01/14/2006 FINAL EXAMINATION	**Math 21b**

Name:

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- Start by printing your name in the above box and check the section in which you are.
- Try to answer each question on the same page as the question is asked. If needed, use the back or next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit. Justify your answers.
- No notes, books, calculators, computers or other electronic aids are allowed.
- You have 180 minutes (3 hours) time to complete your work.

1		20
2		10
3		10
4		10
5		20
6		18
7		25
8		22
9		15
Total:		150

Problem (1) TF questions (20 points) Circle the correct letter and provide a brief justification. (Each problem gives 1 point for a correct answer and 1 point for a correct justification.)

T F (a) Any basis can be made into an orthonormal basis.

T F (b) $f'(x) + 2f''(x) + f'(x)f''(x) = f'(x)f''(x)$ is a linear equation.

T F (c) If A is a 2×2 matrix with eigenvalues 0 and 1, then A is the matrix of an orthogonal projection.

T F (d) If A and B are $n \times n$ diagonalizable matrices with the same eigenvectors, then AB is diagonalizable.

T	F

(e)If λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB.

T	F

(f) If 0 is an eigenvalue of the $n \times n$ matrix A, then the geometric multiplicity of 0 is $n - rank(A)$.

T	F

(g) If $f(x) = -f(-x)$ on $[-\pi, \pi]$, then the Fourier series of f has no constant term and no cosine terms.

T	F

(h) Since the heat equation $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$ is a second degree equation, it is sufficient to have an initial condition and one boundary condition (at one end of the rod) for it to have a unique solution.

T	F

(i) If $\frac{dx}{dt} = xy - sin(y)$ and $\frac{dy}{dt} = xy + y^2$, then the point $(x, y) = (1, 0)$ is a stable equilibrium point.

T	F

(j) If there are no lines through the origin of a phase diagram (or phase portrait) of $\frac{d\overrightarrow{x}}{dt} = A\overrightarrow{x}$, where A is a 2×2 matrix, it means that A has no real eigenvalues.

Space for work

Problem (2) (10 points)

Let A be any n by n matrix. If A has n distinct eigenvalues, use geometric multiplicity to show that there must be an eigenbasis for A.

Space for work

Problem (3) (10 points)

Solve the initial value problem

$$f'''(x) - 3f''(x) + 3f'(x) - f(x) = 0$$

with $f'(0) = f(0) = 1$ and $f''(0) = 5$.

Space for work

Problem (4) (10 points)

Consider the case where $\frac{d\overrightarrow{x}}{dt} = A\overrightarrow{x}$, A is 3x3, and $(-1+i)$ is an eigenvalue of A. If $det(A) = 1$, is the zero state necessarily asymptotically stable? Be sure to justify your answer.

Space for work

Problem (5) (20 points)

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(a) (10 points) Diagonalize A by means of an orthogonal matrix S.

(b) (10 points) Using your work in (a), diagonalize the following matrix: $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ (Hint: Try to see whether the matrix $A - cI_3$ for some well-chosen real number c might help...)

Space for work

Problem (6) (18 points)

Match the following six dynamical systems to six of the nine phase portraits to which they correspond. Justify your answers. You will receive 1 point for each correct match and 2 additional points for a correct justification.

______ $\vec{x}(t+1) = \begin{pmatrix} 1/2 & 0 \\ -3/2 & 2 \end{pmatrix} \vec{x}(t)$

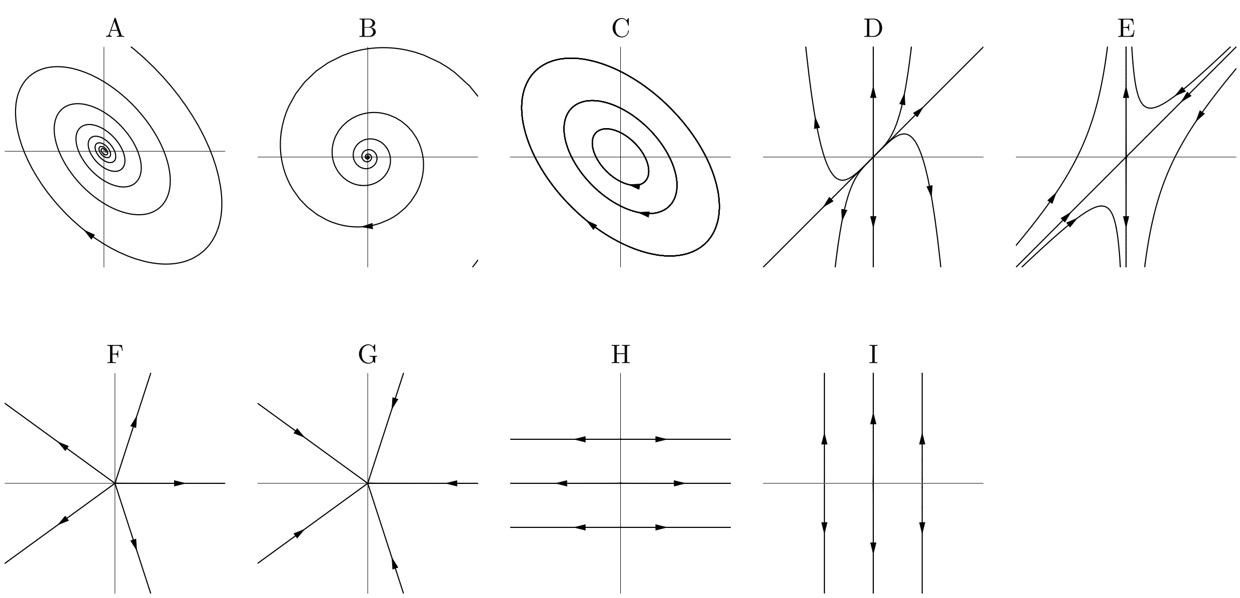
______ $\frac{d\vec{x}}{dt} = \begin{pmatrix} 1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \vec{x}$

______ $\vec{x}(t+1) = \begin{pmatrix} 1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \vec{x}(t)$

______ $\vec{x}(t+1) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \vec{x}(t)$

______ $\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x}$

______ $\vec{x}(t+1) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}(t)$



Space for work

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Problem (7) (20 points)

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and consider the continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$.

(a) (10 points) If $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, find a formula for $\vec{x}(t)$.

(b) (5 points) Draw a phase portrait of the continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$.

(c) (5 points) Is $\vec{0}$ an asymptotically stable equilibrium of the system?

(d) (5 points) Now consider the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with the same initial condition as in (a), namely $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Write down a formula for $\vec{x}(t)$.

Space for work

Problem (8) (22 points)

Consider $P_2(\mathbb{R})$, the linear space of all polynomials with real coefficients, of degree less than or equal to two.

(a) (5 points) Show that 1, x, and x^2 are linearly independent and form a basis for $P_2(\mathbb{R})$.

(b) (7 points) Define the inner product $< p, q >= \int_0^1 p(x)q(x)dx$, where p and q are elements of $P_2(\mathbb{R})$. Make $\{1, x, x^2\}$ into an orthonormal basis of $P_2(\mathbb{R})$ via the Gram-Schmidt process with respect to the given inner product.

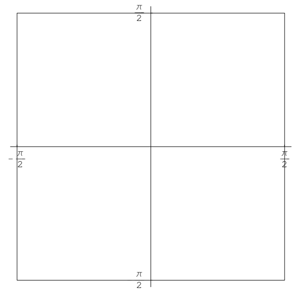
(c) (5 points) Find $proj_{P_2(\mathbb{R})}(x^2 - 2x)$ with respect to the inner product from (b). (Hint: No computation might be necessary.)

(d) (5 points) Apply the theory of least squares with respect to the orthonormal basis you found in (b) to give the "best" quadratic approximation of x^3.

Space for work

Problem (9) (15 points)

Let Ω be the square given by $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



Consider the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y) = 0.$$

Let $k \geq 1$ be an integer.

(a) (5 points) Solve the Laplace equation for a function of the form $u(x, y) = c(y)\, cos(kx)$ on Ω. (Hint: Think about which differential equation $c(y)$ has to satisfy.)

(b) (5 points) If $u(x, y) = c(y)\, cos(kx)$ assumes the following values on the boundary of Ω

$$u(x, y) = f(x) = \begin{cases} cos(kx) & on \quad y = \frac{\pi}{2} & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ cos(kx) & on \quad y = -\frac{\pi}{2} & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & on \quad x = \frac{\pi}{2} & -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\ 0 & on \quad x = -\frac{\pi}{2} & -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \end{cases}$$

describe the values of k for which this particular Laplace equation makes sense on Ω.

(c) (5 points) Is there some $p(x) = \sum_{k=1}^{\infty} a_k\, cos(kx)$ you can write down such that the Laplace equation with values on the boundary of Ω

$$u(x, y) = f(x) = \begin{cases} p(x) & on \quad y = \frac{\pi}{2} & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ p(x) & on \quad y = -\frac{\pi}{2} & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 3 & on \quad x = \frac{\pi}{2} & -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\ -3 & on \quad x = -\frac{\pi}{2} & -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \end{cases}$$

can be solved for a function of the form $u(x, y) = c(y)\, p(x)$? Justify your answer.

Space for work

Space for work