Chess and Mathematics: Knights of the Square Table

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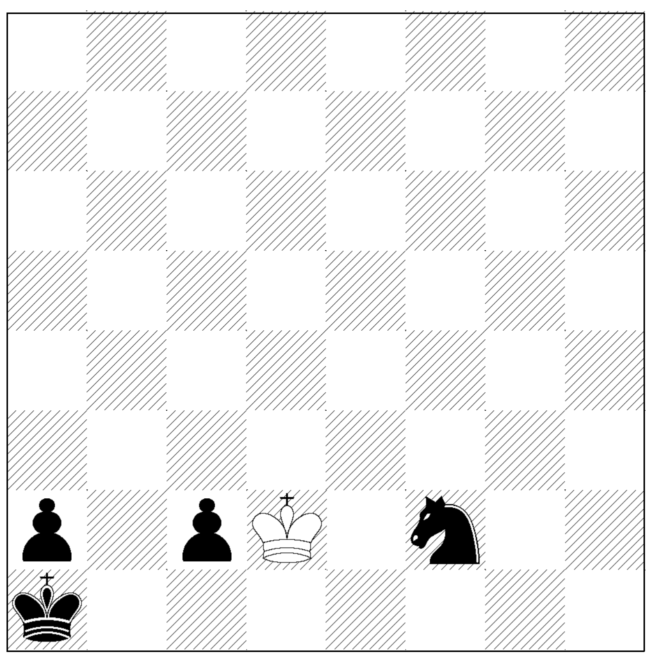
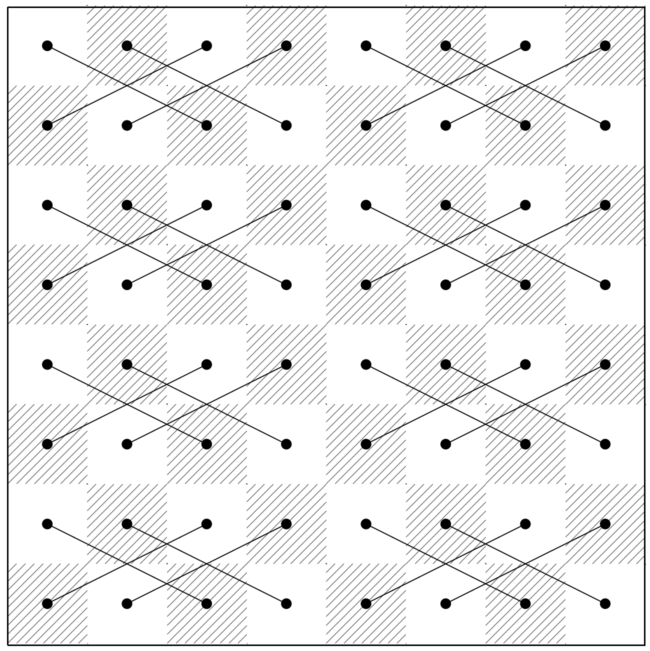


White to move

32 mutually nonattacking knights

I. Newman (1963) asked: Can one do better?

Answered by R. Patenaude 1964:

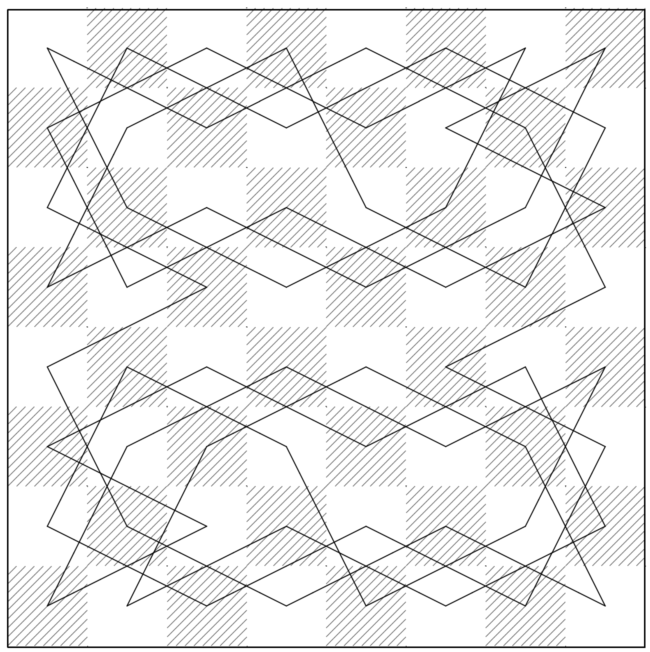


Each pair of squares can hold at most one knight.

Newman also asked: What are all the solutions with 32 knights?

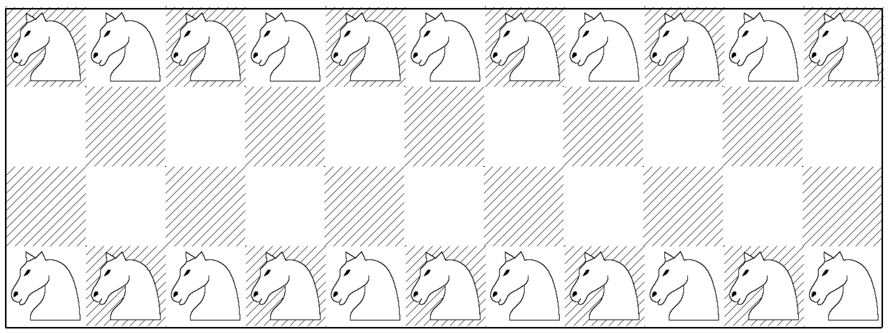
Proved by R. Greenberg, also in 1964:

A symmetrical closed Knight's tour constructed by Euler:



But why do only 8×8? Let us (try to) generalize!

There's a third maximal configuration on a 4×11 (or $4\times$anything) board:



Therefore:

Theorem. There is no closed knight's tour on a $4 \times n$ board for any n. [Attributed to L.Posa]

It is known that there is a closed knight's tour on an $m \times n$ board if and only if ("iff"):

- Neither m nor n is 1, 2, or 4;

- The board area mn is even (why?);

- If $m = 3$ then n is at least 10, and vice versa [equivalently, $mn \geq 30$].

Suppose now that the board allows for closed knight's tours. How many are there?

This problem is too hard! Only partial results are known:

i) For 8×8 (and smaller) boards, the answer has been computed. For example, 9862 on the 6×6, and (according to B.McKay, 1997) the 8×8 chessboard has 13267364410532 closed knight's tours.

ii) If we specify the length of one side, say m, then there is a formula that works for all n. But it's really complicated (except of course when m is one of 1, 2, or 4). The only known case is $m = 3$, and even there the formula [obtained by Knuth and (independently) NDE in 1994] is a ghastly mess:

The number of closed knight's tours on the $m \times (2n)$ board is the X^n coefficient of the "generating function"

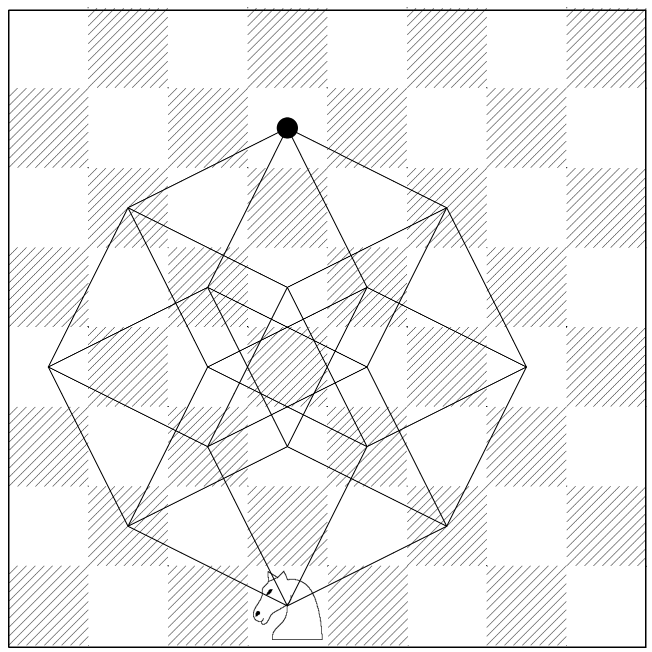
$$16(X^5 + 5X^6 - 34X^7 - 116X^8 + 505X^9 + 616X^{10} - 3179X^{11} - 4X^{12} + 9536X^{13} - 8176X^{14} - 13392X^{15} + 15360X^{16} + 13888X^{17} + 2784X^{18} - 3328X^{19} - 22016X^{20} + 5120X^{21} + 2048X^{22})$$

$$\div$$

$$(1 - 6X - 64X^2 + 200X^3 + 1000X^4 - 3016X^5 - 3488X^6 + 24256X^7 - 23776X^8 - 104168X^9 + 203408X^{10} + 184704X^{11} - 443392X^{12} - 14336X^{13} + 151296X^{14} - 145920X^{15} + 263424X^{16} - 317440X^{17} - 36864X^{18} + 966656X^{19} - 573440X^{20} - 131072X^{21})$$

!

The 4! shortest knight paths from d1 to d7



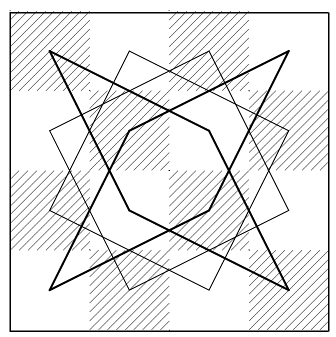
[NB $4! = 1 \times 2 \cdot 3 \cdot 4 = 24$]

Back to the standard 8×8 board...

i) Suppose we allow each knight to be defended at most once. How many more knights can the board then accommodate?

ii) Now suppose we require each knight to be defended *exactly* once. What is the largest number of knights on the 8×8 board satisfying this constraint, and what are all the maximal configurations?

i) Still only 32 knights! Proof: Each 4×4 quarter-board can accommodate at most 8, because...



ii) Again 32 knights — and once more, in an essentially unique configuration!

Can you find it?