Math 229: Introduction to Analytic Number Theory

Incomplete character sums II:
The Burgess bound on short character sums

We observed that nontrivial bounds on individual short character sums

$$S_\chi(n_0, n_0+N) := \sum_{n=1}^{N} \chi(n_0+n) = S_\chi(n_0+N) - S_\chi(n_0).$$

must exploit the structure of $\mathcal{N}$ in the special case $\mathcal{N} = \{1, 2, 3, \ldots, N\}$.

We start from the fact that the set $\{1, 2, 3, \ldots, N\}$ contains many translates of $\{1, 2, 3, \ldots, H\}$ for $H = o(N)$. This means that if a single $|S_\chi(n_0, n_0+N)|$ attains the trivial bound of N then each of $N-H+1$ values of $|S_\chi(n_0', n_0'+H)|$ attains its trivial bound of H. More generally, if instead of $|S_\chi(n_0, n_0+N)| = N$ we assume only that $|S_\chi(n_0, n_0+N)|$ is unusually large then the same is true of $|S_\chi(n_0', n_0'+H)|$ for many choices of n_0'. Indeed we have

$$S_\chi(n_0, n_0+N) = \frac{1}{H} \sum_{h=1}^{N} S_\chi(n_0+h, n_0+h+H) + O(H). \tag{1}$$

By Hölder's inequality[1] we have for $r = 1, 2, 3, \ldots$

$$\left| \sum_{h=1}^{N} S_\chi(n_0+h, n_0+h+H) \right| \le N^{1-\frac{1}{2r}} \left(\sum_{h=1}^{N} |S_\chi(n_0+h, n_0+h+H)|^{2r} \right)^{1/2r}.$$

We can then use

$$\sum_{h=1}^{N} |S_\chi(n_0+h, n_0+h+H)|^{2r} \le \sum_{m=0}^{p-1} |S_\chi(m, m+H)|^{2r} \ll_r pH^r + p^{1/2} H^{2r}.$$

Unfortunately the cost of bounding the sum of $|S_\chi(m, m+H)|^{2r}$ over the N values $m = n_0+h$ by the sum over all p choices is too high: there is no value of r, N, and $H < N$ that makes the resulting upper bound on $|S_\chi(n_0, n_0+N)|$ smaller than both the trivial bound N and the Pólya-Vinogradov bound $O(p^{1/2} \log p)$, even when the $O(H)$ error in (1) is ignored.

[1] This inequality asserts that for $p, q > 1$ with $p^{-1} + q^{-1} = 1$ the upper bound

$$\Bigl| \sum_{i=1}^{n} x_j \overline{y_j} \Bigr| \le \Bigl(\sum_{i=1}^{n} |x_j|^p \Bigr)^{1/p} \Bigl(\sum_{i=1}^{n} |y_j|^q \Bigr)^{1/q}$$

holds for all $x_j, y_j \in \mathbf{C}$, with equality if and only if one of the vectors $(x_j), (y_j) \in \mathbf{C}^n$ is a multiple of the other. The inequality asserts in effect that the l^p and l^q norms on $\mathbf{C}^n$ are each other's dual; the special case $p = q = 2$ is the Cauchy-Schwarz inequality. In the application to the right-hand side of (1) we have $y_j = 1$ for each j, a special case that is equivalent to the inequality of the means: for fixed $x_j \ge 0$, the p-th power mean $(n^{-1} \sum_{i=1}^{n} x_j^p)^{1/p}$ is an increasing function of p. We later apply Hölder's inequality with other choices of y_j.

Burgess's key idea for going beyond both the trivial bound and Pólya-Vinogradov is the observation that $\{1, 2, 3, \ldots, N\}$ contains not just many translates of $\{1, 2, 3, \ldots, H\}$ but even more images of $\{1, 2, 3, \ldots, H\}$ under affine transformations — that is, arithmetic progressions of length H. If $|S_\chi(n_0, n_0 + N)| = N$ then $|\sum_{n=1}^{H} \chi(n_0' + dn)| = H$ for all n_0' and $d > 0$ such that $n_0' + d > n_0$ and $n_0' + Hd \le n_0 + N$. But χ is multiplicative, so

$$\sum_{n=1}^{H} \chi(n_0' + dn) = \chi(d) \sum_{n=1}^{H} \chi(d^{-1}(n_0' + dn)) = \chi(d)\, S_\chi(d^{-1} n_0, d^{-1} n_0 + H),$$

whence $\left|S_\chi(d^{-1} n_0, d^{-1} n_0 + H)\right| = H$ as well, where d^{-1} is the multiplicative inverse of $d \bmod p$. Proceeding as before, we try to bound $S_\chi(n_0, n_0 + N)$ by writing

$$\begin{aligned} S_\chi(n_0, n_0 + N) &= \frac{1}{H} \sum_{h=1}^{N} \sum_{n=1}^{H} \chi(n_0 + h + dn) + O(Hd) \qquad (2) \\ &= \frac{\chi(d)}{H} \sum_{h=1}^{N} S_\chi(d^{-1}(n_0 + h), d^{-1}(n_0 + h) + H) + O(Hd) \end{aligned}$$

for each $d < N/H$. Varying both d and h, we now expect to obtain an upper bound on $S_\chi(n_0, n_0 + N)$ in terms of the sum of N^2/H powers $|S_\chi(n_0', n_0' + H)|^{2r}$ rather than only N, and thus to incur a lower penalty when replacing that sum by $\sum_{m=1}^{p} |S_\chi(m, m + H)|^{2r}$.

To see what would happen, suppose we could ignore the edge effects $O(Hd)$ in (2). We would then have

$$(?) \qquad S_\chi(n_0, n_0 + N) \ll \frac{1}{N} \sum_{d=1}^{N/H} \sum_{h=1}^{N} \left|S_\chi(d^{-1}(n_0 + h), d^{-1}(n_0 + h) + H)\right|,$$

and by Hölder the double sum is

$$\ll \left(\frac{N^2}{H}\right)^{1 - \frac{1}{2r}} \left(\sum_{d=1}^{N/H} \sum_{h=1}^{N} \left|S_\chi(d^{-1}(n_0 + h), d^{-1}(n_0 + h) + H)\right|^{2r}\right)^{1/2r}.$$

We want to bound the sum over (d, h) by

$$\sum_{m=1}^{p-1} \left|S_\chi(m, m + H)\right|^{2r} \ll_r pH^r + p^{1/2} H^{2r}.$$

This would give

$$S_\chi(n_0, n_0 + N) \ll_r \frac{1}{N} \left(\frac{N^2}{H}\right)^{1 - \frac{1}{2r}} (p^{1/2r} H^{1/2} + p^{1/4r} H),$$

and we would take $H \approx p^{1/2r}$ to balance $p^{1/2r}H^{1/2}$ with $p^{1/4r}H$, finally obtaining

$$S_\chi(n_0, n_0+N) \ll_r N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}}. \tag{??}$$

This would be a genuine improvement for suitable N and $r > 1$; for instance if $N \approx p^{1/2}$ we could take $r = 2$ to get $S_\chi(n_0, n_0+N) \ll p^{7/16}$, of lower order than both N and $p^{1/2}\log p$. For any N, the bound (??) for $r = 2$ and $n_0 = 0$ would give $S_\chi(N)/N^{1/2} \ll p^{3/16}$, and thus $L(1/2+it) \ll p^{3/16}\log p$. We note for future reference that if $N = p^\nu$ then (??) improves on both $|S_\chi(n_0, n_0+N)| < N$ and $|S_\chi(n_0, n_0+N)| \ll p^{1/2}\log p$ for $1/4 < \nu - (1/4r) \le 1/2$; in particular for each $\nu > 1/4$ we would get a nontrivial bound by taking r large enough, and would thus prove the existence of a quadratic nonresidue mod p less than $p^{1/4\sqrt{e}+o(1)}$. For $r = 1$, (??) recovers the Pólya-Vinogradov bound without the $\log p$ factor (already hinting that we cannot expect to quite prove (??) as it stands), and as $r \to \infty$ (??) converges to the trivial bound N. Thus for finite integers $r > 1$ we may regard (??) as an interpolation between those two bounds.

But we have two difficulties to overcome before obtaining a bound such as (??). One is the "edge effects" error $O(Hd)$ that we ignored in (?): if we combine all these errors in the average over $d < N/H$, we get $O(N)$, which obviates any improvement over the trivial bound on $S_\chi(n_0, n_0+N)$. Fortunately these error terms are themselves averages over shorter character sums $S_\chi(n_1, n_1+N')$; explicitly we may write

$$\begin{aligned} HS_\chi(n_0, n_0+N) &- \sum_{h=1}^{N}\sum_{n=1}^{H} \chi(n_0+h+dn) \\ &= \sum_{j=1}^{H-1} \big(S_\chi(n_0, n_0+dj) - S_\chi(n_0+N, n_0+N+dj)\big). \end{aligned} \tag{3}$$

So we will be able to use induction on N to reduce the edge effects to a constant factor.

A more serious difficulty is that we cannot bound the sum over $d^{-1}(n_0+h)$ in (?) by a sum over m, because the same m may arise as $d^{-1}(n_0+h)$ in more than one way. There are several ways to overcome this difficulty, at the cost of a fractional power of $\log(p)$ which will not significantly affect the quality of our bound. When $n_0 = 0$ (the case relevant to bounds on $L(1/2+it, \chi)$), we can simply restrict the sum to prime d, because then the rational numbers h/d are all distinct, and remain distinct mod p provided their denominators do not exceed $p^{1/2}$. (Recall that $d \le N/H$ and we shall take $H = p^{1/2r}$ while N will be no greater than $p^{1/2+(1/4r)}$.) This reduces the number of (d,h) pairs by a factor $O(\log p)$, and thus increases the bound on $S_\chi(0,N)$ by $O(\log p)^{1-(1/2r)}$.

For arbitrary n_0, we argue instead as follows. Fix $D < N/H$, and vary d over all integers in $[1, D]$. For m mod p let c_m be the number of representations of m

as $(n_0+h)/d$ with $1 \le h \le N$ and $1 \le d \le D$. Then

$$\frac{1}{N}\sum_{d=1}^{D}\sum_{h=1}^{N}\left|S_\chi(d^{-1}(n_0+h), d^{-1}(n_0+h)+H)\right| = \sum_{m=0}^{p-1} c_m \left|S_\chi(m, m+H)\right|$$

and Hölder bounds the sum over m by

$$\left(\sum_{m=0}^{p-1} c_m^{2r/(2r-1)}\right)^{1-(1/2r)} \left(\sum_{m=0}^{p-1} \left|S_\chi(m, m+H)\right|^{2r}\right)^{1/2r}.$$

If all c_m were 0 or 1, the first factor would be $(ND)^{1-1/(2r)}$, which is what we used to obtain (??). Since some c_m may exceed 1, the actual bound is larger, but we shall show that in fact

$$\left(\sum_{m=0}^{p-1} c_m^{2r/(2r-1)}\right)^{1-(1/2r)} \ll \log(p)^{1/2r}\,(ND)^{1-(1/2r)},$$

and thus that only a fractional power of $\log(p)$ is required to fix (??).

Clearly $\sum_{m=0}^{p-1} c_m = ND$. We shall show that $\sum_{m=0}^{p-1} c_m^2 \ll ND(\log p + (ND/p))$, from which the desired estimate will follow as long as $ND < p$ by another application of Hölder, or using the logarithmic convexity of $\sum_{m=0}^{p-1} c_m^q$ as a function of q. We can certainly assume $ND < p$, because $ND < N^2/H \approx p^{-1/2r}N^2$, which as already noted does not exceed p for any N that makes (??) better than Pólya-Vinogradov. Now $\sum_{m=0}^{p-1} c_m^2$ is the number of solutions of the congruence $(n_0+h)/d \equiv (n_0+h')/d' \bmod p$ in positive integers $h, h' \le N$, $d, d' \le D$. Equivalently, we must have

$$kp = d'(n_0+h) - d(n_0+h') = (d'-d)n_0 + d'h - dh' \tag{4}$$

for h, h', d, d' as above and $k \in \mathbf{Z}$. Fix d, d', and let $d_0 = \gcd(d, d')$. Then $|d'h - dh'| \le ND$, so there are $O(1 + ND/(d_0 p))$ choices for k. Given k, if there is any solution (h_0, h_0') of $kp - (d'-d)n_0 = d'h - dh'$ then the general solution is $(h_0, h_0') + j(d/d_0, d'/d_0)$ for $j \in \mathbf{Z}$, so the number of solutions is $O(Nd_0/\max(d, d'))$. Thus the total number of solutions (k, h, h') of (4) is $O(Nd_0 + N^2D/p)/\max(d, d')$. It remains to sum this over $d, d' \le D$. The sum of $N^2D/(p\max(d, d'))$ is $< 2(ND)^2/p$. Given $d_0 < D$, we have

$$\sum_{\substack{d, d' < D \\ \gcd(d, d') = d_0}} \frac{1}{\max(d, d')} \le \frac{1}{d_0}\sum_{e, e' < D/d_0} \frac{1}{\max(e, e')} < \frac{D}{d_0^2}.$$

Therefore our upper bound on $\sum_{m=0}^{p-1} c_m^2$ is

$$\frac{(ND)^2}{p} + N\sum_{d_0=1}^{D} \frac{D}{d_0} \ll \frac{(ND)^2}{p} + ND\log D,$$

which yields the claimed bound because $\log D < \log p$.

We can now prove:

Theorem. *For every integer $r \geq 2$ there exists C_r such that for any nontrivial character χ modulo a prime p, and integers $n_0 \bmod p$ and $n < N$, we have*

$$|S_\chi(n_0, n_0 + N)| < C_r N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{1/2r}. \tag{5}$$

Remarks: We formulate the result with C_r rather than $\ll_r$ because we shall apply induction on N and must not let C_r change with each induction step. (See Exercise 2 for the dependence of C_r on r.) The Burgess bound is often given with final factor $(\log p)^{1/r}$, not $(\log p)^{1/2r}$; but as far as I can tell, that is needed only for $r = 1$, when it recovers Pólya-Vinogradov (except that the proof does not work as it stands for $r = 1$; see Exercise 3 below for an elementary proof along these lines of the Pólya-Vinogradov inequality without Gauss sums).

Proof: We fix p, χ, r and apply induction on N. We shall show that, for some choice of $C_r > 1$, if (5) holds for all $N < N_0$ then it holds for all $N < 10N_0$. Since the bound is trivial for $N < p^{1/4+1/(4r)}$, it will follow by induction that for each $k = 1, 2, 3, \ldots$ the bound holds for all $N < 10^k p^{1/4+1/(4r)}$, which will establish it for all N.

Take $H = \lfloor p^{1/2r} \rfloor$ and $D = \lfloor N/10H \rfloor$. For all $d \leq D$, the right-hand side of (3) is the sum of fewer than $2H$ sums $S_\chi(n_1, n_1 + dj)$ with $dj < DH < N/10$. Thus by the inductive assumption it has absolute value less than

$$2HC_r(N/10)^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{1/2r}. \tag{6}$$

Now sum over $d \leq D$ to get

$$DH \cdot S_\chi(n_0, n_0 + N) = \sum_{d=1}^{D} \chi(d) \left(\sum_{h=1}^{N} S_\chi(d^{-1}(n_0 + h), d^{-1}(n_0 + h) + H) \right) + E$$

where the sum over d has absolute value at most

$$\sum_{m=0}^{p-1} c_m |S_\chi(m, m + H)|$$

while the error E satisfies

$$|E| < 2DH(N/10)^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{1/2r}.$$

By applying Hölder to $\sum_{m=0}^{p-1} c_m |S_\chi(m, m + H)|$ together with $\sum_m c_m = ND$ and our upper bound on $\sum_m c_m^2$, we have

$$\sum_{m=0}^{p-1} c_m |S_\chi(m, m + H)| < DH \cdot C'_r N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{1/2r}$$

as above, with some constant C'_r that includes an extra but fixed factor, say $11 > N/DH$. Hence (5) holds provided $C_r < C'_r + 2/10^{1-1/r}C_r$, which is possible because $r > 1$. □□

Exercises

1. Prove that if $\alpha_m \geq 0$ and $c_m \geq 0$ then $f(q) = \log \sum_m \alpha_m c_m^q$ is a convex function of $q > 0$, and strictly convex unless $\alpha_m c_m = 0$ for all but one m. [Prove that $2f(q_1 + q_2) \leq f(2q_1) + f(2q_2)$.]

2. Show that the constant C_r of (5) can be taken to be $C_0 r$ for some universal constant C_0.

3. Our proof of the Burgess bound does not work as stated for $r = 1$ because the factors of 2 from (6) accumulate; at best we might show this way that $S_\chi(n_0, n_0 + N) \ll_\epsilon p^{1/2+\epsilon}$ by replacing 10 by a constant larger than $2^{1/\epsilon}$. Show instead that the following variation works, giving an elementary proof of the Pólya-Vinogradov bound for characters modulo a prime. Fix χ, and define $\Delta(n_0, N) = \sum_{n=1}^{N} (N - n)\, \chi(n_0 + n)$. Show that if $N = N_1 + N_2$ then

$$\Delta(n_0, N) = \Delta(n_0, N_1) + \Delta(n_0, N_2) + \sum_{h=0}^{N_1-1} S_\chi(n_0 + h, n_0 + h + N_2).$$

But the sum has absolute value less than $(N_1 N_2 p)^{1/2}$. Since $\Delta(n_0, 1) = 0$, it follows by induction that $\Delta(n_0, N) \ll p^{1/2} N \log N$ for all n_0. Finish by writing $S_\chi(n_0, n_0 + N)$ in terms of $\Delta(n_0, N)$ and $\Delta(n_1, N + 1)$ for suitable n_1.

References

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