Practice Exercises on Differential Equations

What follows are some exerices to help with your studying for the part of the final exam on differential equations. In this regard, keep in mind that the exercises below are not necessarily examples of those that you will see on the final exam. Even so, if you understand how to do these, you should do fine on the differential equation portion of the final. The answers are provided at the end.

Exercises:

1. Find the Fourier series of the function on $[-\pi, \pi]$ that equals x where $x \geq 0$ and zero where $x < 0$.

2. Find the Fourier series of the function on $[-\pi, \pi]$ given by $x \to |\sin(x)|$.

3. Let f(x) denote a function on $[-\pi, \pi]$ with the property that $f(x) = f(-x)$ for all x. Explain why there are no sine functions in the Fourier series of f.

4. By its very definition, the Fourier series of a smooth function $x \to f(x)$ on $[-\pi, \pi]$ has the form $f(x) = a_0 + \sum_{k=1,2,\ldots} (a_k \cos(kx) + b_k \sin(kx))$. When computing the Fourier series of the derivative, $f'(x)$, there is the inevitable temptation to exchange orders of differentiation and summation and so conclude that $f'(x)$ has the Fourier series $\sum_{k=1,2,\ldots} (k\, b_k \cos(kx) - k\, a_k \sin(kx))$. Show that this is the correct answer when $f(\pi) = f(-\pi)$ by computing the relevant integrals.

5. Find a basis for the kernel of the linear operator $f \to f'' + 3f' - 4f$ on the space of smooth functions on [0, 1]. Find an element in the kernel of this operator that obeys $f(0) = 1$ and $f(1) = -1$.

6. Find a basis for the kernel of the linear operator $f \to f'' + 4f$ on the space of smooth functions on $[-\pi, \pi]$. Find an element in the kernel of this operator that obeys $f(0) = 1$ and $f'(0) = -1$.

7. An inner product on the space of continuous functions on $[-\pi, \pi]$ is defined as in the Differential Equation Handout using the rule that has the inner product between functions f and g equal to $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$. Exhibit a non-zero function, g, and an infinite, orthonormal set such that g is orthogonal to each element in this set.

8. Let A denote the linear operator $f \to f'$ and let B denote the linear operator $f \to x\, f$,

where f here is any smooth function on $(-\infty, \infty)$. What is the operator $AB - BA$?

9. A ubiquitous operator in quantum mechanics sends a function, f, on $(-\infty, \infty)$ to the function $T(f) = -f'' + x^2 f - f$.
   a) Suppose that f is a function that obeys $f' + x f = 0$. Prove that $T(f) = 0$.
   b) Write down all functions f that obey $f' + x f = 0$.

10. Reintroduce the inner product from Problem 7 on the space of functions in $[-\pi, \pi]$. Let f be any function on $[-\pi, \pi]$ that vanishes at the endpoints. Write the orthogonal projection of f onto the span of $\{1, x\}$ as $a + bx$. Give the orthogonal projection of $f'(x)$ onto the span of $\{1, x, x^2\}$ in terms of a and b.

11. Let $x \to f(x)$ be a smooth function such that $f(1) = 2$ while $f(x) < 2$ if $x \neq 1$. Let $c > 0$ be a constant.
    a) Prove that the function $u(t, x) = f(x-ct)$ obeys the version of the wave equation given by $u_{tt} - c^2 u_{xx} = 0$.
    b) At what time $t \geq 0$ does the function $x \to u(t, x)$ have its maximum at $x = 10$?

12. Let T denote the operator that sends a smooth function, h, on $(-\infty, \infty)$ to the function $T(h) = h'' + 2h' + h$. Exhibit two different functions in the kernel of T that both equal 1 at $x = 0$.

13. Let $u(t, x)$ denote a solution to the heat equation $u_t = u_{xx}$ on $[-\pi, \pi]$ whose x-derivative is zero at both $x = \pi$ and $x = -\pi$ for all $t \geq 0$.
    a) Explain why the function $t \to \int_{-\pi}^{\pi} u(t,x)\,dx$ is constant.
    b) Explain why the function $t \to \int_{-\pi}^{\pi} u(t, x)^2\,dx$ is non-increasing as a function of t.

    In your explanations to both a) and b), you don't have to justify exchanging orders of differentiation and integration.

14. Let f and g denote any two continuous functions on $[-\pi, \pi]$ and let $\langle f, g\rangle$ denote their inner product, $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$.
    a) Let $\varepsilon > 0$ be any given number. Use the inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$ on the integrand for suitable a and b to prove that $|\langle f, g\rangle| \leq \frac{1}{2}\varepsilon \langle f, f\rangle + \frac{1}{2}\varepsilon^{-1} \langle g, g\rangle$.
    b) Use the preceding inequality with $\varepsilon = \langle g, g\rangle^{1/2} \langle f, f\rangle^{-1/2}$ to prove that $|\langle f, g\rangle|$ is never greater than $\langle f, f\rangle^{1/2} \langle g, g\rangle^{1/2}$.

15. Let T denote the operator that sends a smooth function h on $(-\infty, \infty)$ to the function $T(h) = h'''' - 5h'' + 4h$.

a) Exhibit a basis for the kernel of T.
b) Give the dimension of the subspace of functions in kernel(T) that are equal to 0 at $x = 1$.

16. Let T denote the operator that sends a smooth function h on $[0, \infty)$ to the function $T(h) = h' + h$. Suppose that g is a smooth function on $(-\infty, \infty)$. Denote by h the function $x \to e^{-x} \int_0^x e^s g(s) ds$.
    a) Prove that $T(h) = g$.
    b) What is the range of T?
    c) What is the dimension of the kernel of T?

17. Let n be any positive integer and let $\{f_1, \ldots, f_n\}$ be any finite set of continuous functions on $[-\pi, \pi]$. Explain why there is a non-zero element in the span of the set $\{\sin(x), \sin(2x), \ldots, \sin((n+1)x)\}$ that is orthogonal to each f_k.

18. Exhibit an infinite dimensional subspace in the space of continuous functions on the interval $[-\pi, \pi]$ whose complement, as defined by the inner product in Problems 7 and 14, is also infinite dimensional.

19. Write down a solution, $u(x, y)$, to the equation $u_{xx} + u_{yy} = 0$ where both $x \in [-\pi, \pi]$ and $y \in [-\pi, \pi]$ that obeys the following conditions on the boundary: First, $u = 0$ where $y = \pm\pi$ and also where $x = -\pi$. Meanwhile $u(\pi, y) = \sin(y) + 3 \sin(2y)$.

20. Give a solution, $u(x, y)$, to the equation $u_{xx} + u_{yy} = 0$ where both $x \in [-\pi, \pi]$ and $y \in [-\pi, \pi]$ that obeys the following boundary conditions: First, $u = 0$ where $x = \pi$ and $y = -\pi$. Meanwhile $u(x, -\pi) = \sin x + 2 \sin(3x)$ and $u(x, \pi) = \sin(x) - \sin(3x)$.

21. Let k denote a positive integer. Find all solutions to the Laplace equation, $u_{xx} + u_{yy} = 0$, that have the form $u(x, y) = \cos(kx)\, h(y)$.

22. Exhibit three functions on $(-\infty, \infty)$ with the following properties: The first is a linear combination of the second and third, the first is never zero, the second is zero only at the origin, and the third is zero only at $x = 1$.

23. Suppose that f is never zero on $[-\pi, \pi]$. Prove that every function on $[-\pi, \pi]$ that is orthogonal to f with respect to the inner product in Exercise 14 must be zero at some point in $x \in (-\pi, \pi)$.

24. Explain why there is no function of the form $u(x, y) = x^2\, h(y)$ that solves the Laplace equation $u_{xx} + u_{yy} = 0$ except in the case that h is identically zero.

25. Find a solution to the heat equation $u_t = u_{xx}$ for $t \geq 0$ and $x \in [-\pi, \pi]$ that is zero at both $x = \pi$ and $x = -\pi$ and equals $\sin(x)(1 + \cos(x))$ at $t = 0$.

26. Find a function, $u(t, x)$, that obeys $u_t = 4u_{xx}$ for $t \geq 0$ and $x \in [-\pi, \pi]$ that obeys the initial conditions $u(0, x) = \sin(2x) + \sin(3x) - \sin(4x)$ and vanishes at $x = -\pi$ and $x = \pi$ at all $t \geq 0$.

27. Find the form for the general solution of the equation $f''' + 2f'' - 3f' = 0$. Then write down a function that obeys this equation with $f(0) = 1$, $f'(0) = 2$ and $f''(0) = 3$.

28. Find a function $u(x, y)$ that obeys $u_{xx} + u_{yy} = 0$ where $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$ plus the boundary conditions $u(x, 0) = u(x, \pi) = \sin(x) + 2\sin(2x)$ and $u(0, y) = u(\pi, y) = 0$.

29. Find a solution to the heat equation $u_t = 4u_{xx}$ on the interval $[-\pi, \pi]$ that obeys the initial condition $u(0, x) = 2 \sin(2x) - 3\sin(3x) + 4\sin(4x)$ and obeys for all $t \geq 0$ the boundary conditions $u(t, \pi) = u(t, -\pi) = 0$.

30. Give a basis for the vector space of function that obey $f''' - 3f'' = 0$ and $f(0) = f(1)$.

31. Give two distinct solutions to the equation $f''' - 2f'' + 2f' = 0$ that obey both $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 1$.

32. Written below are four infinite sequence of the form $\{a_1, a_2, \ldots, a_n, \ldots\}$ of real numbers. Three of these sequences have the property that the

$$a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx) + \cdots$$

is the Fourier series of a continuous function on the interval $[-\pi, \pi]$. Meanwhile, there is one sequence that does not have this property. Find the latter sequence.

a) $\{1, \frac{1}{2}, \frac{1}{4}, \ldots, (\frac{1}{2})^n, \ldots\}$.

b) $\{1, 1, \frac{2}{3!}, \frac{2}{4!}, \ldots, (\frac{2}{n!}), \ldots\}$

c) $\{1, -1, 1, -1, \ldots, (-1)^n, \ldots\}$

d) $\{1, -\frac{1}{2}, \frac{1}{4}, \ldots, (-\frac{1}{2})^n, \ldots\}$

33. Write down continuous functions $f(x)$ and $g(x)$ that are defined where $-\pi \leq x \leq \pi$ and have all of the following properties:

a) $f(x) = g(x)$ where $x \leq 0$.

b) $f(0) = g(0) = 1$.

c) The inner product of f and g, $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$, is zero.

34. Give the Fourier series of the function $\sin(\frac{1}{2}x)$ for $-\pi \le x \le \pi$.

35. Define a linear map from the space of functions on $[-\pi, \pi]$ to $\mathbb{R}$ by sending any given function f to $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2x)\,dx$.
   a) Prove that the range of this map is the whole of $\mathbb{R}$.
   b) Exhibit an orthonormal basis for the kernel of this map.

Answers:

1. $\frac{\pi}{4} - \sum_{k=1,3,\ldots} \frac{2}{\pi k^2}\cos(kx) - \sum_{k=1,2,\ldots} (-1)^k \frac{1}{k}\sin(kx)$.

2. $\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=2,4,\ldots} \frac{1}{k^2-1}\cos(kx)$.

3. The coefficient in front of sin(kx) is $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$ and this is zero since the contribution from the $x \ge 0$ part of the integral is minus that from the $x \le 0$ part.

4. The coefficient in front of sin(kx) is $\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(kx)\,dx$ and and so an integration by parts finds this equal to $-\frac{1}{\pi}k\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$ which is $-ka_k$. On the otherhand, the coefficient for cos(kx) is $\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(kx)\,dx$ and an integration by parts finds that this is $\frac{1}{\pi}k\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx + \frac{1}{\pi}(-1)^k (f(\pi) - f(-\pi))$. This equals kb_k given that $f(\pi) = f(-\pi)$. The integral for the constant term is $\frac{1}{\pi\sqrt{2}}(f(\pi) - f(-\pi))$ which is zero.

5. A basis is $\{e^{-4x}, e^x\}$ and the desired element is $f(x) = \frac{1+e}{e-e^{-4}}e^{-4x} - \frac{1+e^{-4}}{e-e^{-4}}e^x$.

6. A basis is {cos(2x), sin(2x)} and the desired element is $f(x) = \cos(2x) - \frac{1}{2}\sin(2x)$.

7. Take the function 1 and the set {sin(x), sin(2x), …, }.

8. This is the identity operator, it acts to send $f \to f$ for any f.

9. a) $T(f) = -(f' + xf)' + x(f' + xf)$.
   b) All such functions are multiples of $e^{-x^2/2}$.

10. The orthogonal projection in this case is $-a\,x - 2b\,x^2$.

11. a) Use the Chain Rule to deduce that $u_t = -cf'|_{x-ct}$ and $u_{tt} = c^2 f''|_{x-ct}$. Meanwhile, $u_x = f'|_{x-ct}$ and $u_{xx} = f''|_{x-ct}$.
    b) This occurs at t = 9/c.

12. One is e^{-x} and another is $(1+x)\, e^{-x}$.

13. a) The time derivative of the indicated integral is $\int_{-\pi}^{\pi} u_t(t,x)\, dx$. Since the heat equation is obeyed, this is $\int_{-\pi}^{\pi} u_{xx}\, dx$. Integrating by parts finds the latter equal to $u_x(t, \pi) - u_x(t, -\pi)$ which is assumed to be zero.
    b) The time derivative of the indicated integral in this case is $2\int_{-\pi}^{\pi} u\, u_t\, dx$. Use the heat equation to write this as $2\int_{-\pi}^{\pi} u\, u_{xx}\, dx$. Then integrate by parts to equate the latter integral with $-2\int_{-\pi}^{\pi} u_x^2\, dx + 2\,(u\, u_x)|_{(t,\pi)} - 2\,(u\, u_x)|_{(t,-\pi)}$. Since this last part is zero and the first part is non-positive, the function of t here can not be increasing.

14. a) Write $a = \varepsilon^{1/2} f(x)$ and $b = \varepsilon^{-1/2} g(x)$ to see that $|f(x)g(x)| \le \frac{1}{2}\varepsilon f^2(x) + \frac{1}{2}\varepsilon^{-1} g^2(x)$. Do this for each value of x to see that $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\, dx$ has absolute value that is no greater than $\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\varepsilon f^2(x)\, dx + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\varepsilon^{-1} g^2(x)\, dx$. This is the desired conclusion.
    b) Taking this value for ε makes $\frac{1}{2}\varepsilon\, \langle f, f\rangle = \frac{1}{2}\varepsilon^{-1}\langle g, g\rangle = \langle f, f\rangle^{1/2}\langle g, g\rangle^{1/2}$.

15. a) The corresponding polynomial is $r^4 - 5r^2 + 4 = (r^2 - 1)(r^2 - 4)$. Thus, the roots are $r = \pm 1$ and $r = \pm 2$. This implies that kernel(T) = Span$\{e^{-x}, e^{x}, e^{-2x}, e^{2x}\}$.
    b) This is a 3-dimensional vector subspace spanned by $\{e^{-x} - e^{-2}e^{x}, e^{-2x} - e^{-3}e^{x}, e^{2x} - ee^{x}\}$.

16. a) Differentiating finds $h' = -e^{-x}\int_0^x e^s g(s)\, ds + e^{-x}(e^x g(s))|_{s=x} = -h + g$.
    b) Let g be any smooth function. Since $h = e^{-x}\int_0^x e^s g(s)\, ds$ obeys T(h) = g, so g is in the range of T. Thus, all smooth functions are in T's range.
    c) The kernel of T is one dimensional, spanned by e^{-x}.

17. The orthogonal projection from the span of {sin(x), sin(2x), …, sin((n+1)x)} to the span of $\{f_1, \ldots, f_n\}$ is a linear map from an (n+1) dimensional space to an n dimensional one, so it must have a positive dimensional kernel. Any function in this kernel is orthogonal to all f_k.

18. {sin(x), sin(2x), …} is orthogonal to $\{\sqrt{\frac{1}{2\pi}}$, cos(x), cos(2x), …}.

19. $u(x, y) = \sinh(2\pi)^{-1} \sinh(x+\pi) \sin(x) + 3 \sinh(6\pi)^{-1} \sinh(3x + 3\pi) \sin(3y)$

20. $u(x, y) = \cosh(\pi)^{-1} \cosh(x) \sin(x)$
$- [\sinh(6\pi)^{-1} \sinh(3x-3\pi) + \sinh(3\pi)^{-1} \sinh(3x)] \sin(3x)$

21. These all have the form $\cos(kx)(a\, e^{ky} + b\, e^{-ky})$ where a and b are constants.

22. $f_1 = 1$, $f_2 = x$ and $f_3 = 1-x$.

23. The function f is either positive or negative but never zero. If g is orthogonal to f, then the product fg must change signs along $[-\pi, \pi]$ and so g must be zero somewhere between $-\pi$ and π.

24. Differentiating u finds that h must obey $h + x^2 h_{yy}$ at all y. Plug in $x = 0$ to see that $h(y) = 0$ for all y.

25. Since $\sin(x)(1 + \cos(x)) = \sin(x) + \frac{1}{2}\sin(2x)$, take $u(t, x) = e^{-t} \sin(x) + \frac{1}{2} e^{-4t} \sin(2x)$.

26. $u(t, x) = \cos(2t) \sin(2x) + \frac{1}{4}\sin(4t) \sin(4x) + \frac{1}{3}\sin(3t) \cos(3x)$.

27. The general form is $f(x) = a + b\, e^x + c\, e^{-3x}$. Of these, $f(x) = -\frac{4}{3} + \frac{9}{4} e^x + \frac{1}{12} e^{-3x}$ obeys the stated conditions.

28. $u(x, y) = \left(\frac{1 - e^{-\pi}}{e^{\pi} - e^{-\pi}} e^y + \frac{e^{\pi} - 1}{e^{\pi} - e^{-\pi}} e^{-y}\right) \sin(x) + 2 \left(\frac{1 - e^{-2\pi}}{e^{2\pi} - e^{-2\pi}} e^{2y} + \frac{e^{2\pi} - 1}{e^{2\pi} - e^{-2\pi}} e^{-2y}\right) \sin(2x)$.

29. $u(t, x) = e^{-16t}\, 2\sin(2x) - e^{-12t}\, 3\sin(3x) + e^{64t}\, 4\sin(4x)$.

30. A basis is $\{1, e^{3x} + (1 - e^3)x\}$.

31. Any function of the form $1 - e^{-x} \cos(x) + c\, e^{-x} \sin(x)$ for $c \in \mathbb{R}$ has the desired properties. Choose any two distinct values for the constant c.

32. If $\{a_1, a_2, \dots\}$ are the coefficients from the sine terms in a Fourier series of some continuous function, then the sum of their squares, $a_1^2 + a_2^2 + \cdots$, must be finite since it is equal to the integral of the square of the function. This means, in particular, that the sequence must converge to zero. Such is not the case for the sequence in c).

33. There are infinitely many such pairs. Here is one: Take $f(x) = 1$. Meanwhile, take $g(x)$ to equal 1 where $x \le 0$ and to equal $1-\frac{4}{\pi}x$ where $x \ge 1$.

34. $\sin(\frac{1}{2}x) = -\frac{16}{\pi} \sum_{k=1,2,\ldots} (-1)^k \frac{k}{4k^2-1} \sin(kx)$

35. a) If $r \in \mathbb{R}$, then the map sends the function $r \sin(2x)$ maps to r.

b) $\{1, \cos(x), \cos(2x), \cos(3x), \ldots, \sin(x), \sin(3x), \sin(4x), \ldots\}$.