number Theorem, it does show that, for instance, if $c < 1 < C$ then there are arbitrarily large x, x' such that $\pi(x) > cx/\log x$ and $\pi(x') < Cx'/\log x'$.

Remarks

Euler's result $\sum_p 1/p = \infty$ underlies for our expectation that p_{n+1} divides $1 + \prod_{i=1}^n p_n$ infinitely often. The residue of $\prod_{i=1}^n p_n \bmod p_{n+1}$ should behave like a random element of $(\mathbf{Z}/p_{n+1}\mathbf{Z})^*$, and thus should equal -1 with probability $1/(p-1)$. The expected value of the number of $n < N$ such that p_{n+1} divides $1 + \prod_{i=1}^n p_n$ is thus $\sum_{n=2}^N 1/(p-1) > \sum_{n=2}^N 1/p \to \infty$ as $N \to \infty$. We expect the same behavior for many other problems of the form "how many primes p are factors of $f(p)$?", notably $f(p) = ((p-1)! + 1)/p$ (the Wilson quotient), $f(p) = (a^p - a)/p$ (the Fermat quotient with fixed base $a > 1$), and $f(p) = p^{-2}\sum_{i=1}^{p-1} 1/i$ (the Wolstenholme quotient). We shall soon see that $\sum_p 1/p$ diverges very slowly: $\sum_{p<x} 1/p = \log\log x + O(1)$. Therefore, while we expect infinitely many solutions of $p|f(p)$ in each case, we expect that these solutions will be very scarce.

Euler's work on the zeta function includes also its evaluation at positive integers: $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, "etc." The silliest proof I know of the infinitude

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[1] We shift the lower limit of integration to $y = 2$ to avoid the spurious singularity of $1/\log y$ at $y = 1$, and suppress the factor s because only the behavior as $s \to 1$ matters, and multiplying by s does not affect it to within $O(1)$. We also made the traditional and convenient choice $s_0 = 2$; the value of s_0 does not matter, as long as $s_0 > 1$, because we are concerned with the behavior near $s = 1$, and by specifying s_0 we can dispense with a distracting subscript in O_{s_0}.

of primes is to fix one such integer s, and observe that if there were finitely many primes then $\zeta(s) = \prod_p (1-p^{-s})^{-1}$, and thus also π^s, would be rational, contradicting Lindemann's theorem (1882) that π is transcendental. It is only a bit less silly to take $s = 2$ and use the irrationality of π^2, which though unknown to Euler was proved a few generations later by Legendre (1794?). This can actually be used to obtain lower bounds on $\pi(x)$, but even with modern "irrationality measures" we can obtain no lower bounds on $\pi(x)$ better than the $\log\log x$ bound already available from Euclid's proof.

Less frivolously, we note that the integral $\int_1^\infty \pi(y) y^{-s}\, dy/y$ appearing in (7) is the *Mellin transform* of $\pi(y)$, evaluated at $-s$. The Mellin transform may not be as familiar as the integral transforms of Fourier and Laplace, but the change of variable $y = e^u$ yields

$$\int_1^\infty \pi(y) y^{-s} \frac{dy}{y} = \int_0^\infty \pi(e^u) e^{-su}\, du,$$

which is the Laplace transform of $\pi(e^u)$, evaluated at s. In general, if $f(u)$ is a nonnegative function whose Laplace transform $\mathcal{L}f(s) := \int_0^\infty f(u)e^{-su}\, du$ converges for $s > s_0$, then the behavior of $\mathcal{L}f(s)$ as $s \to s_0+$ detects the behavior of $f(u)$ as $u \to \infty$. In our case, $s_0 = 1$, so we expect that our estimate on $\int_1^\infty \pi(y) y^{-s}\, dy/y$ for s near 1 will give us information on the behavior of $\pi(x)$ for large x. Moreover, inverting the Laplace transform requires a contour integral over complex s; this suggests that we shall need to consider $\log\zeta(s)$, and thus the solutions of $\zeta(s) = 0$, in the complex plane. We shall return to these ideas and the Mellin transform before long.

Exercises

Concerning the Big Oh (equivalently "$\ll$") notation:

1. If $f \ll g$ and $g \ll h$ then $f \ll h$. If $f_1 = O(g_1)$ and $f_2 = O(g_2)$ then $f_1 f_2 = O(g_1 g_2)$ and $f_1 + f_2 = O(g_1 + g_2) = O(\max(g_1, g_2))$. Given a positive function g, the functions f such that $f = O(g)$ constitute a vector space.

2. If $f \ll g$ on the interval (a, b) or $(a, b]$ then $\int_a^x f(y)\, dy \ll \int_a^x g(y)\, dy$ for all x in the same interval such that the integrals exist. (We already used this to obtain $I(s) = \log(1/(s-1)) + O(1)$ from $I'(s) = 1/(1-s) + O(1)$.) In general differentiation does not commute with "$\ll$" (why?). Nevertheless, prove that $\zeta'(s) [= -\sum_{n=1}^\infty n^{-s} \log n]$ is $-1/(s-1)^2 + O(1)$ on $s \in (1, \infty)$.

3. So far all the implicit constants in the $O(\cdot)$ or $\ll$ we have seen are *effective*: we didn't bother to specify them, but we could if we really had to. Moreover the transformations in exercises 1,2 preserve effectivity: if the input constants are effective then so are the output ones. However, it can happen that we know that $f = O(g)$ without being able to name a constant M such that $|f| \leq Mg$. Here is a prototypical example. Suppose $x_1, x_2, x_3, \ldots$ is a sequence of positive reals which we suspect are all ≤ 1, but all we can show is that if $i \neq j$ then $x_i x_j < x_i + x_j$. Prove that the x_i are bounded, i.e., $x_i = O(1)$, but that as long as we do not find some x_i greater than 1, we cannot use this to exhibit a

specific M such that $x_i < M$ for all i — and indeed if our suspicion that every $x_i \leq 1$ is correct then we shall never be able to find M.

We shall encounter this sort of unpleasant ineffectivity (where it takes at least two outliers to get a contradiction) in Siegel's lower bound on $L(1, \chi)$; it arises elsewhere too, notably in Faltings' proof of the Mordell conjecture, where the number of rational points on a given curve of genus > 1 can be effectively bounded but their size cannot.

Applications of the Euler product for $\zeta(s)$:

4. Complete the proof that for each $c < 1$ there are arbitrarily large x such that $\pi(x) > cx/\log x$ and for each $C > 1$ there are arbitrarily large x' such that $\pi(x') < Cx'/\log x'$.

5. It is known that there exists a constant M such that $|\pi^2 - a/b| \gg 1/b^M$ for all positive integers a, b. Use this together with the Euler product for $\zeta(2)$ to prove that $\pi(x) \gg \log\log x$.

6. Prove that there are $N/\zeta(2) + O(N^{1/2})$ squarefree integers in $[1, N]$. Obtain similar estimates for the number of natural numbers $< N$ not divisible by n^s for any $n > 1$ $(s = 3, 4, 5, \ldots)$. NB this and the next few exercises are not quite as easy at they may seem: remember the final exercise for the previous lecture! A hint as to the solution: use the Euler product for $\zeta(s)$ to obtain a series expansion for $1/\zeta(s)$.

It follows that an integer chosen uniformly at random from $[1, N]$ is squarefree with probability approaching $1/\zeta(2) = 6/\pi^2$ as $N \to \infty$. Informally, "a random integer is squarefree with probability $6/\pi^2$". We shall see that the error estimate $O(N^{1/2})$ can be improved, and that the asymptotic growth of the error hinges on the Riemann Hypothesis.

7. Prove that there are $N^2/\zeta(2) + O(N \log N)$ ordered pairs of relatively prime integers in $[1, N]$. What of relatively prime pairs (x_1, x_2) with $x_1 < N_1$ and $x_2 < N_2$? Generalize.

Again we may informally deduce that two random integers are coprime with probability $6/\pi^2$. Alternatively, we may regard a coprime pair (x_1, x_2) with $x_i \leq N$ as a positive rational number x_1/x_2 of height at most N. Dropping the positivity requirement, we find that there are asymptotically $2N^2/\zeta(2)$ rational numbers of height at most N. This has been generalized to number fields other than $\mathbf{Q}$ by Schanuel [1979]; a function-field analogue, concerning rational functions of bounded degree on a given algebraic curve over a finite field, was announced by Serre [1989, p.19] and proved by DiPippo [1990] and Wan [1992] (independently but in the same way). The function-field result was the starting point of our estimate on the size of the nonlinear codes obtained from rational functions on modular curves [Elkies 2001]. Schanuel also obtained asymptotics for rational points of height at most N in projective space of dimension $s - 1$ over a number field K; when $K = \mathbf{Q}$ this recovers the asymptotic enumeration of coprime s-tuples of integers.

8. Prove that as $N \to \infty$ the number of ordered quadruples (a, b, c, d) of integers in $[1, N]$ such that $\gcd(a, b) = \gcd(c, d)$ is asymptotic to $2N^4/5$.

Can this be proved without invoking the values of $\zeta(2)$ or $\zeta(4)$? This can be regarded as a form of a question attributed to Wagstaff in [Guy 1981, B48]: "Wagstaff asked for an elementary proof (e.g., without using properties of the Riemann zeta-function) that $\prod_p (p^2+1)/(p^2-1) = 5/2$."

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