Solution to the Problem 4, Midterm II

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a) The characteristic polynomial is $f_A(\lambda) = \lambda^2 - 1/2\lambda - 1/2 = (\lambda + 1/2)(\lambda - 1)$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1/2$.

b) The eigenspace for $\lambda_1 = 1$ is the kernel of the matrix $A - I_2 = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix}$.

Hence $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector with eigenvalue 1 (any multiple of v_1 also works).

Similarly, the $\ker(A - 1/2I_2) = \text{span}(\begin{bmatrix} 1 \\ -2 \end{bmatrix})$ and the vector $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is the eigenvector with the eigenvalue $-1/2$.

c) The coordinates of the vector $x(0)$ in the eigenbasis (v_1, v_2) are $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1}x(0)$. Here $S = [v_1 v_2] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ is the change of basis matrix. Simple computation gives $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 24 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$.

Therefore, $x(t) = A^t(17v_1 + 7v_2) = 17(1)^t v_1 + 7(-1/2)^t v_2 = \begin{bmatrix} 17 + 7(-1/2)^t \\ 17 - 7(-1/2)^{t-1} \end{bmatrix}$.

d) Since $\lim_{t\to\infty}(-1/2)^t = 0$, $\lim_{t\to\infty} = \begin{bmatrix} 17 \\ 17 \end{bmatrix}$.

e) The sequence a_t oscillates around 17 converging to 17 in the limit. The phase portrait consists of the straight lines parallel to the vector v_2, because the v_1-coordinate of the vector $x(t)$ does not change with time!