Math 21b
Fall 2006 Midsemester Exam
Solutions to Practice Problems 1 and 2
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Note: These solutions are not extremely detailed and are only meant as a guide to your studying. Let me know if you find any errors or omissions. If you are confused as to how I arrived at any of the conclusions, I will be at the Math Question Center on Sunday night (starting at 8:30 pm in Science Center B10). Good luck studying!

Problem 1

(a) True. $\text{rref}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$, which is an inconsistent matrix.

(b) False. Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and $v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then all conditions hold, but (v_1, v_2, v_3, v_4) is not a linearly independent list.

(c) False. Consider projection from 3-dimensional space onto the xy-plane: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $A^2 = A$.

(d) True. This is just a computation.

(e) False. The statement does not specify that the matrix is square, and matrices like $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ have a trivial kernel but are not invertible because they are not square.

(f) True. Since rref A is obtained from row operations, the span of the rows should not be changed.

(g) False. Let A be rotation by $\frac{2\pi}{3}$ degrees on the plane, so $A^3 = I$.

(h) True. If either A or B had a nontrivial kernel, then AB would fail to have a trivial kernel since A and B are maps from $\mathbb{R}^n$ to itself.

(i) False. The image of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ does not fully specify T. Moreover, since $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is linearly independent of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, for any $\begin{pmatrix} x \\ y \end{pmatrix}$ there is a linear transformation taking $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$. For example, the linear transformation specified in this statement is a shear, but we could also get from $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ by rotating and scaling. Since the norm of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is $\sqrt{5}$ and the norm of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is $\sqrt{17}$, the scale factor would have to be $\sqrt{\frac{17}{5}}$. But if we consider $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$, the scale factor would have to be $\sqrt{\frac{65}{17}} \neq \sqrt{\frac{17}{5}}$.

Problem 2

(a) $Tv_1 = e_1$ and $Tv_2 = e_2$. Also, we know T must be invertible since it sends the basis (v_1, v_2) to the basis (e_1, e_2). Therefore we can say $T^{-1}e_1 = v_1$ and $T^{-1}e_2 = v_2$, and we can easily write down the matrix for T^{-1}: $\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$. We can get T by inverting this matrix: $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$.

(b) $Tu = v_1 \rightarrow u = T^{-1}v_1 = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$.