Here's a nice way to see that if A is any $n \times m$ matrix then A and A^{T} have the same rank (which is part (c) of "Fact 5.3.9" in the textbook, page 215).

We begin by showing an identity that also has other uses (though we won't see it much in 21b):

Let $\vec{x}$ be any vector in $\mathbb{R}^n$, and $\vec{y}$ be any vector in $\mathbb{R}^m$. Then for any $n \times m$ matrix A we have

$$\vec{x} \cdot (A\vec{y}) = (A^{\mathrm{T}}\vec{x}) \cdot \vec{y}.$$

[Note that the first dot product is in $\mathbb{R}^n$, and the second is in $\mathbb{R}^m$.]

The reason for this identity is that both sides equal the single entry of the 1×1 matrix $\vec{x}^{\mathrm{T}} A\vec{y}$. For $\vec{x} \cdot (A\vec{y})$ this is clear from "Fact 5.3.6" in the textbook (page 213). To get $(A^{\mathrm{T}}\vec{x}) \cdot \vec{y}$, note that any 1×1 matrix is symmetrical, so

$$\vec{x}^{\mathrm{T}} A\vec{y} = (\vec{x}^{\mathrm{T}} A\vec{y})^{\mathrm{T}} = \vec{y}^{\mathrm{T}} A^{\mathrm{T}}\vec{x}$$

at which point we can use 5.3.6 again to get the 1×1 matrix $[\vec{y}^{\mathrm{T}} \cdot (A^{\mathrm{T}}\vec{x})] = [(A^{\mathrm{T}}\vec{x}) \cdot \vec{y}]$.

Now consider the linear subspace $V = (\mathrm{im}(A))^{\perp}$, the orthogonal complement of $\mathrm{im}(A)$ in $\mathbb{R}^n$. By "Fact 5.1.8" (page 192) and "Fact 3.3.6" (page 130) we have

$$\dim V = n - \dim(\mathrm{im}(A)) = n - \mathrm{rank}(A).$$

On the other hand, V consists of all $\vec{x}$ in $\mathbb{R}^n$ that are orthogonal to every vector of the form $A\vec{y}$ for some y in $\mathbb{R}^m$; that is, such that $\vec{x} \cdot (A\vec{y}) = 0$ for all $\vec{y} \in \mathbb{R}^m$. But this is the same as $(A^{\mathrm{T}}\vec{x}) \cdot \vec{y} = 0$ by our identity above. So, $A^{\mathrm{T}}\vec{x}$ is to be a vector in $\mathbb{R}^m$ orthogonal to *every* vector in $\mathbb{R}^m$! The only such vector is $\vec{0}$. (Do you see why?) Therefore V is none other than $\ker A^{\mathrm{T}}$. We have thus given another argument for "Fact 5.4.1" (page 219):

$$(\mathrm{im}(A))^{\perp} = \ker(A^{\mathrm{T}}).$$

But by the Rank-Nullity Theorem ("Fact 3.3.7", page 131) we know that

$$\dim \ker(A^{\mathrm{T}}) = n - \mathrm{rank}(A^{\mathrm{T}}).$$

Since $\ker(A^{\mathrm{T}}) = V$, we have thus shown that $n - \mathrm{rank}(A) = n - \mathrm{rank}(A^{\mathrm{T}})$. It follows that $\mathrm{rank}(A) = \mathrm{rank}(A^{\mathrm{T}})$, as claimed.