Math 21b Practice Midterm 2

CA Answer Key to Problem 1 and True/False

November 19, 2006

1 Problem 1

1.1 Part a

$\det(A) = 0$, so A is not invertible. Thus, its rank is less than its number of columns, so solutions of $A\vec{x} = \vec{b}$ will either have no solutions or infinitely many solutions.

1.2 Part b

We can find $\text{rref}(A|\vec{b}_1)$, where $A|\vec{b}_1$ is the augmented matrix:

$$\text{rref}(A|\vec{b}_1) = \text{rref}\left(\left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 3 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Thus, $A\vec{x} = \vec{b}_1$ has no solutions.

1.3 Part c

Similarly,

$$\text{rref}(A|\vec{b}_2) = \text{rref}\left(\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, $A\vec{x} = \vec{b}_2$ has infinitely many solutions.

1.4 Part d

The image of A is just the span of columns of A, but note that the third column of A is the sum of the first two. Thus, $\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}\right)$. Applying the Gram-Schmidt Algorithm to these two vectors, we get the orthonormal basis $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

1.5 Part e

Applying this least-squares (normal) equation $A^T A\vec{x} = A^T\vec{b}_1$ (because A is not invertible), we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 4 \\ 2 & 8 & 10 \\ 4 & 10 & 14 \end{bmatrix}\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

We find rref of the augmented matrix, as before, and we get $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2/3 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Thus, $\vec{x} = \begin{bmatrix} 2/3 - t \\ 1/3 - t \\ t \end{bmatrix}$ for $t \in \mathbb{R}$. We get a family of solutions here. One of them is obtained when $t = 0$: $\vec{x} = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix}$. Note that you only need to find one solution to answer this problem; by analyzing the rref augmented matrix, you could have just concluded that the rightmost column equals 2/3 times the first column plus 1/3 times the second, giving you an answer directly.

1.6 Part f

We saw in Part c that the equation has exact solutions. Thus we use $\text{rref}(A|\vec{b}_2) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$, and conclude $\vec{x} = \begin{bmatrix} 3 - t \\ 1/2 - t \\ t \end{bmatrix}$ for $t \in \mathbb{R}$. One solutions of this family is obtained when $t = 0$: $\vec{x} = \begin{bmatrix} 3 \\ 1/2 \\ 0 \end{bmatrix}$. Note again that you only need to find one solution to answer this problem.

2 True or False?

- True. Expand using $||\vec{a}||^2 = \vec{a} \cdot \vec{a}$ and simplify. Note: this equality is known as the "parallelogram law" and may be encountered elsewhere in your mathematical endeavors.
- True. The least-squares solutions are given by $A^T A\vec{x} = A^T \vec{b}$, and $\vec{x} = \vec{0}$ is always a solution to $A^T A\vec{x} = \vec{0}$.
- False. $\dim(V) + \dim(V^\perp) = \dim(\mathbb{R}^3)$.
- False. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Note that $\text{tr}(A) = 0$ but $\det(A) = -1$, so A is invertible.
- False. Expand by minors, down the 4th column and then third column to get that the determinant equals $11 \cdot 7 \cdot (1 \cdot 4 - 2 \cdot 3)$, and note that $(1 \cdot 4 - 2 \cdot 3) < 0$.
- True. Fact 7.2.7.
- True. Note that $\det(A^m) = (\det(A))^m = 0$ implies that $\det(A) = 0$, which implies that A is not invertible.
- True. Suppose λ in an eigenvalue of A with an eigenvector $\vec{v}$. $A^m\vec{v} = \lambda^m\vec{v} = \vec{0}$ implies that λ must equal 0 because $\vec{v}$, an eigenvector, cannot equal $\vec{0}$.

3 Good luck!

Be sure to ask us if you have any questions!