Chapter Two

FUNCTIONS

Contents

2.1 INPUT AND OUTPUT

Finding Output Values: Evaluating a Function

Evaluating a function means calculating the value of a function's output from a particular value of the input.

In the housepainting example on page **??**, the notation $n = f(A)$ indicates that the number of gallons of paint n is a function of area A. The expression $f(A)$ represents the output of the function—specifically, the amount of paint required to cover an area of A ft^2. For example, $f(20{,}000)$ represents the number of gallons of paint required to cover an area of 20,000 ft^2.

Example 1 Using the fact that 1 gallon of paint covers 250 ft^2, evaluate the expression $f(20{,}000)$.

Solution To evaluate $f(20{,}000)$, calculate the number of gallons required to cover 20,000 ft^2:

$$f(20{,}000) = \frac{20{,}000 \text{ ft}^2}{250 \text{ ft}^2/\text{gallon}} = 80 \text{ gallons of paint.}$$

Evaluating a Function Using a Formula

If we have a formula for a function, we evaluate it by substituting the input value into the formula.

Example 2 The formula for the area of a circle of radius r is $A = q(r) = \pi r^2$. Use the formula to evaluate $q(10)$ and $q(20)$. What do your results tell you about circles?

Solution In the expression $q(10)$, the value of r is 10, so

$$q(10) = \pi \cdot 10^2 = 100\pi \approx 314.$$

Similarly, substituting $r = 20$, we have

$$q(20) = \pi \cdot 20^2 = 400\pi \approx 1257.$$

The statements $q(10) \approx 314$ and $q(20) \approx 1257$ tell us that a circle of radius 10 cm has an area of approximately 314 cm^2 and a circle of radius 20 cm has an area of approximately 1257 cm^2.

Example 3 Let $g(x) = \dfrac{x^2+1}{5+x}$. Evaluate the following expressions.

(a) $g(3)$ (b) $g(-1)$ (c) $g(a)$

Solution (a) To evaluate $g(3)$, replace every x in the formula with 3:

$$g(3) = \frac{3^2+1}{5+3} = \frac{10}{8} = 1.25.$$

(b) To evaluate $g(-1)$, replace every x in the formula with (-1):

$$g(-1) = \frac{(-1)^2+1}{5+(-1)} = \frac{2}{4} = 0.5.$$

(c) To evaluate $g(a)$, replace every x in the formula with a:

$$g(a) = \frac{a^2+1}{5+a}.$$

Evaluating a function may involve algebraic simplification, as the following example shows.

Example 4 Let $h(x) = x^2 - 3x + 5$. Evaluate and simplify the following expressions.

(a) $h(2)$ (b) $h(a-2)$ (c) $h(a) - 2$ (d) $h(a) - h(2)$

Solution Notice that x is the input and $h(x)$ is the output. It is helpful to rewrite the formula as

$$\text{Output} = h(\text{Input}) = (\text{Input})^2 - 3 \cdot (\text{Input}) + 5.$$

(a) For $h(2)$, we have Input $= 2$, so

$$h(2) = (2)^2 - 3 \cdot (2) + 5 = 3.$$

(b) In this case, Input $= a - 2$. We substitute and multiply out

$$\begin{aligned} h(a-2) &= (a-2)^2 - 3(a-2) + 5 \\ &= a^2 - 4a + 4 - 3a + 6 + 5 \\ &= a^2 - 7a + 15. \end{aligned}$$

(c) First input a, then subtract 2:

$$\begin{aligned} h(a) - 2 &= a^2 - 3a + 5 - 2 \\ &= a^2 - 3a + 3. \end{aligned}$$

(d) Since we found $h(2) = 3$ in part (a), we subtract from $h(a)$:

$$\begin{aligned} h(a) - h(2) &= a^2 - 3a + 5 - 3 \\ &= a^2 - 3a + 2. \end{aligned}$$

Finding Input Values: Solving Equations

Given an input, we evaluate the function to find the output. Sometimes the situation is reversed: we know the output and we want to find a corresponding input. If the function is given by a formula, the input values are solutions to an equation.

Example 5 Use the cricket function $T = \frac{1}{4}R + 40$, introduced on page **??**, to find the rate, R, at which the snowy tree cricket chirps when the temperature, T, is 76°F.

Solution We want to find R when $T = 76$. Substitute $T = 76$ into the formula and solve the equation

$$\begin{aligned} 76 &= \frac{1}{4}R + 40 \\ 36 &= \frac{1}{4}R && \text{subtract 40 from both sides} \\ 144 &= R. && \text{multiply both sides by 4} \end{aligned}$$

The cricket chirps at a rate of 144 chirps per minute when the temperature is 76°F.

Example 6 Suppose $f(x) = \dfrac{1}{\sqrt{x-4}}$.

(a) Find an x-value that results in $f(x) = 2$.
(b) Is there an x-value that results in $f(x) = -2$?

Solution (a) To find an x-value that results in $f(x) = 2$, solve the equation

$$2 = \frac{1}{\sqrt{x-4}}.$$

Square both sides:

$$4 = \frac{1}{x-4}.$$

Now multiply by $(x-4)$:

$$\begin{aligned} 4(x-4) &= 1 \\ 4x-16 &= 1 \\ x &= \frac{17}{4} = 4.25. \end{aligned}$$

The x-value is 4.25. (Note that the simplification $(x-4)/(x-4) = 1$ in the second step was valid because $x - 4 \neq 0$.)

(b) Since $\sqrt{x-4}$ is nonnegative if it is defined, its reciprocal, $f(x) = \dfrac{1}{\sqrt{x-4}}$ is also nonnegative if it is defined. Thus, $f(x)$ is not negative for any x input, so there is no x-value that results in $f(x) = -2$.

In the next example, we solve an equation for a quantity that is being used to model a physical quantity; we must choose the solutions that make sense in the context of the model.

Example 7 Let $A = q(r)$ be the area of a circle of radius r, where r is in cm. What is the radius of a circle whose area is 100 cm^2?

Solution The output $q(r)$ is an area. Solving the equation $q(r) = 100$ for r gives the radius of a circle whose area is 100 cm^2. Since the formula for the area of a circle is $q(r) = \pi r^2$, we solve

$$\begin{aligned} q(r) = \pi r^2 &= 100 \\ r^2 &= \frac{100}{\pi} \\ r &= \pm\sqrt{\frac{100}{\pi}} = \pm 5.642. \end{aligned}$$

We have two solutions for r, one positive and one negative. Since a circle cannot have a negative radius, we take $r = 5.642$ cm. A circle of area 100 cm^2 has a radius of 5.642 cm.

Finding Output and Input Values From Tables and Graphs

The following two examples use function notation with a table and a graph respectively.

Example 8 Table 2.1 shows the revenue, $R = f(t)$, received by the National Football League,[1] from network TV as a function of the year, t, since 1975. Contracts are signed with all networks at once for periods ranging from 2 to 8 years. The table shows the annual revenue under the contract holding that year.

(a) Evaluate and interpret $f(20)$.   (b) Solve and interpret $f(t) = 2600$.

Table 2.1

t (years)	0	5	10	15	20	25	30	35
R (m. \$)	55	162	420	900	1097	2600	2600	3735

Solution

(a) Table 2.1 shows $f(20) = 1097$. Since $t = 20$ in the year 1995, we know that NFL's revenue from TV was \$1097 million in the year 1995.

(b) Solving $f(t) = 2600$ means finding a year in which TV revenues were \$2600 million. There are two values, $t = 25$ and $t = 30$. In other words, in 2000 and 2005, NFL's TV revenues were \$2600 million. (The same contract was in effect in 2000 and 2005.)

[1] John Vrooman, "The economic structure of the NFL," in *The Economics of the National Football League*, K.G. Quinn (Ed.), Springer, 2011.

Example 9 A man drives from his home to a store and back. The entire trip takes 30 minutes. Figure 2.1 gives his velocity $v(t)$ (in mph) as a function of the time t (in minutes) since he left home. A negative velocity indicates that he is traveling away from the store back to his home.

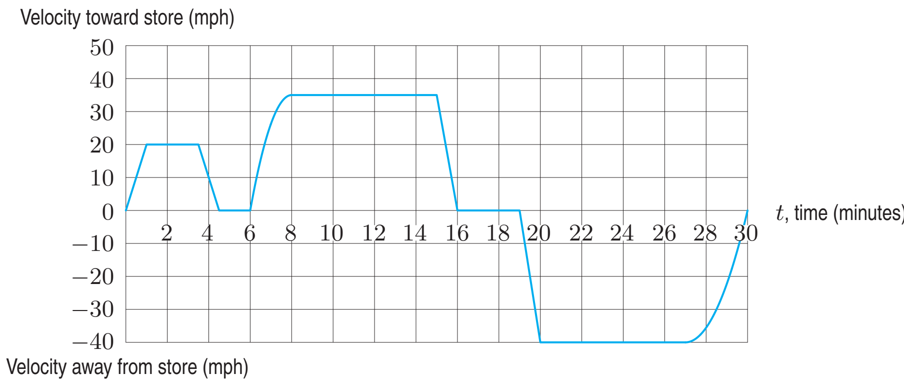


Figure 2.1: Velocity of a man on a trip to the store and back

Evaluate and interpret:

(a) $v(5)$ (b) $v(24)$ (c) $v(8) - v(6)$ (d) $v(-3)$

Solve for t and interpret:

(e) $v(t) = 15$ (f) $v(t) = -20$ (g) $v(t) = v(7)$ (h) $v(t) > 30$

Solution

(a) To evaluate $v(5)$, look on the graph where $t = 5$ minutes. Five minutes after he left home, his velocity is 0 mph. Thus, $v(5) = 0$. Perhaps he had to stop at a light.
(b) The graph shows that $v(24) = -40$ mph. After 24 minutes, he is traveling at 40 mph away from the store, back to his home.
(c) From the graph, $v(8) = 35$ mph and $v(6) = 0$ mph. Thus, $v(8) - v(6) = 35 - 0 = 35$. This shows that the man's speed increased by 35 mph in the interval between $t = 6$ minutes and $t = 8$ minutes.
(d) The quantity $v(-3)$ is not defined since the graph only gives velocities for nonnegative times.
(e) To solve for t when $v(t) = 15$, look on the graph where the velocity is 15 mph. This occurs at $t \approx 0.75$ minute, 3.75 minutes, 6.5 minutes, and 15.5 minutes. At each of these four times the man's velocity was 15 mph.
(f) To solve $v(t) = -20$ for t, we see that the velocity is -20 mph (that is, 20 mph toward home) at $t \approx 19.5$ and $t \approx 29$ minutes.
(g) First we evaluate $v(7) \approx 30$. To solve $v(t) = 30$, we look for the values of t making the velocity 30 mph. One such t is of course $t = 7$; the other t is $t \approx 15$ minutes. These are the two times when the velocity is the same as it is at 7 minutes.
(h) To solve for t when $v(t) > 30$, look on the graph where the velocity toward the store is greater than 30 mph. This occurs for t such that $7 < t < 15$. Therefore, between 7 and 15 minutes after he left home, the man was driving faster than 30 mph toward the store. Note that there are also times when the man is driving faster than 30 mph away from the store; however, these times are not solutions to the inequality $v(t) > 30$, since $v(t)$ is positive only when the man is driving toward the store.

If we are given a formula for a function, we can use the graph of the function to find input values that satisfy a certain equation or inequality.

Example 10 Let $f(x) = x^3 - 3x$. Find approximate values of the input x such that

(a) $f(x) = 1$ (b) $f(x) > 1$.

Solution (a) We can determine for what values of x the output $f(x)$ is equal to 1 by using a calculator or computer to graph the curve $y = f(x)$. See Figure 2.2.

To find solutions to $f(x) = 1$, we look for points on the curve that lie on the line $y = 1$. Using a calculator, we find that these points are $(-1.532, 1)$, $(-0.347, 1)$, and $(1.879, 1)$. By zooming out to observe the long-term behavior of the function, we see that these are the only three points on the curve for which $y = 1$. Therefore, the approximate solutions of $f(x) = 1$ are $x = -1.532$, $x = -0.347$, and $x = 1.879$.

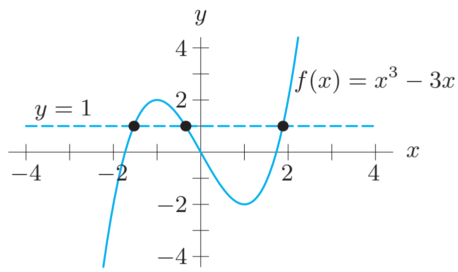


Figure 2.2: The function $f(x) = x^3 - 3x$

(b) We can solve $f(x) > 1$ by referring to Figure 2.2. We look for points on the graph where the y-coordinate is greater than 1; these are points on the graph that lie above the line $y = 1$.

We know that the graph crosses the line $y = 1$ at $x = -1.532$, $x = -0.347$, and $x = 1.879$. We see from the figure that the graph of $f(x)$ lies above the line $y = 1$ for all x such that $-1.532 < x < -0.347$ or $x > 1.879$. Therefore, the approximate solutions of $f(x) > 1$ are

$$-1.532 < x < -0.347 \quad \text{or} \quad x > 1.879.$$

Occasionally, looking at a graph helps us locate exact solutions to an equation or inequality.

Example 11 Let $f(x) = 1 + \frac{1}{2}x$ and $g(x) = 3/(1 + x^2)$. Find all values of the input x such that $f(x) \leq g(x)$.

Solution We begin by using a calculator or computer to graph the curves $y = f(x)$ and $y = g(x)$, as in Figure 2.3.

The two graphs appear to intersect at the point $(1, 3/2)$. We can confirm that this is an intersection point by computing $f(1)$ and $g(1)$:

$$f(1) = 1 + \tfrac{1}{2}(1) = \tfrac{3}{2} \quad \text{and} \quad g(1) = \frac{3}{1 + 1^2} = \tfrac{3}{2}$$

Thus the point $(1, 3/2)$ is the intersection point of the two graphs. Since the graph of $f(x)$ lies below the graph of $g(x)$ only to the left of $x = 1$, the solution to the inequality $f(x) \leq g(x)$ is $x \leq 1$.

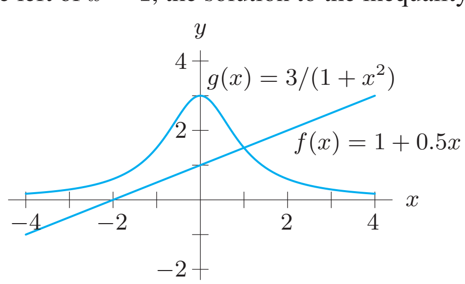


Figure 2.3: The functions $f(x) = 1 + \frac{1}{2}x$ and $g(x) = 3/(1 + x^2)$

Exercises and Problems for Section 2.1

Skill Refresher

For Exercises S1–S6, expand and simplify.

S1. $5(x-3)$

S2. $a(2a+5)$

S3. $(m-5)(4(m-5)+2)$

S4. $(x+2)(3x-8)$

S5. $3\left(1+\frac{1}{x}\right)$

S6. $3+2\left(\frac{1}{x}\right)^2-x$

Solve the equations in Exercises S7–S10.

S7. $x^2-9=0$

S8. $\sqrt{2x-1}+3=9$

S9. $\frac{21}{z-5}-\frac{13}{z^2-5z}=3$

S10. $2x^{3/2}-1=7$

Exercises

1. If $g(x)=-\frac{1}{2}x^{1/3}$, find $g(-27)$.

2. Let $f(x)=\frac{2x+1}{x+1}$. For what value of x is $f(x)=0.3$?

3. If $f(t)=t^2-4$, **(a)** Find $f(0)$ **(b)** Solve $f(t)=0$.

4. If $g(x)=x^2-5x+6$, **(a)** Find $g(0)$ **(b)** Solve $g(x)=0$.

5. If $g(t)=\frac{1}{t+2}-1$, **(a)** Find $g(0)$ **(b)** Solve $g(t)=0$.

6. If $h(x)=ax^2+bx+c$, find $h(0)$.

If $p(r)=r^2+5$, evaluate the expressions in Exercises 7–8.

7. $p(7)$

8. $p(x)+p(8)$

In Figure 2.4, mark the point(s) representing the statements in Exercises 9–12 and label their coordinates.

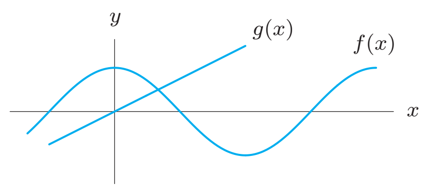


Figure 2.4

9. $f(0)=2$

10. $f(-3)=f(3)=f(9)=0$

11. $f(2)=g(2)$

12. $g(x)>f(x)$ for $x>2$

In Exercises 13–16, let $f(x)=x^2+2x-2$ and $g(x)=2x+2$. Find all x values satisfying the following.

13. $g(x)=2$

14. $g(x)<2$

15. $f(x)<x^2$

16. $f(x)>g(x)$

Problems

17. Let $F=g(t)$ be the number of foxes in a park as a function of t, the number of months since January 1. Evaluate $g(9)$ using Table **??** on page **??**. What does this tell us about the fox population?

18. Let $F=g(t)$ be the number of foxes in month t in the national park described in Example **??** on page **??**. Solve the equation $g(t)=75$. What does your solution tell you about the fox population?

19. Table 2.2 shows the base salary of members of Congress[2], $S=f(t)$, in thousands of dollars, as a function of the year, t, since 1992.

(a) Evaluate and interpret $f(12)$.
(b) Solve and interpret $f(t)=169.3$.

Table 2.2

Year, t	0	4	8	12	16	20
Salary, S	129.5	133.6	141.3	158.1	169.3	174.0

20. Let $f(x)=3+2x^2$. Find $f\left(\frac{1}{3}\right)$ and $\frac{f(1)}{f(3)}$. Are they equal?

21. If $g(x)=x\sqrt{x}+100x$, evaluate without a calculator

(a) $g(100)$ **(b)** $g(4/25)$ **(c)** $g(1.21\cdot 10^4)$

22. Let $g(x)=x^2+x$. Find formulas for the following functions. Simplify your answers.

(a) $g(-3x)$ **(b)** $g(1-x)$ **(c)** $g(x+\pi)$
(d) $g(\sqrt{x})$ **(e)** $g(1/(x+1))$ **(f)** $g(x^2)$

23. Let $f(x)=\frac{x}{x-1}$.

(a) Find and simplify

(i) $f\left(\frac{1}{t}\right)$ (ii) $f\left(\frac{1}{t+1}\right)$

(b) Solve $f(x)=3$.

[2] https://hobnobblog.com/congress-by-the-numbers/pay-and-perquisites-of-members-of-congress-including-a-history-of-house-and-senate-salaries/, accessed October 24, 2013.

24. Let $f(x) = x^2$. Find and simplify the following.

(a) $f(2)$ **(b)** $f(2+h)$

(c) $f(2+h) - f(2)$ **(d)** $\dfrac{f(2+h) - f(2)}{h}$

25. Let $g(t) = 2t - 1$. Find and simplify

$$\frac{g(t+h) - g(t)}{h}.$$

26. **(a)** Using Figure 2.5, fill in Table 2.3.

Table 2.3

x	-2	-1	0	1	2	3
$h(x)$						

(b) Evaluate $h(3) - h(1)$ (c) Evaluate $h(2) - h(0)$

(d) Evaluate $2h(0)$ (e) Evaluate $h(1) + 3$

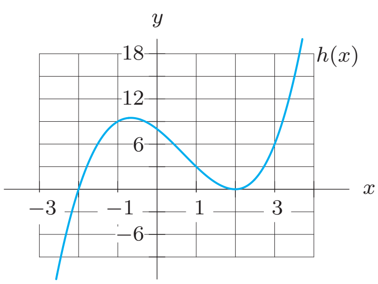


Figure 2.5

27. A ball is thrown up from the ground with initial velocity 64 ft/sec. Its height at time t is

$$h(t) = -16t^2 + 64t.$$

(a) Evaluate $h(1)$ and $h(3)$. What does this tell us about the height of the ball?

(b) Sketch this function. Using a graph, determine when the ball hits the ground and the maximum height of the ball.

28. The profit, in dollars, made by a theater when n tickets are sold is $P(n) = 20n - 500$.

(a) Calculate $P(0)$, and explain what this number means for the theater.

(b) Under what circumstances will the profit equal 0?

(c) What is the meaning of the quantity $P(100)$? What are its units?

29. Let $v(t) = t^2 - 2t$ be the velocity, in ft/sec, of an object at time t, in seconds.

(a) What is the initial velocity, $v(0)$?

(b) When does the object have a velocity of zero?

(c) What is the meaning of the quantity $v(3)$? What are its units?

30. Let $s(t) = 11t^2 + t + 100$ be the position, in miles, of a car driving on a straight road at time t, in hours. The car's velocity at any time t is given by $v(t) = 22t + 1$.

(a) Use function notation to express the car's position after 2 hours. Where is the car then?

(b) Use function notation to express the question, "When is the car going 65 mph?"

(c) Where is the car when it is going 67 mph?

31. Let $v = f(t)$ be the speed of a braking car, in feet per second, t seconds after the brakes are first applied. A graph of f is given in Figure 2.6. Complete the following, and explain the meaning of your answers in terms of the car.

(a) If $t = 1$, estimate $f(t+5)$.

(b) If $t = 1$, estimate $f(t) + 5$.

(c) Solve $f(t+2) = 40$ for t.

(d) Solve $f(t) + 10 = 40$ for t.

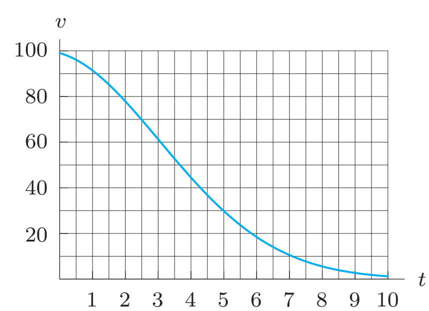


Figure 2.6

32. Use the letters a, b, c, d, e, h in Figure 2.7 to answer the following questions.

(a) What are the coordinates of the points P and Q?

(b) Evaluate $f(b)$.

(c) Solve $f(x) = e$ for x.

(d) Suppose $c = f(z)$ and $z = f(x)$. What is x?

(e) Suppose $f(b) = -f(d)$. What additional information does this give you?

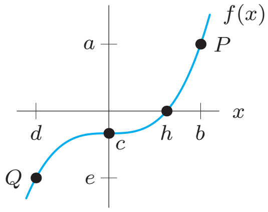


Figure 2.7

33. New York state income tax is based on taxable income, which is part of a person's total income. The tax owed to the state is calculated using the taxable income (not total income). In 2013, for a single person with a taxable income between \$77,150 and \$205,850, the tax owed was \$4650.63 plus 6.65% of the taxable income over \$77,150.[3]

(a) Compute the tax owed by a lawyer whose taxable income is \$88,000.
(b) Consider a lawyer whose taxable income is 80% of her total income, x, where x is between \$100,000 and \$150,000. Write a formula for $T(x)$, the taxable income.
(c) Write a formula for $L(x)$, the amount of tax owed by the lawyer in part (b).
(d) Use $L(x)$ to evaluate the tax owed for $x = 110{,}000$ and compare your results to part (a).

34. Table 2.4 shows $N(s)$, the number of sections of Economics 101, as a function of s, the number of students in the course. If s is between two numbers listed in the table, then $N(s)$ is the higher number of sections.

(a) Evaluate and interpret:

(i) $N(150)$ (ii) $N(80)$ (iii) $N(55.5)$

(b) Solve for s and interpret:

(i) $N(s) = 4$ (ii) $N(s) = N(125)$

Table 2.4

s	50	75	100	125	150	175	200
$N(s)$	4	4	5	5	6	6	7

35. (a) Complete Table 2.5 using

$$f(x) = 2x(x-3) - x(x-5) \quad \text{and} \quad g(x) = x^2 - x.$$

What do you notice? Graph these two functions. Are the two functions the same? Explain.

(b) Complete Table 2.6 using

$$h(x) = x^5 - 5x^3 + 6x + 1 \quad \text{and} \quad j(x) = 2x + 1.$$

What do you notice? Graph these two functions. Are the two functions the same? Explain.

Table 2.5

x	-2	-1	0	1	2
$f(x)$					
$g(x)$					

Table 2.6

x	-2	-1	0	1	2
$h(x)$					
$j(x)$					

(b) Describe in full sentences the patterns you see in the values for each function.
(c) Assuming that the patterns you observed in part (b) hold true for all values of x, calculate $f(5)$, $f(-2)$, $g(5)$, and $g(-2)$.
(d) Find possible formulas for $f(x)$ and $g(x)$.

Table 2.7

x	-1	0	1	2	3	4
$f(x)$	-4	-1	2	5	8	11
$g(x)$	4	1	0	1	4	9

Problems 37–38 concern $v = r(s)$, the eyewall wind profile of a hurricane at landfall, where v is the eyewall wind speed (in mph) as a function of s, the height (in meters) above the ground. (The eyewall is the band of clouds that surrounds the eye of the storm.) Let s_0 be the height at which the wind speed is greatest, and let $v_0 = r(s_0)$. Interpret the following in terms of the hurricanes.

37. $r(0.5s_0)$ **38.** $r(s) = 0.75v_0$

39. Let $h(x) = x^2 + bx + c$. Evaluate and simplify:

(a) $h(1)$ **(b)** $h(b+1)$

In Problems 40–42, if $f(x) = \dfrac{ax}{a+x}$, find and simplify

40. $f(a)$ **41.** $f(1-a)$ **42.** $f\left(\dfrac{1}{1-a}\right)$

43. Figure 2.8 shows $y = f(x)$. Label the coordinates of any points on the graph where

(a) $f(c) = 0$.
(b) $f(0) = d$.

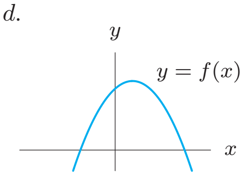


Figure 2.8

44. An epidemic of influenza spreads through a city. Figure 2.9 is the graph of $I = f(w)$, where I is the number of individuals (in thousands) infected w weeks after the epidemic begins.

(a) Evaluate $f(2)$ and explain its meaning in terms of the epidemic.
(b) Approximately how many people were infected at the height of the epidemic? When did that occur? Write your answer in the form $f(a) = b$.
(c) Solve $f(w) = 4.5$ and explain what the solutions mean in terms of the epidemic.

(d) The graph used $f(w) = 6w(1.3)^{-w}$. Use the graph to estimate the solution of the inequality $6w(1.3)^{-w} \geq 6$. Explain what the solution means in terms of the epidemic.

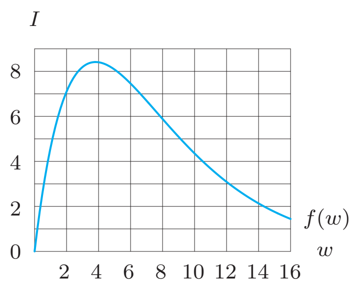


Figure 2.9

Problems 45–46 concern studies suggesting that as carbon dioxide (CO_2) levels rise, hurricanes will become more intense.[4] Hurricane intensity is measured in terms of the minimum central pressure P (in mb): the lower the pressure, the more powerful the storm. Since warm ocean waters fuel hurricanes, P is a decreasing function of H, sea surface temperature in °C. Let $P = n(H)$ be the hurricane-intensity function for present-day CO_2 levels, and let $P = N(H)$ be the hurricane-intensity function for future projected CO_2 levels. If H_0 is the average temperature in the Caribbean Sea, what do the following quantities tell you about hurricane intensity?

45. $N(H_0) - n(H_0)$

46. $n(H_0 + 1) - n(H_0)$

2.2 DOMAIN AND RANGE

In Example **??** on page **??**, we defined R to be the average monthly rainfall at Chicago's O'Hare airport in month t. Although R is a function of t, the value of R is not defined for every possible value of t. For instance, it makes no sense to consider the value of R for $t = -3$, or $t = 8.21$, or $t = 13$ (since a year has 12 months). Thus, although R is a function of t, this function is defined only for $t = 1, t = 2, t = 3, \ldots, t = 12$. Notice also that R, the output value of this function, takes only the values $\{1.8, 2.1, 2.4, 2.5, 2.7, 3.1, 3.2, 3.4, 3.5, 3.7\}$.

A function is often defined only for certain values of the independent variable. Also, the dependent variable often takes on only certain values. This leads to the following definitions:

> If $Q = f(t)$, then the
> - **Domain** of f is the set of input values, t, which yield an output value.
> - **Range** of f is the corresponding set of output values, Q.

If the domain of a function is not specified, we usually assume that it is as large as possible—that is, all numbers that make sense as inputs for the function. For example, if there are no restrictions, the domain of the function $f(x) = x^2$ is the set of all real numbers, because we can substitute any real number into the formula $f(x) = x^2$. Sometimes, however, we may restrict the domain to suit a particular application. If the function $f(x) = x^2$ is used to represent the area of a square of side x, we restrict the domain to positive numbers.

Example 1 The house-painting function $n = f(A)$ in Example **??** on page **??** has domain $A > 0$ because all houses have some positive paintable area. There is a practical upper limit to A because houses cannot be infinitely large, but in principle, A can be as large or as small as we like, as long as it is positive. Therefore we take the domain of f to be $A > 0$.

The range of this function is $n \geq 0$, because we cannot use a negative amount of paint.

[4] *Journal of Climate*, September 14, 2004, pages 3477–3495 or www.theguardian.com/environment/2013/aug/23/climate-change-carbon-emissions-ipcc-extreme, accessed May 21, 2014.

Choosing Realistic Domains and Ranges

When a function is used to model a real situation, we may need to modify the domain and range.

Example 2 Algebraically speaking, the formula

$$T = \frac{1}{4}R + 40$$

can be used for all values of R. If we know nothing more about this function than its formula, its domain is all real numbers. The formula for $T = \frac{1}{4}R + 40$ can return any value of T when we choose an appropriate R-value. (See Figure 2.10.) Thus, the range of the function is also all real numbers. However, if we use this formula to represent the temperature, T, as a function of a cricket's chirp rate, R, as we did in Example **??** on page **??**, some values of R cannot be used. For example, it does not make sense to talk about a negative chirp rate. Also, there is some maximum chirp rate R_{max} that no cricket can physically exceed. Thus, to use this formula to express T as a function of R, we must restrict R to the interval $0 \le R \le R_{max}$ shown in Figure 2.11.

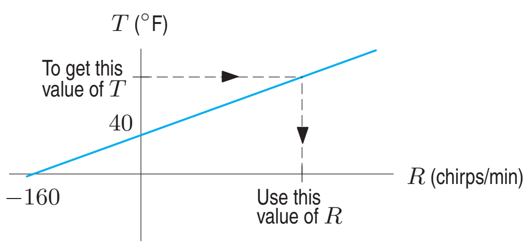


Figure 2.10: Graph showing that any T value can be obtained from some R value

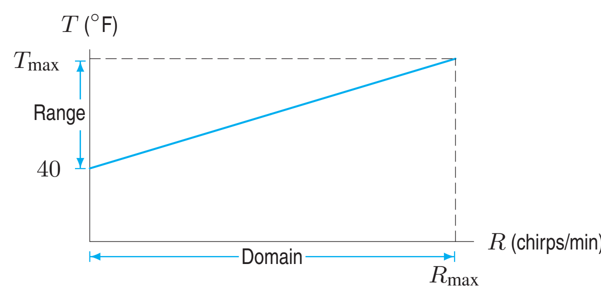


Figure 2.11: Graph showing that if $0 \le R \le R_{max}$, then $40 \le T \le T_{max}$

The range of the cricket function is also restricted. Since the chirp rate is nonnegative, the smallest value of T occurs when $R = 0$. This happens at $T = 40$. On the other hand, if the temperature gets too hot, the cricket will not be able to keep chirping faster. If the temperature T_{max} corresponds to the chirp rate R_{max}, then the values of T are restricted to the interval $40 \le T \le T_{max}$.

Using a Graph to Find the Domain and Range of a Function

A good way to estimate the domain and range of a function is to examine its graph. The domain is the set of input values on the horizontal axis that give rise to a point on the graph; the range is the corresponding set of output values on the vertical axis.

Example 3 A sunflower plant is measured every day t, for $t \ge 0$. The height, $h(t)$ centimeters, of the plant[5] can be modeled by the *logistic function*, which is graphed in Figure 2.12:

$$h(t) = \frac{250}{1 + 24(0.9)^t}.$$

(a) What is the domain of the function in this model?

[5] Adapted from H.S. Reed and R.H. Holland, "Growth of an Annual Plant Helianthus," *Proc. Nat. Acad. Sci.*, 5, 1919.

(b) What is the range of the function? What does this tell you about the height of the sunflower?

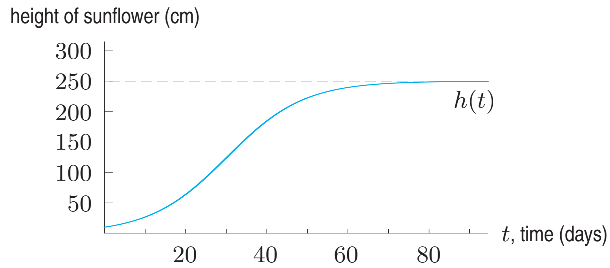


Figure 2.12: Height of sunflower as a function of time

Solution

(a) Measurements can be made at any time after $t = 0$. The graph starts on the vertical axis and extends to the right, so the domain of the function is $t \geq 0$. If we consider the fact that the sunflower dies on some day T, then the domain is $0 \leq t \leq T$.

(b) To find the range, notice that the smallest value of h occurs at $t = 0$. Evaluating gives

$$h(0) = \frac{250}{1 + 24(0.9)^0} = 10 \text{ cm}.$$

This means that the plant was 10 cm high when it was first measured on day $t = 0$. As t increases, the plant grows and $h(t)$ increases. The values of $h(t)$ approach, but never reach, 250. This suggests that the range is $10 \leq h(t) < 250$. This information tells us that sunflowers typically grow to a height of about 250 cm.

Using Formulas and Graphs to Find Domains and Ranges

When a function is defined by a formula, its domain and range can often be determined by examining the formula algebraically.

Example 4 State the domain and range of g, where

$$g(x) = \frac{1}{x - 2}.$$

Solution The domain is all real numbers except those which do not yield an output value. The expression $1/(x - 2)$ is defined for any real number x except $x = 2$ (division by 0 is undefined). Therefore,

$$\text{Domain: all real } x, \quad x \neq 2.$$

The range is all real numbers that the formula can return as output values. It is not possible for $g(x)$ to equal zero, since 1 divided by a real number is never zero. All real numbers except 0 are possible output values, since all nonzero real numbers have reciprocals. To see this algebraically, suppose we ask if a particular value of y is in the range. Then we solve for y in

$$y = \frac{1}{x - 2} \qquad \text{giving} \qquad x = \frac{1}{y} + 2,$$

so any value of y is possible except $y = 0$. Thus

$$\text{Range: all real values}, \quad g(x) \neq 0.$$

The graph in Figure 2.13 reflects this the domain and this range.

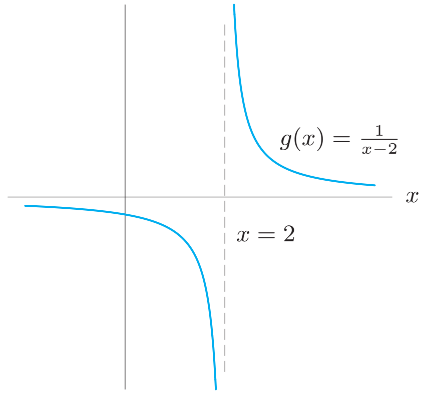


Figure 2.13: Domain and range of $g(x) = 1/(x-2)$

Example 5 Find the domain of the function $f(x) = \dfrac{1}{\sqrt{x-4}}$ by examining its formula and its graph.

Solution The domain is all real numbers except those for which the function is undefined. The square root of a negative number is undefined (if we restrict ourselves to real numbers), and so is division by zero. Therefore we need

$$x - 4 > 0.$$

Thus, the domain is all real numbers greater than 4.

$$\text{Domain:} \quad x > 4.$$

From the formula $f(x) = 1/\sqrt{x-4}$ and the graph in Figure 2.14, we see that

$$\text{Range:} \quad f(x) > 0.$$

Notice that in Example 6 on page 71 we saw that the output of $f(x) = 1/\sqrt{x-4}$ cannot be negative. Exercise 21 asks for an algebraic argument.

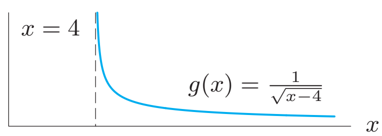


Figure 2.14: Domain and range of $f(x) = 1/\sqrt{x-4}$

Example 6 Find the domain of the function $h(x) = \dfrac{1}{\sqrt{4-x^2}}$ by examining its formula.

Solution As in Example 5, the domain consists of all x-values making the denominator positive, that is:

$$4 - x^2 > 0.$$

Thus, to find the domain, we solve for the x-values satisfying $4 > x^2$. That is, we want all the x-values whose squares are smaller then 4. These are the numbers between -2 and 2, so we have

$$\text{Domain:} \quad -2 < x < 2.$$

Rational functions can lead to more complicated domains and ranges, such as those resulting from "holes" in the graph. See Section **??**, page **??**.

Exercises and Problems for Section 2.2

Skill Refresher

In Exercises S1–S4, for what value(s), if any, are the functions undefined?

S1. $f(x)=\dfrac{x-2}{x-3}$

S2. $g(x)=\dfrac{1}{x(x-3)}$

S3. $h(x)=\sqrt{x-15}$

S4. $k(x)=\sqrt{15-x}$

Solve the inequalities in Exercises S5–S12.

S5. $x-8>0$

S6. $-x+5>0$

S7. $-3(n-4)>12$

S8. $12\le 24-4a$

S9. $x^2-25>0$

S10. $36-x^2\ge 0$

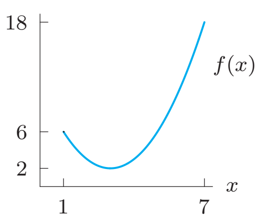
S11. $12-2a^2\le a^2$

S12. $y^2-3>15-y^2$

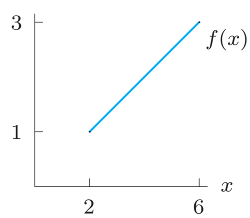
Exercises

In Exercises 1–4, estimate the domain and range of the function. Assume the entire graph is shown.

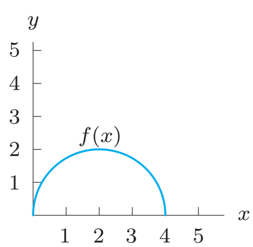
1.



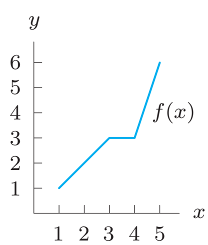
2.



3.



4.



In Exercises 5–8, use a graph to find the range of the function on the given domain.

5. $f(x)=\dfrac{1}{x},\quad -2\le x\le 2$

6. $f(x)=\dfrac{1}{x^2},\quad -1\le x\le 1$

7. $f(x)=x^2-4,\quad -2\le x\le 3$

8. $f(x)=\sqrt{9-x^2},\quad -3\le x\le 1$

In Exercises 9–18, find the domain of the function algebraically.

9. $f(x)=\dfrac{1}{x+3}$

10. $p(t)=\dfrac{1}{t^2-4}$

11. $f(t)=\dfrac{t-3}{3t+9}$

12. $n(q)=\dfrac{1}{q^4+2}$

13. $f(x)=\dfrac{1}{\sqrt{9+x}}$

14. $y(t)=\dfrac{1}{t^4}$

15. $f(x)=\sqrt{x^2-4}$

16. $q(r)=\sqrt[3]{r^2-16}$

17. $m(x)=x^2-9$

18. $t(a)=\sqrt[4]{a-2}$

In Exercises 19–22, find the domain and range of the function algebraically.

19. $m(q)=\dfrac{1}{5}q-4$

20. $f(x)=\sqrt{15-4x}$

21. $f(x)=\dfrac{1}{\sqrt{x-4}}$

22. $f(x)=\dfrac{1}{\sqrt{9-x}}$

Problems

In Problems 23–26, you are given the domain D of the function. Where applicable, find possible values for the constants a and b.

23. $f(x)=\dfrac{1}{x-a}$, D: all real numbers $\neq 3$

24. $p(t)=\dfrac{1}{(2t-a)(t+b)}$, D: all real numbers except 4 and 5.

25. $n(q)=\sqrt{q^2+a}$, D: all real numbers

26. $m(r)=\sqrt{r-a}$, D: all real numbers ≥ -3

27. Give a formula for a function that is undefined for $x=-2$ and for $x<-4$, but is defined everywhere else.

28. Give a formula for a function whose domain is all negative values of x except $x=-5$.

29. A restaurant is open from 2 pm to 2 am each day, and a maximum of 200 clients can fit inside. If $f(t)$ is the number of clients in the restaurant t hours after 2 pm each day, what are a reasonable domain and range for $f(t)$?

30. A car gets the best mileage at intermediate speeds. Graph the gas mileage as a function of speed. Determine a reasonable domain and range for the function and justify your reasoning.

31. A movie theater seats 200 people. For any particular show, the amount of money the theater makes is a function of the number of people, n, in attendance. If a ticket costs \$4.00, find the domain and range of this function. Sketch its graph.

32. **(a)** Table 2.8 shows the relationship between the number of calories used per minute of walking and a person's weight in pounds.[6] Use the table to

(i) Determine the number of calories that a person weighing 200 lb uses in a half-hour of walking.

(ii) Describe in words the relationship between weight and the number of calories used. Identify the dependent and independent variables and specify whether the function is increasing or decreasing.

(b) The function described in part (a)(ii) is approximately linear.

(i) Estimate the equation for this function and graph it.

(ii) Give a reasonable domain and range for your function based on the data in the table.

(iii) Use your function to estimate how many calories per minute a person who weighs 135 lb uses per minute of walking.

Table 2.8 *Calories per minute as a function of weight*

Weight	100 lb	120 lb	150 lb	170 lb	200 lb	220 lb
Walking	2.7	3.2	4.0	4.6	5.4	5.9

In Exercises 33–34, find the domain and range of the function.

33. $g(x) = a + 1/x$, where a is a constant

34. $q(x) = (x - b)^{1/2} + 6$, where b is a constant

35. The last digit, d, of a phone number is a function of n, its position in the phone book. Table 2.9 gives d for the first 10 listings in the 2009 New York State telephone directory.[7] The table shows that the last digit of the first listing is 1, the last digit of the second listing is 5, and so on. In principle we could use a phone book to figure out other values of d. For instance, if $n = 300$, we could count down to the 300^{th} listing in order to determine d. So we write $d = f(n)$.

(a) What is the value of $f(6)$?

(b) Explain how you could use the phone book to find the domain of f.

(c) What is the range of f?

Table 2.9

n	1	2	3	4	5	6	7	8	9	10
d	1	2	1	5	9	9	0	1	1	7

36. Let t be time in seconds and let $r(t)$ be the rate, in gallons/second, that water enters a reservoir:

$$r(t) = 800 - 40t.$$

(a) Evaluate the expressions $r(0)$, $r(15)$, $r(25)$, and explain their physical significance.

(b) Graph $y = r(t)$ for $0 \leq t \leq 30$, labeling the intercepts. What is the physical significance of the slope and the intercepts?

(c) For $0 \leq t \leq 30$, when does the reservoir have the most water? When does it have the least water?

(d) What are the domain and range of $r(t)$?

37. In month $t = 0$, a small group of rabbits escapes from a ship onto an island where there are no rabbits. The island rabbit population, $p(t)$, in month t is given by

$$p(t) = \frac{1000}{1 + 19(0.9)^t}, \quad t \geq 0.$$

(a) Evaluate $p(0)$, $p(10)$, $p(50)$, and explain their meaning in terms of rabbits.

(b) Graph $p(t)$ for $0 \leq t \leq 100$. Describe the graph in words. Does it suggest the growth in population you would expect among rabbits on an island?

(c) Estimate the range of $p(t)$. What does this tell you about the rabbit population?

(d) Explain how you can find the range of $p(t)$ from its formula.

38. Bronze is an alloy or mixture of the metals copper and tin. The properties of bronze depend on the percentage of copper in the mix. A chemist decides to study the properties of a given alloy of bronze as the proportion of copper is varied. She starts with 9 kg of bronze that contain 3 kg of copper and 6 kg of tin and either adds or removes copper. Let $f(x)$ be the percentage of copper in the mix if x kg of copper are added ($x > 0$) or removed ($x < 0$).

(a) State the domain and range of f. What does your answer mean in the context of bronze?

(b) Find a formula in terms of x for $f(x)$.

(c) If the formula you found in part (b) was not intended to represent the percentage of copper in an alloy of bronze, but instead simply defined an abstract mathematical function, what would be the domain and range of this function?

[6] Source: 1993 World Almanac. Speeds assumed are 3 mph for walking, 10 mph for bicycling, and 2 mph for swimming.
[7] www6.oft.state.ny.us/telecom/phones/. Accessed February 10, 2010.

39. The surface area of a cylindrical aluminum can is a measure of how much aluminum the can requires. If the can has radius r and height h, its surface area A and its volume V are given by the equations:

$$A = 2\pi r^2 + 2\pi rh \quad \text{and} \quad V = \pi r^2 h.$$

(a) The volume, V, of a 12-oz cola can is 355 cm^3. A cola can is approximately cylindrical. Express its surface area A as a function of its radius r, where r is measured in centimeters. [Hint: First solve for h in terms of r.]

(b) Graph $A = s(r)$, the surface area of a cola can whose volume is 355 cm^3, for $0 \leq r \leq 10$.

(c) What is the domain of $s(r)$? Based on your graph, what, approximately, is the range of $s(r)$?

(d) The manufacturers wish to use the smallest amount of aluminum (in cm^2) necessary to make a 12-oz cola can. Use your answer in (c) to find the minimum amount of aluminum needed. State the values of r and h that minimize the amount of aluminum used.

(e) The radius of a real 12-oz cola can is about 3.25 cm. Show that real cola cans use more aluminum than necessary to hold 12 oz of cola. Why do you think real cola cans are made in this way?

2.3 PIECEWISE-DEFINED FUNCTIONS

A function may employ different formulas on different parts of its domain. Such a function is said to be *piecewise defined*. For example, the function graphed in Figure 2.15 has the following formulas:

$$\begin{array}{ll} y = x^2 & \text{for } x \leq 2 \\ y = 6 - x & \text{for } x > 2 \end{array} \quad \text{or more compactly} \quad y = \begin{cases} x^2 & \text{for } x \leq 2 \\ 6 - x & \text{for } x > 2. \end{cases}$$

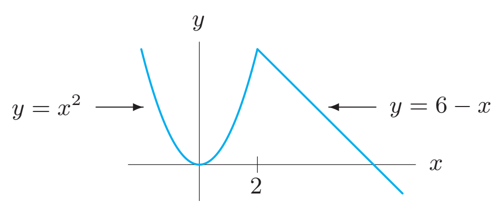


Figure 2.15: Piecewise defined function

Example 1 Graph the function $y = g(x)$ given by the following formulas:

$$g(x) = x + 1 \quad \text{for} \quad x \leq 2 \qquad \text{and} \qquad g(x) = 1 \quad \text{for} \quad x > 2.$$

Using bracket notation, this function is written:

$$g(x) = \begin{cases} x + 1 & \text{for } x \leq 2 \\ 1 & \text{for } x > 2. \end{cases}$$

Solution For $x \leq 2$, graph the line $y = x + 1$. The solid dot at the point $(2, 3)$ shows that it is included in the graph. For $x > 2$, graph the horizontal line $y = 1$. See Figure 2.16. The open circle at the point $(2, 1)$ shows that it is not included in the graph. (Note that $g(2) = 3$, and $g(2)$ cannot have more than one value.)

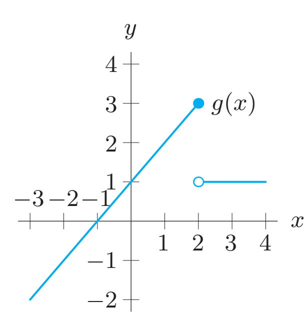


Figure 2.16: Graph of the piecewise defined function g

Example 2 A data plan for a smart phone charges 99 cents per day for data usage up to 20 megabytes and 7 cents for each additional megabyte or part of a megabyte.

(a) Use bracket notation to write a formula for the daily cost, C, of the data plan as a function of the number of megabytes m used.
(b) Graph the function.
(c) State the domain and range of the function.

Solution (a) For $0 \leq m \leq 20$, the value of C is 99 cents. If $m > 20$, we subtract 20 to find the additional number of megabytes used and multiply by the rate, 7 cents per megabyte.[8] The cost function in cents is thus

$$C = f(m) = \begin{cases} 99 & \text{for } \ 0 \leq m \leq 20 \\ 99 + 7(m-20) & \text{for } \ m > 20, \end{cases}$$

or, after simplifying,

$$C = f(t) = \begin{cases} 99 & \text{for } \ 0 \leq m \leq 20 \\ 7m - 41 & \text{for } \ m > 20. \end{cases}$$

(b) See Figure 2.17.
(c) Because negative data usage does not make sense, the domain is $m \geq 0$. From the graph, we see that the range is $C \geq 99$.

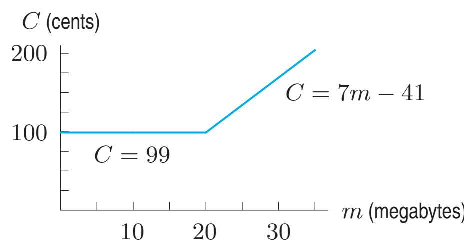


Figure 2.17: Daily cost of a data plan

Example 3 The Ironman Triathlon is a race that consists of three parts: a 2.4-mile swim followed by a 112-mile bike race and then a 26.2-mile marathon. A participant swims steadily at 2 mph, cycles steadily at 20 mph, and then runs steadily at 9 mph.[9] Assuming that no time is lost during the transition from one stage to the next, find a formula for the distance covered, d, in miles, as a function of the elapsed time t in hours, from the beginning of the race. Graph the function.

Solution For each leg of the race, we use the formula Distance $=$ Rate $\cdot$ Time. First, we calculate how long it took for the participant to cover each of the three parts of the race. The first leg took $2.4/2 = 1.2$ hours, the second leg took $112/20 = 5.6$ hours, and the final leg took $26.2/9 \approx 2.91$ hours. Thus, the participant finished the race in $1.2 + 5.6 + 2.91 = 9.71$ hours.

During the first leg, $t \leq 1.2$ and the speed is 2 mph, so

$$d = 2t \qquad \text{for} \qquad 0 \leq t \leq 1.2.$$

During the second leg, $1.2 < t \leq 1.2 + 5.6 = 6.8$ and the speed is 20 mph. The length of time spent in the second leg is $(t - 1.2)$ hours. Thus, by time t,

$$\text{Distance covered in the second leg} = 20(t-1.2) \quad \text{for } 1.2 < t \leq 6.8.$$

When the participant is in the second leg, the total distance covered is the sum of the distance covered in the first leg (2.4 miles) plus the part of the second leg that has been covered by time t:

$$\begin{aligned} d &= 2.4 + 20(t-1.2) \\ &= 20t - 21.6 \qquad \text{for } 1.2 < t \leq 6.8. \end{aligned}$$

[8] In actuality, most plans round data usage to whole megabytes or blocks of megabytes.
[9] Personal communication Susan Reid, Athletics Department, University of Arizona.

In the third leg, $6.8 < t \leq 9.71$ and the speed is 9 mph. Since 6.8 hours were spent on the first two parts of the race, the length of time spent on the third leg is $(t - 6.8)$ hours. Thus, by time t,

$$\text{Distance covered in the third leg} = 9(t - 6.8) \quad \text{for } 6.8 < t \leq 9.71.$$

When the participant is in the third leg, the total distance covered is the sum of the distances covered in the first leg (2.4 miles) and the second leg (112 miles), plus the part of the third leg that has been covered by time t:

$$\begin{aligned} d &= 2.4 + 112 + 9(t - 6.8) \\ &= 9t + 53.2 \qquad \text{for } 6.8 < t \leq 9.71. \end{aligned}$$

The formula for d is different on different intervals of t:

$$d = \begin{cases} 2t & \text{for} \quad 0 \leq t \leq 1.2 \\ 20t - 21.6 & \text{for} \quad 1.2 < t \leq 6.8 \\ 9t + 53.2 & \text{for} \quad 6.8 < t \leq 9.71. \end{cases}$$

Figure 2.18 gives a graph of the distance covered, d, as a function of time, t. Notice the three pieces.

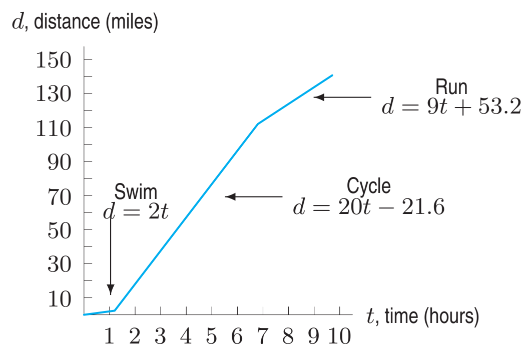


Figure 2.18: Ironman Triathlon: d as a function of t

The Absolute Value Function

The absolute value of x, written $|x|$, is defined piecewise:

$$\text{For nonnegative } x, \quad |x| = x.$$

$$\text{For negative } x, \quad |x| = -x.$$

(Remember that $-x$ is a positive number if x is a negative number.) For example, if $x = -3$, then

$$|-3| = -(-3) = 3.$$

For $x = 0$, we have $|0| = 0$. This leads to the following two-part definition:

The **absolute value function** is defined by

$$f(x) = |x| = \begin{cases} x & \text{for} \quad x \geq 0 \\ -x & \text{for} \quad x < 0 \end{cases}.$$

Table 2.10 gives values of $f(x) = |x|$ and Figure 2.19 shows a graph of $f(x)$.

Table 2.10 *Absolute value function*

x	$\|x\|$
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

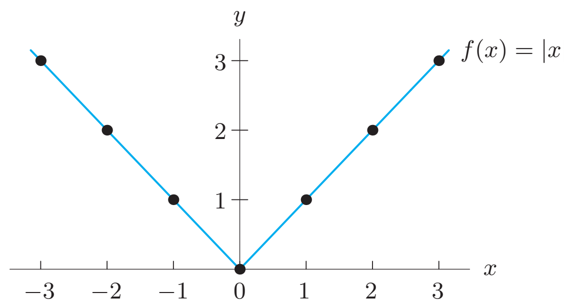


Figure 2.19: Graph of absolute value function

Visualizing Absolute Value on the Number Line

Notice that the absolute value of a number is always the distance between that number and zero on the number line. For examples, see Figure 2.20.

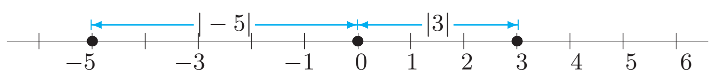


Figure 2.20: Distance from 3 to 0 is $|3|$. Distance from -5 to 0 is $|-5|$

Similarly, we can think of $|x - 4|$ as the distance between x and 4, as in Figure 2.21.

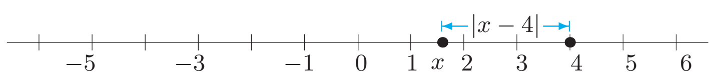


Figure 2.21: Distance from x to 4 is $|x - 4|$

In general:

> If a and b are real numbers, then $|a - b|$ is the distance between a and b on the number line.

Example 4 Let $f(x) = |2x - 7|$.

(a) What are the domain and range of $f(x)$? (b) Find all values of x such that $f(x) = 3$.

Solution

(a) The function $f(x)$ gives the distance between $2x$ and 7 on the number line. This distance can be calculated for any real number x, so the domain is all real numbers. The distance can be any nonnegative real number, so the range of is all real numbers greater than or equal to 0.

(b) We want to find all numbers x such that $|2x - 7| = 3$. That is, we want the distance between $2x$ and 7 to be 3. Thus, $2x$ must be three units to the left or three units to the right of 7; that is,

$$2x = 4 \qquad \text{or} \qquad 2x = 10.$$

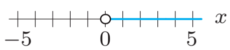
In the first case, $x = 2$; in the second, $x = 5$. So the values of x such that $f(x) = 3$ are $x = 2$ and $x = 5$.

Exercises and Problems for Section 2.3

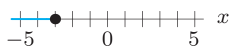
Skill Refresher

In Exercises S1–S6, write all the possible values for x that match the graph.

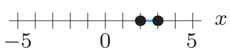
S1.



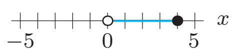
S2.



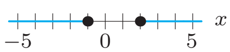
S3.



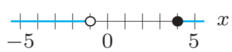
S4.



S5.

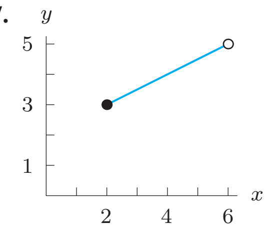


S6.

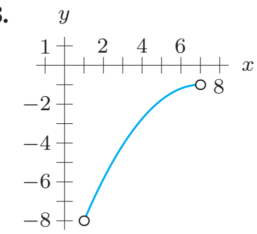


In Exercises S7–S10, determine the domain and range of the function.

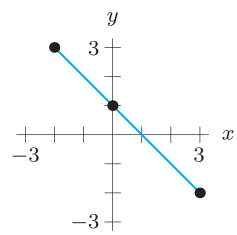
S7.



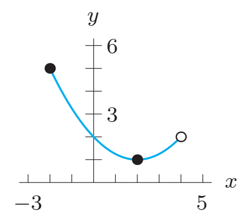
S8.



S9.



S10.



Exercises

In Exercises 1–4, graph the piecewise defined function. Use an open circle to represent a point that is not included and a solid dot to indicate a point that is included in the graph.

1. $f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

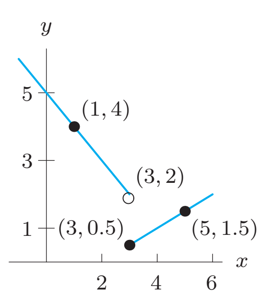
2. $f(x) = \begin{cases} x+1, & -2 \le x < 0 \\ x-1, & 0 \le x < 2 \\ x-3, & 2 \le x < 4 \end{cases}$

3. $f(x) = \begin{cases} x+4, & x \le -2 \\ 2, & -2 < x < 2 \\ 4-x, & x \ge 2 \end{cases}$

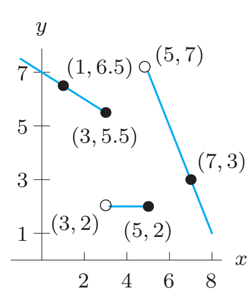
4. $f(x) = \begin{cases} x^2, & x \le 0 \\ \sqrt{x}, & 0 < x < 4 \\ x/2, & x \ge 4 \end{cases}$

In Exercises 5–8, write formulas for the functions.

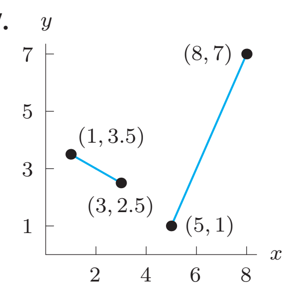
5.



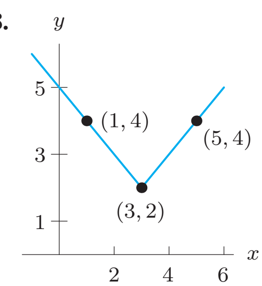
6.



7.



8.



For Exercises 9–10, find the domain and range.

9. $G(x) = \begin{cases} x+1 & \text{for} \quad x < -1 \\ x^2+3 & \text{for} \quad x \ge -1 \end{cases}$

10. $F(x) = \begin{cases} x^3 & \text{for} \quad x \le 1 \\ 1/x & \text{for} \quad x > 1 \end{cases}$

Find all x values satisfying the equations in Exercises 11–16.

11. $|x| = 5$

12. $2|y| - 20 = 0$

13. $|x-3| = 7$

14. $20 = 10|x-5|$

15. $|y-3| = 8 - |y-3|$

16. $|2y-6| = 8$

Problems

17. Figure 2.22 shows an entire graph. An open circle represents a point that is not included.

(a) Is y a function of x? Explain.
(b) Is x a function of y? Explain.
(c) If you identified a function in part (a) or part (b), what is the range of that function?

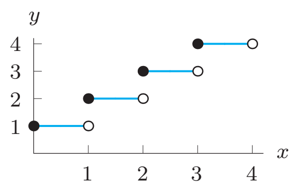


Figure 2.22

18. Many people believe that $\sqrt{x^2} = x$. We will investigate this claim graphically and numerically.

(a) Graph the two functions x and $\sqrt{x^2}$ in the window $-5 \le x \le 5$, $-5 \le y \le 5$. Based on what you see, do you believe that $\sqrt{x^2} = x$? What function does the graph of $\sqrt{x^2}$ remind you of?
(b) Complete Table 2.11. Based on this table, do you believe that $\sqrt{x^2} = x$? What function does the table for $\sqrt{x^2}$ remind you of? Is this the same function you found in part (a)?

Table 2.11

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\sqrt{x^2}$											

(c) Explain how you know that $\sqrt{x^2}$ is the same as the function $|x|$.
(d) Graph the function $\sqrt{x^2} - |x|$ in the window $-5 \le x \le 5$, $-5 \le y \le 5$. Explain what you see.

19. **(a)** Graph $u(x) = |x|/x$ in the window $-5 \le x \le 5$, $-5 \le y \le 5$. Explain what you see.
(b) Complete Table 2.12. Does this table agree with what you found in part (a)?

Table 2.12

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\lvert x\rvert/x$											

(c) Identify the domain and range of $u(x)$.
(d) Comment on the claim that $u(x)$ can be written as

$$u(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

20. Let $g(x) = \begin{cases} -1 & \text{for} \quad x < 0 \\ x^3 & \text{for} \quad x \ge 0 \end{cases}$

(a) Find $g(-2)$, $g(2)$, and $g(0)$.
(b) Find the domain and range of $g(x)$.

21. Let $f(x) = \begin{cases} 3x & \text{for} \quad -1 \le x \le 1 \\ -x+4 & \text{for} \quad 1 < x \le 5 \end{cases}$

(a) Find $f(0)$ and $f(3)$.
(b) Find the domain and range of $f(x)$.

22. Let $f(x) = \begin{cases} 1/x & \text{for} \quad x < -1 \\ x^2 & \text{for} \quad -1 \le x \le 1 \\ \sqrt{x} & \text{for} \quad x > 1. \end{cases}$

(a) Evaluate $f(-2)$ and $f(2)$.
(b) What is the range of f?

23. Use bracket notation to write a formula for the piecewise function that is defined by $y = x^2$ for negative values of x and by $y = x - 1$ for x values that are greater than or equal to zero.

24. A floor-refinishing company charges \$1.83 per square foot to strip and refinish a tile floor for up to 1000 square feet. There is an additional charge of \$350 for toxic waste disposal for any job that includes more than 150 square feet of tile.

(a) Express the cost, y, of refinishing a floor as a function of the number of square feet, x, to be refinished.
(b) Graph the function. Give the domain and range.

25. The charge for a taxi ride in New York City is \$2.50 upon entry and \$0.40 for each $1/5$ of a mile traveled (rounded up to the nearest $1/5$ mile), when the taxicab is traveling at 6 mph or more. In addition, a New York State Tax Surcharge of \$0.50 is added to the fare.[10]

(a) Make a table showing the cost of a trip as a function of its length. Your table should start at zero and go up to two miles in $1/5$-mile intervals.
(b) What is the cost for a 1.2-mile trip?
(c) How far can you go for \$5.80?
(d) Graph the cost function in part (a).

[10] www.nyc.gov/html/tlc/html/passenger/taxicab_rate.shtml, accessed February 11, 2010.

26. A museum charges \$40 for a group of 10 or fewer people. A group of more than 10 people must, in addition to the \$40, pay \$2 per person for the number of people above 10. For example, a group of 12 pays \$44 and a group of 15 pays \$50. The maximum group size is 50.

(a) Draw a graph that represents this situation.
(b) What are the domain and range of the cost function?

27. At a supermarket checkout, a scanner records the prices of the foods you buy. In order to protect consumers, the state of Michigan passed a "scanning law" that says something similar to the following:

> If there is a discrepancy between the price marked on the item and the price recorded by the scanner, the consumer is entitled to receive 10 times the difference between those prices; this amount given must be at least \$1 and at most \$5. Also, the consumer will be given the difference between the prices, in addition to the amount calculated above.

For example: If the difference is 5¢, you should receive \$1 (since 10 times the difference is only 50¢ and you are to receive at least \$1), plus the difference of 5¢. Thus, the total you should receive is $\$1.00 + \$0.05 = \$1.05$.
If the difference is 25¢, you should receive 10 times the difference in addition to the difference, giving $(10)(0.25) + 0.25 = \$2.75$.
If the difference is 95¢, you should receive \$5 (because $10(.95) = \$9.50$ is more than \$5, the maximum penalty), plus 95¢, giving $5 + 0.95 = \$5.95$.

(a) What is the lowest possible refund?
(b) Suppose x is the difference between the price scanned and the price marked on the item, and y is the amount refunded to the customer. Write a formula for y in terms of x. [Hint: Look at the sample calculations.]
(c) What would the difference between the price scanned and the price marked have to be in order to obtain a \$9.00 refund?
(d) Graph y as a function of x.

28. Seattle City Light[11] charges residents for electricity on a daily basis. There is a basic daily charge of 15.70 cents. In addition, for each day the first 10 kWh cost 4.66 cents per kWh and any additional kWhs are 10.71 cents per kWh. (A kWh is a unit of energy.)

(a) Make a table showing the cost in dollars of usage from 0 to 40 kWh in increments of 5 kWh.
(b) Write a piecewise defined function to describe the usage rate.
(c) What is the cost for 33 kWh?
(d) How many kWh can you burn on a day for \$3?

29. Gore Mountain is a ski resort in the Adirondack mountains in upstate New York. Table 2.13 shows the cost of a weekday ski-lift ticket for various ages and dates.

(a) Graph cost as a function of age for each time period given. (One graph will serve for times when rates are identical.)
(b) For which age group does the date affect cost?
(c) Graph cost as a function of date for the age group mentioned in part (b).
(d) Why does the cost fluctuate as a function of date?

Table 2.13 *Ski-lift ticket prices at Gore Mountain, 1998–1999*[12]

Age	Opening-Dec 12	Dec 13-Dec 24	Dec 25-Jan 3	Jan 4-Jan 15	Jan 16-Jan 18
Up to 6	Free	Free	Free	Free	Free
7–12	\$19	\$19	\$19	\$19	\$19
13–69	\$29	\$34	\$39	\$34	\$39
70+	Free	Free	Free	Free	Free
Age	**Jan 19-Feb 12**	**Feb 13-Feb 21**	**Feb 22-Mar 28**	**Mar 29-Closing**	
Up to 6	Free	Free	Free	Free	
7–12	\$19	\$19	\$19	\$19	
13–69	\$34	\$39	\$34	\$29	
70+	Free	Free	Free	Free	

In Problems 30–31:

(a) Use the definition of absolute value to write a piecewise formula for f.
(b) Graph f.

30. $f(x) = |x^2 - 4|$

31. $f(x) = |2x - 6|$

2.4 PREVIEW OF TRANSFORMATIONS: SHIFTS

Suppose we shift the graph of a function vertically or horizontally, giving the graph of a new function. In this section we see the relationship between the formulas for the original function and the new function.

[11] http://www.seattle.gov, accessed April 2013
[12] The Olympic Regional Development Authority.

Vertical Shift: The Heating Schedule for an Office Building

We start with an example of a vertical shift in the context of the heating schedule for a building.

Example 1 To save money, an office building is kept warm only during business hours. Figure 2.23 shows the temperature, H, in °F, as a function of time, t, in hours after midnight. At midnight ($t = 0$), the building's temperature is 50°F. This temperature is maintained until 4 am. Then the building begins to warm up so that by 8 am the temperature is 70°F. At 4 pm the building begins to cool. By 8 pm, the temperature is again 50°F.

Suppose that the building's superintendent decides to keep the building 5°F warmer than before. Sketch a graph of the new function.

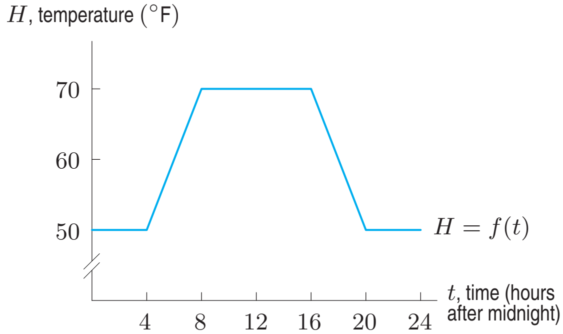


Figure 2.23: The heating schedule at an office building, $H = f(t)$

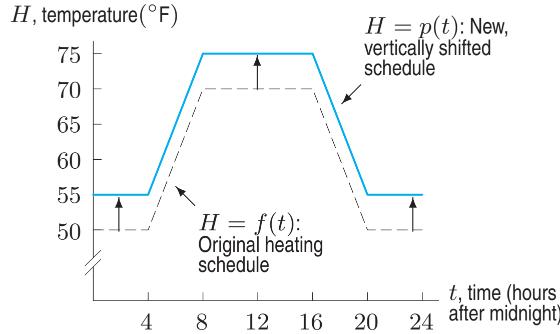


Figure 2.24: Graph of new heating schedule, $H = p(t)$, obtained by shifting original graph, $H = f(t)$, upward by 5 units

Solution The graph of f, the heating schedule function of Figure 2.23, is shifted upward by 5 units. The new heating schedule, $H = p(t)$, is graphed in Figure 2.24. The building's overnight temperature is now 55°F instead of 50°F and its daytime temperature is 75°F instead of 70°F. The 5°F increase in temperature corresponds to the 5-unit vertical shift in the graph.

Example 2 What is the relationship between the formula for $f(t)$, the original heating schedule and $p(t)$, the new heating schedule?

Solution The temperature under the new schedule, $p(t)$, is always 5°F warmer than the temperature under the old schedule, $f(t)$. Thus, at any time t,

$$\underbrace{\text{New temperature}}_{p(t)} = \underbrace{\text{Old temperature}}_{f(t)} + 5.$$

So the relationship between p and f is given by the equation

$$p(t) = f(t) + 5.$$

We can get information from the relationship $p(t) = f(t) + 5$ even though we do not have an explicit formula for f or p.

Suppose we want to know the temperature at 6 am under the schedule $p(t)$. The graph of $f(t)$ shows that under the old schedule $f(6) = 60$. Substituting $t = 6$ into the equation relating f and p gives

$$p(6) = f(6) + 5 = 60 + 5 = 65.$$

Thus, at 6 am the temperature under the new schedule is 65°F.

Example 3 Find $q(t)$, the formula for the heating schedule if at each time the temperature is 2°F lower than the original temperature.

Solution At any time t

$$\underbrace{\text{New temperature}}_{q(t)} = \underbrace{\text{Old temperature}}_{f(t)} - 2.$$

So we have

$$q(t) = f(t) - 2.$$

The graph of $q(t)$ is the graph of f shifted down by 2 units.

Generalizing these observations to any function g:

> If $g(x)$ is a function and k is a positive constant, then the graph of
> - $y = g(x) + k$ is the graph of $y = g(x)$ shifted vertically upward by k units.
> - $y = g(x) - k$ is the graph of $y = g(x)$ shifted vertically downward by k units.

Horizontal Shift: The Heating Schedule

Example 4 The superintendent then changes the original heating schedule to start two hours earlier. The building now begins to warm at 2 am instead of 4 am, reaches 70°F at 6 am instead of 8 am, begins cooling off at 2 pm instead of 4 pm, and returns to 50°F at 6 pm instead of 8 pm. How are these changes reflected in the graph of the heating schedule?

Solution Figure 2.25 gives a graph of $H = r(t)$, the new heating schedule, which is obtained by shifting the graph of the original heating schedule, $H = f(t)$, two units to the left.

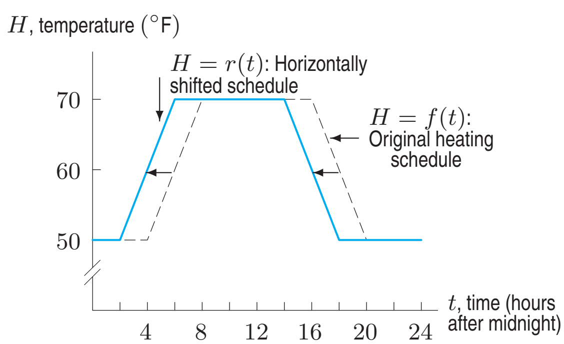


Figure 2.25: Graph of new heating schedule, $H = r(t)$, found by shifting f, the original graph, 2 units to the left

Notice that the upward shift in Example 1 results in a warmer temperature, whereas the leftward shift in Example 4 results in an earlier schedule.

Example 5 In Example 4 the heating schedule was changed to 2 hours earlier, shifting the graph horizontally 2 units to the left. Find a formula for r, this new schedule, in terms of f, the original schedule.

Solution The old schedule always reaches a given temperature 2 hours after the new schedule. For example, at 4 am the temperature under the new schedule reaches 60°. The temperature under the old schedule reaches 60° at 6 am, 2 hours later. In general, we see that

$$\begin{array}{c}\text{Temperature under new schedule} \\ \text{at time } t\end{array} = \begin{array}{c}\text{Temperature under old schedule} \\ \text{at time } (t+2)\text{, two hours later.}\end{array}$$

Algebraically, we have

$$r(t) = f(t+2).$$

This is a formula for r in terms of f.

Let's check the formula from Example 5 by using it to calculate $r(14)$, the temperature under the new schedule at 2 pm. The formula gives

$$r(14) = f(14+2) = f(16).$$

Figure 2.23 shows that $f(16) = 70$. Thus, $r(14) = 70$. This agrees with Figure 2.25.

Example 6 Suppose now the heating schedule is made 1 hour later than it was originally. Find the function $s(t)$ that describes this schedule.

Solution The new schedule reaches a particular temperature 1 hour later than the original. For example, under the old schedule, the temperature reaches 60°F at 6 am, while it reaches 60°F at 7 am under the new schedule. Thus

$$\begin{array}{c} \text{Temperature under new schedule} \\ \text{at time } t \end{array} = \begin{array}{l} \text{Temperature under old schedule} \\ \text{at time } (t-1)\text{, one hour earlier.} \end{array}$$

Thus, we have

$$s(t) = f(t-1).$$

The graph of $s(t)$ is the graph of f shifted to the right by 1 unit.

Generalizing these observations to any function g:

> If $g(x)$ is a function and h is a positive constant, then the graph of
> - $y = g(x+h)$ is the graph of $y = g(x)$ shifted horizontally to the left by h units.
> - $y = g(x-h)$ is the graph of $y = g(x)$ shifted horizontally to the right by h units.

A vertical or horizontal shift of the graph of a function is called a *translation* because it does not change the shape of the graph, but simply translates it to another position in the plane. Shifts and translations are examples of *transformations* of a function. We will see others in Chapter **??**.

Inside Versus Outside Changes

Since the horizontal shift in the heating schedule, $q(t) = f(t+2)$, involves a change to the input value, it is called an *inside change* to f. Similarly the vertical shift, $p(t) = f(t) + 5$, is called an *outside change* because it involves changes to the output value.

Example 7 If $n = f(A)$ gives the number of gallons of paint needed to cover a house of area A ft^2, explain the meaning of the expressions $f(A+10)$ and $f(A)+10$ in the context of painting.

Solution These two expressions are similar in that they both involve adding 10. However, for $f(A+10)$, the 10 is added on the inside, so 10 is added to the area, A. Thus,

$$n = f(\underbrace{A+10}_{\text{Area}}) = \begin{array}{c} \text{Amount of paint needed} \\ \text{to cover an area of } (A+10) \text{ ft}^2 \end{array} = \begin{array}{c} \text{Amount of paint needed to cover} \\ \text{an area 10 ft}^2 \text{ larger than } A. \end{array}$$

The expression $f(A) + 10$ represents an outside change. We are adding 10 to $f(A)$, which represents an amount of paint, not an area. We have

$$n = \underbrace{f(A)}_{\substack{\text{Amount}\\\text{of paint}}} + 10 = \begin{matrix}\text{Amount of paint needed}\\\text{to cover region of area } A\end{matrix} + 10 \text{ gals} = \begin{matrix}\text{10 gallons more paint than}\\\text{amount needed to cover area } A.\end{matrix}$$

In $f(A + 10)$, we added 10 square feet on the inside of the function, which means that the area to be painted is now 10 ft^2 larger. In $f(A) + 10$, we added 10 gallons to the outside, which means that we have 10 more gallons of paint than we need.

Combining Horizontal and Vertical Shifts

We have seen how a function's formula changes when we shift its graph horizontally or vertically. What happens when we shift it both horizontally and vertically?

Example 8 Let g be the transformation of the heating schedule function, $H = f(t)$, given by

$$g(t) = f(t - 2) - 5.$$

(a) Sketch the graph of $H = g(t)$.
(b) Describe in words the heating schedule determined by g.

Solution

(a) To graph g, we break the transformation into two steps. First, we sketch $H = f(t - 2)$. This is an inside change to the function f, which shifts the graph of f to the right 2 units. Next, we sketch $H = f(t - 2) - 5$ by shifting our sketch of $H = f(t - 2)$ down 5 units. The result is shown in Figure 2.26.
(b) The function g represents a schedule that is both 2 hours later and 5 degrees cooler than the original schedule.

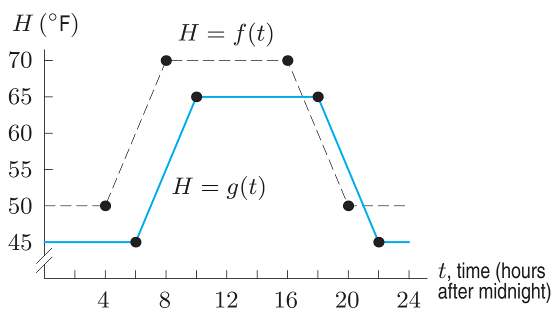


Figure 2.26: Graph of $g(t) = f(t - 2) - 5$ is graph of $H = f(t)$ shifted right by 2 and down by 5

Exercises and Problems for Section 2.4

Skill Refresher

In Exercises S1–S4, evaluate each function at $x = 4$.

S1. $f(x) = \sqrt{x}$

S2. $g(x) = \sqrt{x} + 6$

S3. $h(x) = \sqrt{x} - 3$

S4. $k(x) = \sqrt{x + 5}$

In Exercises S5–S8, solve for x.

S5. $x^2 = 4$

S6. $x^2 - 5 = 4$

S7. $(x - 1)^2 = 4$

S8. $(x + 1)^2 = 4$

Exercises

1. Using Table 2.14, complete the tables for g, h, k, m, where:

(a) $g(x) = f(x-1)$ **(b)** $h(x) = f(x+1)$
(c) $k(x) = f(x) + 3$ **(d)** $m(x) = f(x-1)+3$

Explain how the graph of each function relates to the graph of $f(x)$.

Table 2.14

x	-2	-1	0	1	2
$f(x)$	-3	0	2	1	-1

x	-1	0	1	2	3
$g(x)$					

x	-3	-2	-1	0	1
$h(x)$					

x	-2	-1	0	1	2
$k(x)$					

x	-1	0	1	2	3
$m(x)$					

2. Figure 2.27 shows $f(x)$. Graph $y = f(x+3)+3$. Label all important features.

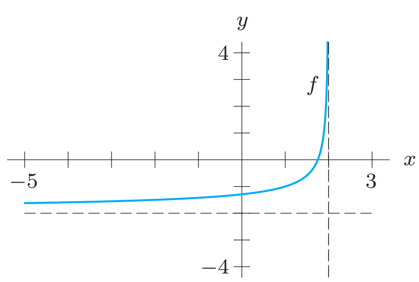


Figure 2.27

In Exercises 3–6, use Figure 2.28 to graph the transformation of f.

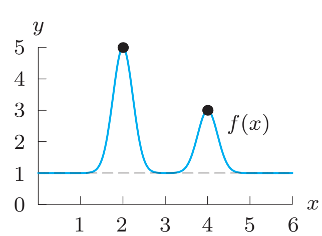


Figure 2.28

3. $y = f(x+2)$

4. $y = f(x) + 2$

5. $y = f(x-1) - 5$

6. $y = f(x+6) - 4$

7. The graph of $f(x)$ contains the point $(3, -4)$. What point must be on the graph of

(a) $f(x) + 5$? **(b)** $f(x+5)$?
(c) $f(x-3) - 2$?

8. The domain of the function $g(x)$ is $-2 < x < 7$. What is the domain of $g(x-2)$?

9. The range of the function $R(s)$ is $100 \le R(s) \le 200$. What is the range of $R(s) - 150$?

10. **(a)** Using Table 2.15, evaluate

(i) $f(x)$ for $x = 6$
(ii) $f(5) - 3$
(iii) $f(5-3)$
(iv) $g(x) + 6$ for $x = 2$
(v) $g(x+6)$ for $x = 2$
(vi) $3g(x)$ for $x = 0$
(vii) $f(3x)$ for $x = 2$
(viii) $f(x) - f(2)$ for $x = 8$
(ix) $g(x+1) - g(x)$ for $x = 1$

(b) Using the values in the table, solve

(i) $g(x) = 6$ (ii) $f(x) = 574$
(iii) $g(x) = 281$

(c) The values in the table were obtained using the formulas $f(x) = x^3 + x^2 + x - 10$ and $g(x) = 7x^2 - 8x - 6$. Use the table to find two solutions to the equation $x^3 + x^2 + x - 10 = 7x^2 - 8x - 6$.

Table 2.15

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	-10	-7	4	29	74	145	248	389	574	809
$g(x)$	-6	-7	6	33	74	129	198	281	378	489

Problems

11. Figure 2.29 shows $y = f(x)$. Give a formula in terms of f for the function in Figure 2.30. Your formula should be of the form $y = f(x - h) + k$ for appropriate constants h and k.

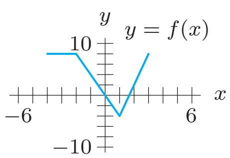


Figure 2.29

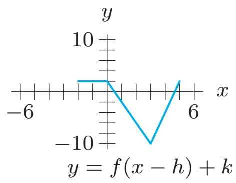


Figure 2.30

12. Figure 2.31 shows $y = m(r)$. Each graph in parts (a)–(d) is a translation of the graph of $y = m(r)$. Give a formula for each of these functions in terms of m.

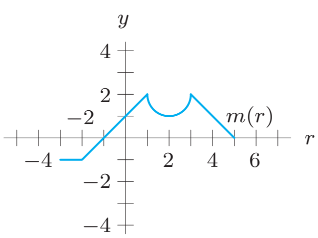
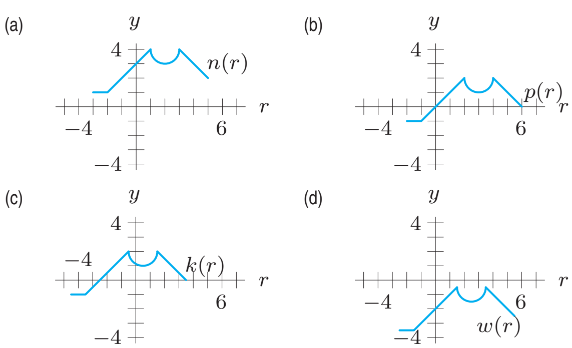


Figure 2.31



13. The graph of $g(x)$ contains the point $(-2, 5)$. Write a formula for a translation of g whose graph contains the point

(a) $(-2, 8)$ **(b)** $(0, 5)$

14. Table 2.16 contains values of $f(x)$. Each function in parts (a)–(c) is a translation of $f(x)$. Find a possible formula for each of these functions in terms of f. For example, given the data in Table 2.17, you could say that $k(x) = f(x) + 1$.

Table 2.16

x	0	1	2	3	4	5	6	7
$f(x)$	0	0.5	2	4.5	8	12.5	18	24.5

Table 2.17

x	0	1	2	3	4	5	6	7
$k(x)$	1	1.5	3	5.5	9	13.5	19	25.5

(a)

x	0	1	2	3	4	5	6	7
$h(x)$	−2	−1.5	0	2.5	6	10.5	16	22.5

(b)

x	0	1	2	3	4	5	6	7
$g(x)$	0.5	2	4.5	8	12.5	18	24.5	32

(c)

x	0	1	2	3	4	5	6	7
$i(x)$	−1.5	0	2.5	6	10.5	16	22.5	30

15. Tables 2.18 and 2.19 give values of functions v and w. Given that $w(x) = v(x - h) + k$, find the constants h and k.

Table 2.18

x	$v(x)$
−2	11
−1	17
0	20
1	17
2	11

Table 2.19

x	$w(x)$
3	4
4	10
5	13
6	10
7	4

In Problems 16–17, let $s(t)$ denote the average weight (in pounds) of a baby at age t months.

16. The weight, V, of a particular baby named Jonah is related to the average weight function $s(t)$ by the equation

$$V = s(t) + 2.$$

Find Jonah's weight at ages $t = 3$ and $t = 6$ months. What can you say about Jonah's weight in general?

17. The weight, W, of another baby named Ben is related to $s(t)$ by the equation
$$W = s(t+4).$$
What can you say about Ben's weight at age $t = 3$ months? At $t = 6$ months? Assuming that babies increase in weight over the first year of life, decide if Ben is of average weight for his age, above average, or below average.

18. The function $P(t)$ gives the number of people in a certain population in year t. Interpret in terms of population:

(a) $P(t) + 100$ **(b)** $P(t + 100)$

In Problems 19–24, explain in words the effect of the transformation on the graph of $q(z)$ for positive constants a, b.

19. $q(z) + 3$

20. $q(z) - a$

21. $q(z + 4)$

22. $q(z - a)$

23. $q(z + b) - a$

24. $q(z - 2b) + ab$

25. Let $T(d)$ give the average temperature in your hometown on the d^{th} day of last year (so $d = 1$ is January 1, etc).

(a) Graph $T(d)$ for $1 \leq d \leq 365$.

(b) Give a possible value for each of the following: $T(6)$; $T(100)$; $T(215)$; $T(371)$.

(c) What is the relationship between $T(d)$ and $T(d + 365)$? Explain.

(d) If you graph $w(d) = T(d + 365)$ on the same axes as $T(d)$, how would the two graphs compare?

(e) Do you think the function $T(d) + 365$ has any practical significance? Explain.

26. Let $S(d)$ give the height of high tide in Seattle on a specific day, d, of the year. Use shifts of the function $S(d)$ to find formulas for each of the following functions:

(a) $T(d)$, the height of high tide in Tacoma on day d, if we assume that high tide in Tacoma is always one foot higher than high tide in Seattle.

(b) $A(d)$, the height of high tide in Astoria on day d, if we assume that high tide in Astoria is the same height as the previous day's high tide in Seattle.

27. Let $H(t)$ be the thermometer reading (Celsius) of a person t hours after onset of an illness. Normal body temperature is 37°C. Find a formula for the fever, the number $f(t)$ of degrees greater than normal, after t hours.

2.5 PREVIEW OF COMPOSITE AND INVERSE FUNCTIONS

Composition of Functions

Two functions may be connected by the fact that the output of one is the input of the other. For example, to find the cost, C, in dollars, to paint an area A square feet, we need to know the number, n, of gallons of paint required. Since one gallon covers 250 square feet, we have the function $n = f(A) = A/250$. If paint is \$30.50 a gallon, we have the function $C = g(n) = 30.5n$. We substitute $n = f(A)$ into $g(n)$ to find the cost C as a function of A.

Example 1 Find a formula for cost, C, as a function of area, A, to be painted.

Solution Since we have
$$C = 30.5n \quad \text{and} \quad n = \frac{A}{250},$$
substituting for n in the formula for C gives
$$C = 30.5\frac{A}{250} = 0.122A.$$

We say that C is a "function of a function", or *composite function*. If the function giving C in terms of A is called h, so $C = h(A)$, then we write
$$C = h(A) = g(f(A)).$$
The function h is said to be the *composition* of the functions f and g. We say f is the *inside* function and g is the *outside* function. In this example, the composite function $C = h(A) = g(f(A))$ tells us the cost of painting an area of A square feet.

For two functions $f(t)$ and $g(t)$, the function $f(g(t))$ is said to be a **composition** of f with g. The function $f(g(t))$ is defined by using the output of the function g as the input to f.

The composite function $f(g(t))$ is defined only for values in the domain of g whose $g(t)$ values are in the domain of f.

Example 2 The air temperature, T, in °F, is given in terms of the chirp rate, R, in chirps per minute, of a snowy tree cricket by the function

$$T = f(R) = \frac{1}{4}R + 40.$$

If the chirp rate varies with the number of hours since midnight, x, according to the function

$$R = g(x) = 20 + x^2,$$

find how temperature varies with time by obtaining a formula for h, where $T = h(x)$.

Solution Since $f(R)$ is a function of R and $R = g(x)$, we see that g is the inside function and f is the outside function. Thus we substitute $R = g(x)$ into f:

$$T = f(R) = f(g(x)) = \frac{1}{4}g(x) + 40 = \frac{1}{4}(20 + x^2) + 40 = \frac{1}{4}x^2 + 45.$$

Thus, for $0 \le x \le 10$, we have

$$T = h(x) = \frac{1}{4}x^2 + 45.$$

Example 3 shows another example of composition.

Example 3 Let $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Calculate $f(g(3))$ and $g(f(3))$.
(b) Find formulas for $f(g(x))$ and $g(f(x))$.

Solution (a) We want

$$f(g(3)).$$

We start by evaluating $g(3)$. The formula for g gives $g(3) = 3^2 - 3 = 6$, so

$$f(g(3)) = f(6).$$

The formula for f gives $f(6) = 2 \cdot 6 + 1 = 13$, so

$$f(g(3)) = 13.$$

To calculate $g(f(3))$, we have

$$\begin{aligned} g(f(3)) &= g(7) && \text{Because } f(3) = 2 \cdot 3 + 1 = 7 \\ &= 46 && \text{Because } g(7) = 7^2 - 3 \end{aligned}$$

Notice that, $f(g(3)) \neq g(f(3))$. The functions $f(g(x))$ and $g(f(x))$ are different.

(b) In the formula for $f(g(x))$,

$$\begin{aligned} f(\underbrace{g(x)}_{\text{Input for } f}) &= f(x^2 - 3) && \text{Because } g(x) = x^2 - 3 \\ &= 2(x^2 - 3) + 1 && \text{Because } f(\text{Input}) = 2 \cdot \text{Input} + 1 \\ &= 2x^2 - 5. \end{aligned}$$

Check this formula by evaluating $f(g(3))$, which we know to be 13:

$$f(g(3)) = 2 \cdot 3^2 - 5 = 13.$$

In the formula for $g(f(x))$,

$$\begin{aligned} g(\underbrace{f(x)}_{\text{Input for } g}) &= g(2x+1) && \text{Because } f(x) = 2x+1 \\ &= (2x+1)^2 - 3 && \text{Because } g(\text{Input}) = \text{Input}^2 - 3 \\ &= 4x^2 + 4x - 2. \end{aligned}$$

Check this formula by evaluating $g(f(3))$, which we know to be 46:

$$g(f(3)) = 4 \cdot 3^2 + 4 \cdot 3 - 2 = 46.$$

Inverse Functions

The roles of a function's input and output can sometimes be reversed. For example, the population, P, of birds on an island is given, in thousands, by $P = f(t)$, where t is the number of years since 2007. In this function, t is the input and P is the output. If the population is increasing, knowing the population enables us to calculate the year. Thus we can define a new function, $t = g(P)$, which tells us the value of t given the value of P instead of the other way round. For this function, P is the input and t is the output.The functions f and g are called *inverses* of each other. A function which has an inverse is said to be *invertible*.

The fact that f and g are inverse functions means that they go in "opposite directions." The function f takes t as input and outputs P, while g takes P as input and outputs t.

Inverse Function Notation

In the preceding discussion, there was nothing about the names of the two functions that stressed their special relationship. If we want to emphasize that g is the inverse of f, we call it f^{-1} (read "f-inverse"). To express the fact that the population of birds, P, is a function of time, t, we write

$$P = f(t).$$

To express the fact that the time t is also determined by P, so that t is a function of P, we write

$$t = f^{-1}(P).$$

The symbol f^{-1} is used to represent the function that gives the output t for a given input P.

Warning: The -1 that appears in the symbol f^{-1} for the inverse function is not an exponent. Unfortunately, the notation $f^{-1}(x)$ might lead us to interpret it as $(f(x))^{-1} = \frac{1}{f(x)}$. The two expressions are not the same in general: $f^{-1}(x)$ is the output when x is fed into the inverse of f, while $(f(x))^{-1} = \frac{1}{f(x)}$ is the reciprocal of the number we get when x is fed into f.

Example 4 Using $P = f(t)$, where P represents the population, in thousands, of birds on an island and t is the number of years since 2007:

(a) What does $f(4)$ represent? (b) What does $f^{-1}(4)$ represent?

Solution

(a) The expression $f(4)$ is the bird population (in thousands) in the year 2011.

(b) Since f^{-1} is the inverse function, f^{-1} is a function which takes population as input and returns time as output. Therefore, $f^{-1}(4)$ is the number of years after 2007 at which there were $4{,}000$ birds on the island.

Example 5 Suppose that g is an invertible function, with $g(10) = -26$ and $g^{-1}(0) = 7$. What other values of g and g^{-1} do you know?

Solution Because $g(10) = -26$, we know that $g^{-1}(-26) = 10$; because $g^{-1}(0) = 7$, we know that $g(7) = 0$.

Not all functions are invertible. We revisit this issue in Chapter **??** where we see how to tell whether a function has an inverse.

Finding a Formula for the Inverse Function

In the next example, we find the formula for an inverse function.

Example 6 The cricket function, which gives temperature, T, in terms of chirp rate, R, is

$$T = f(R) = \frac{1}{4} \cdot R + 40.$$

Find a formula for the inverse function, $R = f^{-1}(T)$.

Solution The inverse function gives the chirp rate in terms of the temperature, so we solve the following equation for R:

$$T = \frac{1}{4} \cdot R + 40,$$

giving

$$\begin{aligned} T - 40 &= \frac{1}{4} \cdot R \\ R &= 4(T - 40). \end{aligned}$$

Thus, $R = f^{-1}(T) = 4(T - 40)$.

Domain and Range of an Inverse Function

The input values of the inverse function f^{-1} are the output values of the function f. Thus, the domain of f^{-1} is the range of f. For the cricket function, $T = f(R) = \frac{1}{4}R + 40$, if a realistic domain is $0 \leq R \leq 160$, then the range of f is $40 \leq T \leq 80$. The domain of f^{-1} is then $40 \leq T \leq 80$ and the range is $0 \leq R \leq 160$. See Figure 2.32.

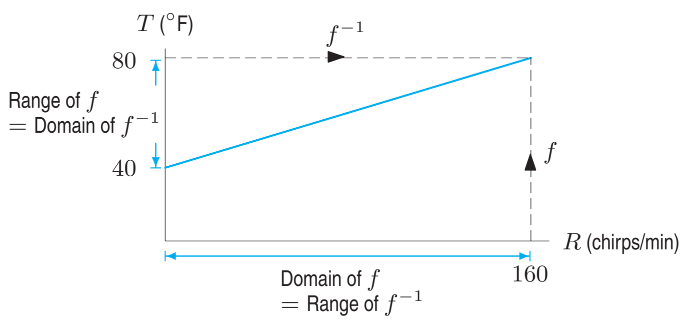


Figure 2.32: Domain and range of an inverse function

Relation Between Composition and Inverses: A Function and Its Inverse Undo Each Other

Composing a function with its inverse gives a striking result, as we see in the next example.

Example 7 Calculate the composite functions $f^{-1}(f(R))$ and $f(f^{-1}(T))$ for the cricket example. Interpret the results.

Solution Since $f(R) = \frac{1}{4}R + 40$. and $f^{-1}(T) = 4(T - 40)$, we have

$$f^{-1}(f(R)) = f^{-1}\left(\frac{1}{4} \cdot R + 40\right) = 4\left(\left(\frac{1}{4} \cdot R + 40\right) - 40\right) = R.$$

$$f(f^{-1}(T)) = f(4(T - 40)) = \frac{1}{4}(4(T - 40)) + 40 = T.$$

To interpret these results, we use the fact that $f(R)$ gives the temperature corresponding to chirp rate R, and $f^{-1}(T)$ gives the chirp rate corresponding to temperature T. Thus $f^{-1}(f(R))$ gives the chirp rate at temperature $f(R)$, which is R. Similarly, $f(f^{-1}(T))$ gives the temperature at chirp rate $f^{-1}(T)$, which is T.

In Example 7, we see that $f^{-1}(f(R)) = R$ and $f(f^{-1}(T)) = T$. This illustrates the following result, which we see is true in general in Chapter **??**.

> The functions f and f^{-1} are called inverses because they "undo" each other when composed.

Example 8 Let $y = f(x) = 2x + 8$.

(a) Find a formula for the inverse function $x = f^{-1}(y)$.
(b) Show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

Solution (a) We have $y = 2x + 8$, so we solve this equation for x:

$$\begin{aligned} y &= 2x + 8 \\ y - 8 &= 2x \\ 0.5y - 4 &= x \end{aligned}$$

We see that $x = f^{-1}(y) = 0.5y - 4$.

(b) We have:

$$f^{-1}(f(x)) = f^{-1}(2x+8) = 0.5(2x+8) - 4 = (x+4) - 4 = x.$$

and

$$f(f^{-1}(y)) = f(0.5y - 4) = 2(0.5y - 4) + 8 = (y - 8) + 8 = y.$$

We see that this result is always true in Chapter **??**.

Exercises and Problems for Section 2.5

Skill Refresher

Solve the equations in Exercises S1–S7 for y.

S1. $x = 3y - 4$

S2. $4x - 3y = 7$

S3. $x = \dfrac{2y+1}{y-2}$

S4. $x = \dfrac{4y}{1-y}$

S5. $x = \sqrt{y} - 2$

S6. $x = y^3 - 4$

S7. $x = 5 - (2y)^3$

Simplify the expressions in Exercises S8–S11.

S8. $5\left(\frac{1}{5}x - 1\right) + 5$

S9. $(2x+1)^2 - 4$

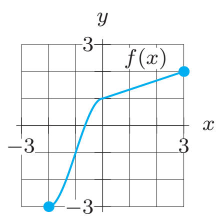
S10. $3(y-2)^2 - 7$

S11. $(1-t)^2 - (1-t)$

Exercises

In Exercises 1–4, use the complete graph of the invertible function $f(x)$ shown to estimate the domain of $f^{-1}(y)$.

1.



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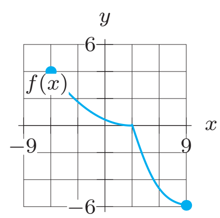

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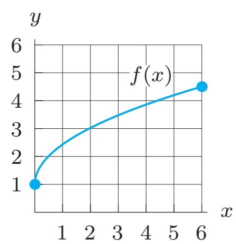

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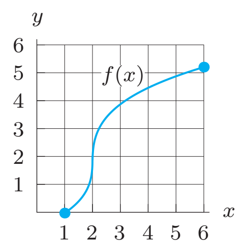


In Exercises 5–12, use $f(x) = 3x - 1$ and $g(x) = 1 - x^2$.

5. $f(g(0))$

6. $g(f(0))$

7. $g(f(2))$

8. $f(g(2))$

9. $f(g(x))$

10. $g(f(x))$

11. $f(f(x))$

12. $g(g(x))$

In Exercises 13–15, give the meaning and units of the composite function.

13. $A(f(t))$, where $r = f(t)$ is the radius, in centimeters, of a circle at time t minutes, and $A(r)$ is the area, in square centimeters, of a circle of radius r centimeters.

14. $R(f(p))$, where $Q = f(p)$ is the number of barrels of oil sold by a company when the price is p dollars/barrel and $R(Q)$ is the revenue earned in millions of dollars from a sale of Q barrels.

15. $C(A(d))$, where $A(d)$ is the area of a circular pizza with diameter d, and $C(x)$ is the price of pizza with area x.

In Exercises 16–20, give the meaning and units of the inverse function. (Assume f is invertible.)

16. $P = f(t)$ is population in millions in year t.

17. $T = f(H)$ is time in minutes to bake a cake at H°F.

18. $N = f(t)$ is number of inches of snow in the first t days of January.

19. $x = C(w)$ is calories for w ounces of almonds.

20. $c = P(d)$ is the price, in dollars, for a pizza of diameter d inches.

In Exercises 21–27, find the inverse function.

21. $y = f(t) = 2t + 3$

22. $y = f(x) = 3x - 7$

23. $y = g(x) = x^3 + 1$

24. $Q = f(x) = x^3 + 3$

25. $A = f(r) = \pi r^2, r \geq 0$

26. $y = g(s) = 1 + \dfrac{1}{s}$

27. $P = f(x) = \dfrac{3x}{2x+1}$

28. For $f(x) = 5x$, evaluate $f(4)$ and $f^{-1}(15)$.

29. For $k(x) = 4x - 10$, evaluate $k(3)$ and $k^{-1}(14)$.

30. For $h(x) = \dfrac{x}{4} + 2$, evaluate $h\left(-\frac{1}{2}\right)$ and $h^{-1}(8)$.

31. For $g(x) = 2x^3 - 1$, evaluate $g(\frac{3}{2})$ and $g^{-1}(-17)$.

32. Use the graph in Figure 2.33 to fill in the missing values:

(a) $f(0) = ?$ **(b)** $f(?) = 0$
(c) $f^{-1}(0) = ?$ **(d)** $f^{-1}(?) = 0$

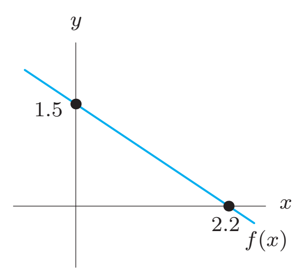


Figure 2.33

33. Use the graph in Figure 2.34 to fill in the missing values:

(a) $f(0) = ?$ **(b)** $f(?) = 0$
(c) $f^{-1}(0) = ?$ **(d)** $f^{-1}(?) = 0$

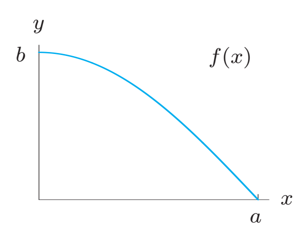


Figure 2.34

Problems

34. The cost (in dollars) of producing x air conditioners is $C = g(x) = 600 + 45x$. Find a formula for the inverse function $g^{-1}(C)$.

In Problems 35–37, let $n = f(A) = A/250$, where n is the number of gallons of paint needed for an area of A square feet.

35. Find a formula for the inverse function $A = f^{-1}(n)$.

36. Interpret and evaluate $f(100)$ and $f^{-1}(100)$.

37. Calculate the composite functions $f^{-1}(f(A))$ and $f(f^{-1}(n))$. Explain the results.

38. The daily cost (in dollars) of renting a car and driving up to 500 miles a day is $C = f(m) = 32 + 0.19m$, where m is the number of miles driven.

(a) Find the domain and range of $f^{-1}(C)$.
(b) Find a formula for the inverse function $f^{-1}(C)$.

39. Table 2.20 gives values of an invertible function, f.

(a) Using the table, fill in the missing values:

(i) $f(0) = ?$ (ii) $f(?) = 0$
(iii) $f^{-1}(0) = ?$ (iv) $f^{-1}(?) = 0$

(b) How do the answers to (i)–(iv) in part (a) relate to one another? In particular, how could you have obtained the answers to (iii) and (iv) from the answers to (i) and (ii)?

Table 2.20

x	-2	-1	0	1	2
$f(x)$	5	4	2	0	-3

40. Suppose that $j(x) = h^{-1}(x)$ and that both j and h are defined for all values of x. Let $h(4) = 2$ and $j(5) = -3$. Evaluate if possible:

(a) $j(h(4))$ **(b)** $j(4)$ **(c)** $h(j(4))$
(d) $j(2)$ **(e)** $h^{-1}(-3)$ **(f)** $j^{-1}(-3)$
(g) $h(5)$ **(h)** $(h(-3))^{-1}$ **(i)** $(h(2))^{-1}$

In Problems 41–44, find the domain and range of the function by first finding the formula for the inverse function.

41. $t(a) = \sqrt[3]{a+1}$ **42.** $n(r) = r^3 + 2$

43. $m(x) = \dfrac{1}{\sqrt{x-2}}$ **44.** $p(x) = \dfrac{1}{\sqrt{3-x}}$

45. The gross domestic product (GDP) of the US is given by $G(t)$ where t is the number of years since 2000 and the units of G are billions of dollars.[13]

(a) What is meant by $G(13) = 16{,}011.2$?
(b) What is meant by $G^{-1}(13{,}776) = 7$?

46. The cost of producing q thousand loaves of bread is $C(q)$ dollars. Interpret the following statements in terms of bread; give units.

(a) $C(5) = 653$
(b) $C^{-1}(80) = 0.62$
(c) The solution to $C(q) = 790$ is 6.3
(d) The solution to $C^{-1}(x) = 1.2$ is 150

47. The perimeter of a square of side s is given by $P = f(s) = 4s$. Find and interpret

(a) $f(3)$ **(b)** $f^{-1}(20)$ **(c)** $f^{-1}(P)$

48. The cost, C, in thousands of dollars, of producing q kg of a chemical is given by $C = f(q) = 100 + 0.2q$. Find and interpret

(a) $f(10)$ **(b)** $f^{-1}(200)$ **(c)** $f^{-1}(C)$

In Problems 49–51, let $H = f(t) = \frac{5}{9}(t - 32)$, where H is temperature in degrees Celsius and t is in degrees Fahrenheit.

49. Find and interpret the inverse function, $f^{-1}(H)$.

50. Using the results of Problem 49, evaluate and interpret:

(a) $f(0)$ **(b)** $f^{-1}(0)$
(c) $f(100)$ **(d)** $f^{-1}(100)$

51. The temperature, $t = g(n) = 68 + 10 \cdot 2^{-n}$, in degrees Fahrenheit of a room is a function of the number, n, of hours that the air conditioner has been running. Find and interpret $f(g(n))$. Give units.

52. The period, T, of a pendulum of length l is given by $T = f(l) = 2\pi\sqrt{l/g}$, where g is a constant. Find a formula for $f^{-1}(T)$ and explain its meaning.

53. The area, in square centimeters, of a circle whose radius is r cm is given by $A = \pi r^2$.

(a) Write this formula using function notation, where f is the name of the function.
(b) Evaluate $f(0)$.
(c) Evaluate and interpret $f(r+1)$.
(d) Evaluate and interpret $f(r) + 1$.
(e) What are the units of $f^{-1}(4)$?

[13] http://www.businessinsider.com/us-q1-2013-gdp-first-estimate-2013-4. Accessed April, 2013.

54. The radius, r, in centimeters, of a melting snowball is given by $r = 50 - 2.5t$, where t is time in hours. The snowball is spherical, with volume $V = \frac{4}{3}\pi r^3$ cm^3. Find a formula for $V = f(t)$, the volume of the snowball as a function of time.

55. The area, $A = f(d)$ in^2, of a circular pizza is a function of the diameter d, in inches. A package of pepperoni costs \$2.99 and covers 250 square inches of pizza.

(a) Write the formula for $f(d)$.
(b) Find a formula for $C = g(A)$, the cost in dollars of adding pepperoni to a pizza of area A in^2.
(c) Find and interpret $C = g(f(d))$.
(d) Evaluate and interpret $f(11)$ and $g(11)$.

56. A circular oil slick is expanding with radius, r in yards, at time t in hours given by $r = 2t - 0.1t^2$, for $0 \leq t \leq 10$. Find a formula for the area in square yards, $A = f(t)$, as a function of time.

57. Table 2.21 gives values for $d = f(v)$, where d, in meters, is the stopping distance for a car traveling at velocity v kilometers per hour.

(a) Evaluate and interpret $f(60)$.
(b) Estimate $f(70)$.
(c) Evaluate and interpret $f^{-1}(70)$.

Table 2.21

v (km/hr)	20	30	40	50	60	80	90	100	120
d (m)	6	10	16	23	30	50	60	70	100

58. Let $f(a)$ be the cost in dollars of a pounds of organic apples at Fresh Abundance[14] in November 2009. What do the following statements tell you? What are the units of each of the numbers?

(a) $f(2) = 2.80$ **(b)** $f(0.5) = 0.70$
(c) $f^{-1}(0.35) = 0.25$ **(d)** $f^{-1}(7) = 5$

59. Carbon dioxide is one of the greenhouse gases that are believed to affect global warming. Between 2004 and 2008, the concentration of carbon dioxide in the earth's atmosphere increased steadily from 375 parts per million (ppm) to 383 ppm.[15] Let $C(t)$ be the concentration in ppm of carbon dioxide t years after 2004 during these four years.

(a) State the domain and range of $C(t)$.
(b) What is the practical meaning of $C(4)$? What is its value?
(c) What does $C^{-1}(381)$ represent?

2.6 CONCAVITY

Concavity and Rates of Change

The graph of a linear function is a straight line because the average rate of change is constant. However, not all graphs are straight lines; they may bend up or down. The function giving salary S, in \$1000s, as a function of t, time in years since being hired, is shown in Table 2.22 and Figure 2.35. Since the rate of change increases with time, the slope of the graph increases as t increases, so the graph bends upward. We say such graphs are *concave up*.

[14] www.freshabundance.com/index.php?main_page=product_info&products_id=254, accessed November 5, 2009.
[15] *The World Almanac and Book of Facts 2010,* page 297.

Table 2.22 *Salary: Increasing rate of change*

t (years)	S (\$1000s)	Rate of change $\Delta S/\Delta t$
0	40	
		3.2
10	72	
		5.6
20	128	
		10.2
30	230	
		18.1
40	411	

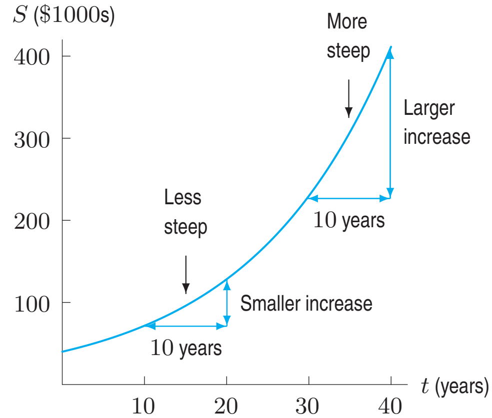


Figure 2.35: Graph of salary function is concave up because rate of change increases

The next example shows that a decreasing function can also be concave up.

Example 1 Table 2.23 shows Q, the quantity of carbon-14 (in μg) in a 200 μg sample remaining after t thousand years. We see from Figure 2.36 that Q is a decreasing function of t, so its rate of change is always negative. What can we say about the concavity of the graph, and what does this mean about the rate of change of the function?

Table 2.23 *Carbon-14: Increasing rate of change*

t (thousand years)	Q (μg)	Rate of change $\Delta Q/\Delta t$
0	200	
		-18.2
5	109	
		-9.8
10	60	
		-5.4
15	33	

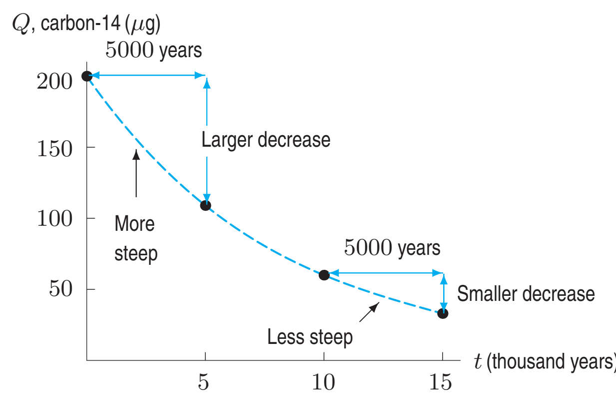


Figure 2.36: Graph of the quantity of carbon-14 is concave up

Solution The graph bends upward, so it is concave up. Table 2.24 shows that the rate of change of the function is increasing, because the rate is becoming less negative. Figure 2.36 shows how the increasing rate of change can be visualized on the graph: the slope is negative and increasing.

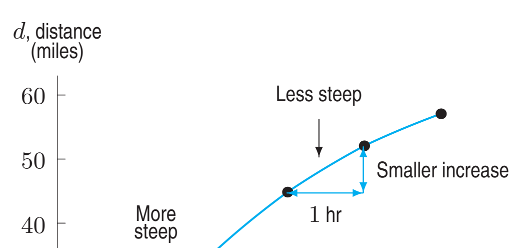
Graphs can bend downward; we call such graphs *concave down*.

Example 2 Table 2.24 gives the distance traveled by a cyclist, Karim, as a function of time. What is the concavity of the graph? Was Karim's speed (that is, the rate of change of distance with respect to time) increasing, decreasing, or constant?

Solution Table 2.24 shows Karim's speed was decreasing throughout the trip. Figure 2.37 shows how the decreasing speed leads to a decreasing slope and a graph which bends downward; thus the graph is concave down.

Table 2.24 *Karim's distance as a function of time, with the average speed for each hour*

t, time (hours)	d, distance (miles)	Average speed, $\Delta d/\Delta t$ (mph)
0	0	
		20 mph
1	20	



Summary: Increasing and Decreasing Functions; Concavity

Knowing that a function is either increasing or decreasing and either concave up or concave down at a point allows us to determine the basic shape of the graph at that point. The four basic shapes are in Figures 2.38–2.41:

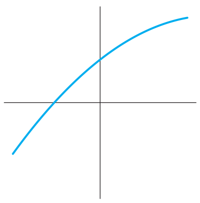


Figure 2.38: Increasing and concave down

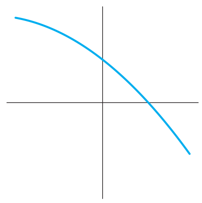


Figure 2.39: Decreasing and concave down

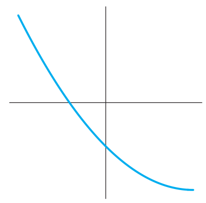


Figure 2.40: Decreasing and concave up

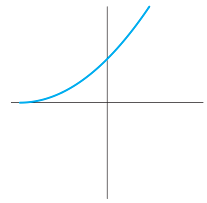


Figure 2.41: Increasing and concave up

Figures 2.38–2.41: Relationship between concavity and rate of change:

- If f is a function whose rate of change increases (gets less negative or more positive as we move from left to right[16]), then the graph of f is **concave up**. That is, the graph bends upward.
- If f is a function whose rate of change decreases (gets less positive or more negative as we move from left to right), then the graph of f is **concave down**. That is, the graph bends downward.

If a function has a constant rate of change, its graph is a line and it is neither concave up nor concave down.

Exercises and Problems for Section 2.6

Exercises

Do the graphs of the functions in Exercises 1–8 appear to be concave up, concave down, or neither?

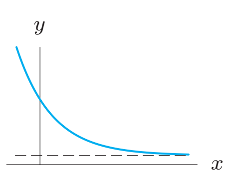
1.

x	0	1	3	6
$f(x)$	1.0	1.3	1.7	2.2

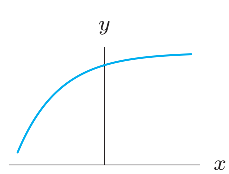
2.

t	0	1	2	3	4
$f(t)$	20	10	6	3	1

3.



4.



5. $y = x^2$

6. $y = -x^2$

7. $y = x^3, x > 0$

8. $y = x^3, x < 0$

9. Calculate successive rates of change for the function, $H(x)$, in Table 2.25 to decide whether you expect the graph of $H(x)$ to be concave up or concave down.

Table 2.25

x	12	15	18	21
$H(x)$	21.40	21.53	21.75	22.02

10. Calculate successive rates of change for the function, $R(t)$, in Table 2.26 to decide whether you expect the graph of $R(t)$ to be concave up or concave down.

Table 2.26

t	1.5	2.4	3.6	4.8
$R(t)$	-5.7	-3.1	-1.4	0

11. Sketch a graph that is everywhere negative, increasing, and concave down.

12. Sketch a graph that is everywhere positive, increasing, and concave up.

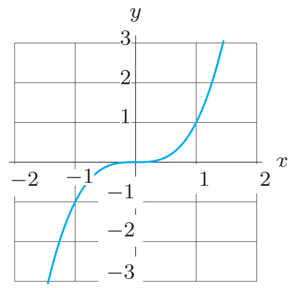
[16] In fact, we need to take the average rate of change over an arbitrarily small interval.

Problems

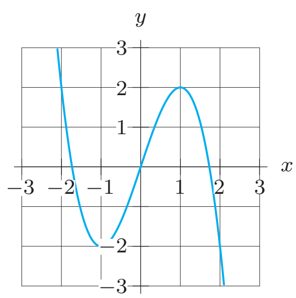
In Problems 13–16, estimate the intervals where the graph is

(a) Increasing and concave down
(b) Decreasing and concave down
(c) Increasing and concave up
(d) Decreasing and concave up

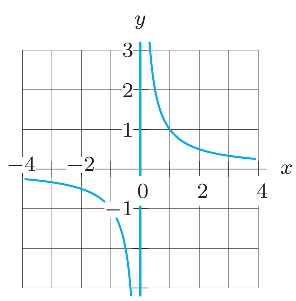
13.



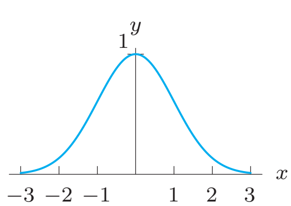
14.



15.



16.



Are the functions in Problems 17–21 increasing or decreasing? What does the scenario tell you about the concavity of the graph modeling it?

17. When a drug is injected into a person's bloodstream, the amount of the drug present in the body increases rapidly at first. If the person receives daily injections, the body metabolizes the drug so that the amount of the drug present in the body continues to increase, but at a decreasing rate. Eventually, the quantity levels off at a saturation level.

18. After a cup of hot chocolate is poured, the temperature cools off very rapidly at first, and then cools off more slowly, until the temperature of the hot chocolate eventually reaches room temperature.

19. When a rumor begins, the number of people who have heard the rumor increases slowly at first. As the rumor spreads, the rate of increase gets greater (as more people continue to tell their friends the rumor), and then slows down again (when almost everyone has heard the rumor).

20. When money is deposited in the bank, the amount of money increases slowly at first. As the size of the account increases, the amount of money increases more rapidly, since the account is earning interest on the new interest, as well as on the original amount.

21. When a new product is introduced, the number of people who use the product increases slowly at first, and then the rate of increase is faster (as more and more people learn about the product). Eventually, the rate of increase slows down again (when most people who are interested in the product are already using it).

22. Graph $f(x)$ with all of these properties:

- $f(0) = 4$
- f is decreasing and concave up for $-\infty < x < 0$
- f is increasing and concave up for $0 < x < 6$
- f is increasing and concave down for $6 < x < 8$
- f is decreasing and concave down for $x > 8$

23. An incumbent politician running for reelection declared that the number of violent crimes is no longer rising and is presently under control. Does the graph shown in Figure 2.42 support this claim? Why or why not?

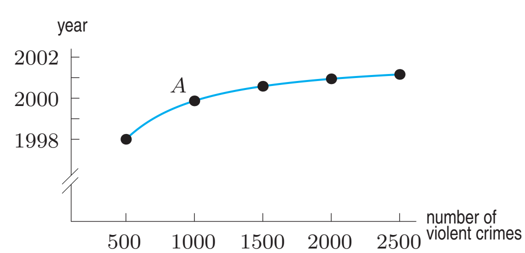


Figure 2.42

24. The rate at which water is entering a reservoir is given for time $t > 0$ by the graph in Figure 2.43. A negative rate means that water is leaving the reservoir. In parts (a)–(d), give the largest interval on which:

(a) The volume of water is increasing.
(b) The volume of water is constant.
(c) The volume of water is increasing fastest.
(d) The volume of water is decreasing.

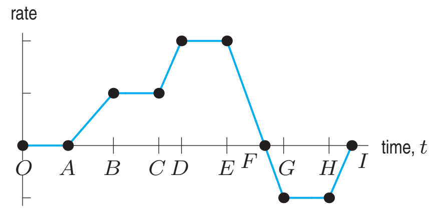


Figure 2.43

25. Match each of the following descriptions with an appropriate graph and table of values.

(a) The weight of your jumbo box of Fruity Flakes decreases by an equal amount every week.
(b) The machinery depreciated rapidly at first, but its value declined more slowly as time went on.
(c) In free fall, your distance from the ground decreases faster and faster.
(d) For a while it looked as if the decline in profits was slowing down, but then they began declining ever more rapidly.

(E)

x	0	1	2	3	4	5
y	400	384	336	256	144	0

(F)

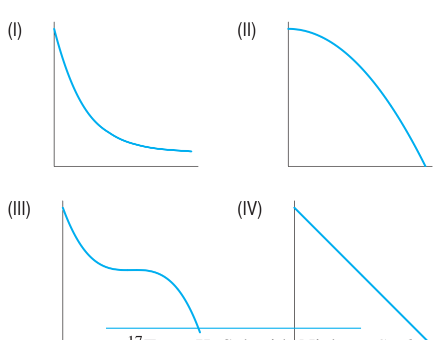
x	0	1	2	3	4	5
y	400	320	240	160	80	0

(G)

x	0	1	2	3	4	5
y	400	184	98	63	49	43

(H)

x	0	1	2	3	4	5
y	412	265	226	224	185	38



26. Match each story with the table and graph which best represent it.

(a) When you study a foreign language, the number of new verbs you learn increases rapidly at first, but slows almost to a halt as you approach your saturation level.
(b) You board an airplane in Philadelphia heading west. Your distance from the Atlantic Ocean, in kilometers, increases at a constant rate.
(c) The interest on your savings plan is compounded annually. At first your balance grows slowly, but its rate of growth continues to increase.

(E)

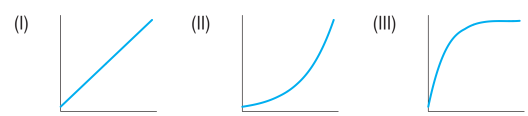
x	0	5	10	15	20	25
y	20	275	360	390	395	399

(F)

x	0	5	10	15	20	25
y	20	36	66	120	220	400

(G)

x	0	5	10	15	20	25
y	20	95	170	245	320	395



27. The relationship between the swimming speed U (in cm/sec) of a salmon to the length l of the salmon (in cm) is given by the function[17]

$$U = 19.5\sqrt{l}.$$

(a) If one salmon is 4 times the length of another salmon, how are their swimming speeds related?
(b) Graph the function $U = 19.5\sqrt{l}$. Describe the graph using words such as increasing, decreasing, concave up, concave down.
(c) Using a property that you described in part (b), answer the question "Do larger salmon swim faster than smaller ones?"
(d) Using a property that you described in part (b), answer the question "Imagine four salmon—two small and two large. The smaller salmon differ in length by 1 cm, as do the two larger. Is the difference in speed between the two smaller fish, greater than, equal to, or smaller than the difference in speed between the two larger fish?"

28. The graph of f is concave down for $0 \le x \le 6$. Which is bigger: $\dfrac{f(3) - f(1)}{3 - 1}$ or $\dfrac{f(5) - f(3)}{5 - 3}$? Why?

[17] From K. Schmidt-Nielsen, *Scaling, Why is Animal Size so Important?* (Cambridge: CUP, 1984).

CHAPTER SUMMARY

- **Input and Output**
  Evaluating functions: finding $f(a)$ for given a.
  Solving equations: finding x if $f(x) = b$ for given b.
- **Domain and Range**
  Domain: set of input values.
  Range: set of output values.
  Piecewise defined functions: different formulas on different intervals.
- **Horizontal and Vertical Shifts**
- **Composite Functions**
  Finding $f(g(x))$ given $f(x)$ and $g(x)$.
- **Inverse Functions**
  If $y = f(x)$, then $f^{-1}(y) = x$.
  Evaluating $f^{-1}(b)$. Interpretation of $f^{-1}(b)$.
  Formula for $f^{-1}(y)$ given formula for $f(x)$.
- **Concavity**
  Concave up: increasing rate of change.
  Concave down: decreasing rate of change.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER TWO

Exercises

In Exercises 1–2, evaluate the function for $x = -7$.

1. $f(x) = x/2 - 1$

2. $f(x) = x^2 - 3$

For Exercises 3–6, calculate exactly the values of y when $y = f(4)$ and of x when $f(x) = 6$.

3. $f(x) = \dfrac{6}{2 - x^3}$

4. $f(x) = \sqrt{20 + 2x^2}$

5. $f(x) = 4x^{3/2}$

6. $f(x) = x^{-3/4} - 2$

7. If $f(x) = 2x + 1$, **(a)** Find $f(0)$ **(b)** Solve $f(x) = 0$.

8. If $f(x) = \dfrac{x}{1 - x^2}$, find $f(-2)$.

9. If $P(t) = 170 - 4t$, find $P(4) - P(2)$.

10. Let $h(x) = 1/x$. Find

(a) $h(x + 3)$ **(b)** $h(x) + h(3)$

11. **(a)** Using Table 2.27, evaluate $f(1)$, $f(-1)$, and $-f(1)$.
(b) Solve $f(x) = 0$ for x.

Table 2.27

x	-1	0	1	2
$f(x)$	0	-1	2	1

12. **(a)** In Figure 2.44, estimate $f(0)$.
(b) For what x-value(s) is $f(x) = 0$?
(c) For what x-value(s) is $f(x) > 0$?

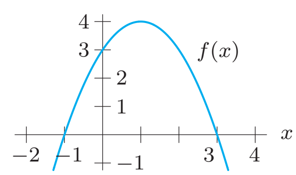


Figure 2.44

Find the domain and range of functions in Exercises 13–18 algebraically.

13. $q(x) = \sqrt{x^2 - 9}$

14. $m(r) = \dfrac{1}{\sqrt{r^2 - 1}}$

15. $m(x) = 9 - x$

16. $n(x) = 9 - x^4$

17. $m(t) = \dfrac{t}{3} + 2$

18. $s(q) = \dfrac{2q + 3}{5 - 4q}$

19. **(a)** How can you tell from the graph of a function that an x-value is not in the domain? Sketch an example.
(b) How can you tell from the formula for a function that an x-value is not in the domain? Give an example.

20. Let $g(x) = x^2 + x$. Evaluate and simplify the following.

(a) $-3g(x)$ **(b)** $g(1) - x$
(c) $g(x) + \pi$ **(d)** $\sqrt{g(x)}$
(e) $g(1)/(x + 1)$ **(f)** $(g(x))^2$

21. Let $f(x) = 1 - x$. Evaluate and simplify the following.

(a) $2f(x)$ **(b)** $f(x)+1$ **(c)** $f(1-x)$
(d) $(f(x))^2$ **(e)** $f(1)/x$ **(f)** $\sqrt{f(x)}$

In Exercises 22–23, let $f(x) = 3x - 7$ and $g(x) = x^3 + 1$ to find a formula for the function.

22. $f(g(x))$ **23.** $g(f(x))$

In Exercises 24–25, give the meaning and units of the composite function.

24. $a(g(w))$, where $F = g(w)$ is the force, in newtons, on a rocket when the wind speed is w meters/sec and $a(F)$ is the acceleration, in meters/sec^2, when the force is F newtons.

25. $P(f(t))$, where $l = f(t)$ is the length, in centimeters, of a pendulum at time t minutes, and $P(l)$ is the period, in seconds, of a pendulum of length l.

In Exercises 26–33, use $f(x) = x^2 + 1$ and $g(x) = 2x + 3$.

26. $f(g(0))$ **27.** $f(g(1))$ **28.** $g(f(0))$ **29.** $g(f(1))$

30. $f(g(x))$ **31.** $g(f(x))$ **32.** $f(f(x))$ **33.** $g(g(x))$

In Exercises 34–35, give the meaning and units of the inverse function. (Assume f is invertible.)

34. $V = f(t)$ is the speed in km/hr of an accelerating car t seconds after starting.

35. $I = f(r)$ is the interest earned, in dollars, on a \$10,000 deposit at an interest rate of $r\%$ per year, compounded annually.

In Exercises 36–37, find the domain and range of the function.

36. $h(x) = \dfrac{a}{\sqrt{x}}$, where a is a constant

37. $p(x) = |x - b| + 6$, where b is a constant

In Exercises 38–39, find the inverse function.

38. $y = g(t) = \sqrt{t} + 1$ **39.** $P = f(q) = 14q - 2$

In Exercises 40–42, let $P = f(t)$ be the population, in millions, of a country at time t in years and let $E = g(P)$ be the daily electricity consumption, in megawatts, when the population is P. Give the meaning and units of the function. Assume both f and g are invertible.

40. $g(f(t))$ **41.** $f^{-1}(P)$ **42.** $g^{-1}(E)$

43. Calculate successive rates of change for the function, $p(t)$, in Table 2.28 to decide whether you expect the graph of $p(t)$ to be concave up or concave down.

Table 2.28

t	0.2	0.4	0.6	0.8
$p(t)$	−3.19	−2.32	−1.50	−0.74

44. If $p(x) = \dfrac{12}{\sqrt{x}}$, evaluate $p(8)$ and $p^{-1}(\sqrt{2})$

45. For $f(x) = 12 - \sqrt{x}$, evaluate $f(16)$ and $f^{-1}(3)$.

In Exercises 46–47, graph the function.

46. $f(x) = \begin{cases} x^2 & \text{for} \quad x \le 1 \\ 2 - x & \text{for} \quad x > 1 \end{cases}$

47. $g(x) = \begin{cases} x+5 & \text{for} \quad x < 0 \\ x^2+1 & \text{for} \quad 0 \le x \le 2 \\ 3 & \text{for} \quad x > 2 \end{cases}$

Problems

48. If $V = \frac{1}{3}\pi r^2 h$ gives the volume of a cone, what is the value of V when $r = 3$ inches and $h = 2$ inches? Give units.

49. Let $q(x) = 3 - x^2$. Evaluate and simplify:

(a) $q(5)$ **(b)** $q(a)$
(c) $q(a-5)$ **(d)** $q(a) - 5$
(e) $q(a) - q(5)$

50. Let $p(x) = x^2 + x + 1$. Find $p(-1)$ and $-p(1)$. Are they equal?

51. Using Figure 2.45, find a formula for $g(x)$ in terms of $f(x)$.

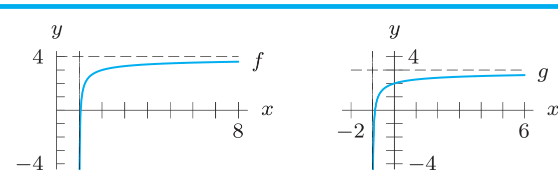


Figure 2.45

52. Chicago's average monthly rainfall, $R = f(t)$ inches, is given as a function of month, t, in Table 2.29. (January is $t = 1$.) Solve and interpret:

(a) $f(t) = 3.7$ **(b)** $f(t) = f(2)$

Table 2.29

t	1	2	3	4	5	6	7	8
R	1.8	1.8	2.7	3.1	3.5	3.7	3.5	3.4

53. Let $f(x) = \sqrt{x^2 + 16} - 5$.

(a) Find $f(0)$.
(b) For what values of x is $f(x)$ zero?
(c) Find $f(3)$.
(d) What is the vertical intercept of the graph of $f(x)$?
(e) Where does the graph cross the x-axis?

54. Use the graph of $f(x)$ in Figure 2.46 to estimate:

(a) $f(0)$ **(b)** $f(1)$ **(c)** $f(b)$ **(d)** $f(c)$ **(e)** $f(d)$

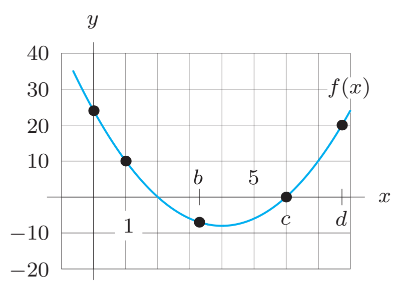


Figure 2.46

55. Let f be the function shown in Figure 2.47.

(a) Estimate all intervals on which f is increasing.
(b) Estimate all intervals on which f is concave up.

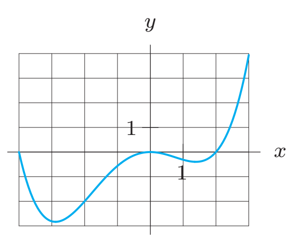


Figure 2.47

56. Use the graph in Figure 2.48 to fill in the missing values:

(a) $f(0) = ?$ **(b)** $f(?) = 0$
(c) $f^{-1}(0) = ?$ **(d)** $f^{-1}(?) = 0$

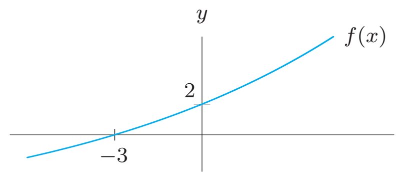


Figure 2.48

57. In Figure 2.49, the values c and d are labeled on the x-axis. On the y-axis, locate the following quantities:

(a) $h(c)$ **(b)** $h(d)$
(c) $h(c+d)$ **(d)** $h(c) + h(d)$

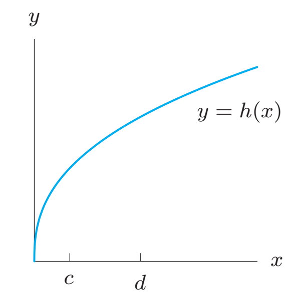


Figure 2.49

58. Use the values of the invertible function in Table 2.30 to find as many values of g^{-1} as possible.

Table 2.30

t	1	2	3	4	5
$y = g(t)$	7	12	13	19	22

59. The formula $V = f(r) = \frac{4}{3}\pi r^3$ gives the volume of a sphere of radius r. Find a formula for the inverse function, $f^{-1}(V)$, giving radius as a function of volume.

60. The formula for the volume of a cube with side s is $V = s^3$. The formula for the surface area of a cube is $A = 6s^2$.

(a) Find and interpret the formula for the function $s = f(A)$.
(b) If $V = g(s)$, find and interpret the formula for $g(f(A))$.

61. The area, $A = f(s)$ ft^2, of a square wooden deck is a function of the side s feet. A can of stain costs \$29.50 and covers 200 square feet of wood.

(a) Write the formula for $f(s)$.
(b) Find a formula for $C = g(A)$, the cost in dollars of staining an area of A ft^2.
(c) Find and interpret $C = g(f(s))$.
(d) Evaluate and interpret, giving units:

(i) $f(8)$ (ii) $g(80)$ (iii) $g(f(10))$

62. Table 2.31 shows the cost, $C(m)$, in dollars, of a taxi ride as a function of the number of miles, m, traveled.

(a) Estimate and interpret $C(3.5)$ in practical terms.
(b) Assume C is invertible. What does $C^{-1}(3.5)$ mean in practical terms? Estimate $C^{-1}(3.5)$.

Table 2.31

m	0	1	2	3	4	5
$C(m)$	0	2.50	4.00	5.50	7.00	8.50

63. The perimeter, in meters, of a square whose side is s meters is given by $P = 4s$.

(a) Write this formula using function notation, where f is the name of the function.
(b) Evaluate $f(s+4)$ and interpret its meaning.
(c) Evaluate $f(s)+4$ and interpret its meaning.
(d) What are the units of $f^{-1}(6)$?

64. **(a)** Find the side, $s = f(d)$, of a square as a function of its diagonal d.
(b) Find the area, $A = g(s)$, of a square as a function of its side s.
(c) Find the area $A = h(d)$ as a function of d.
(d) What is the relation between f, g, and h?

65. Suppose that $f(x)$ is invertible and that both f and f^{-1} are defined for all values of x. Let $f(2) = 3$ and $f^{-1}(5) = 4$. Evaluate the following expressions, or, if the given information is insufficient, write unknown.

(a) $f^{-1}(3)$ (b) $f^{-1}(4)$ (c) $f(4)$

66. Let $k(x) = 6 - x^2$.

(a) Find a point on the graph of $k(x)$ whose x-coordinate is -2.
(b) Find two points on the graph whose y-coordinates are -2.
(c) Graph $k(x)$ and locate the points in parts (a) and (b).
(d) Let $p = 2$. Calculate $k(p) - k(p-1)$.

67. **(a)** Find a point on the graph of $h(x) = \sqrt{x+4}$ whose x-coordinate is 5.
(b) Find a point on the graph whose y-coordinate is 5.
(c) Graph $h(x)$ and mark the points in parts (a) and (b).
(d) Let $p = 2$. Calculate $h(p+1) - h(p)$.

68. Let $t(x)$ be the time required, in seconds, to melt 1 gram of a compound at x°C.

(a) Express the following statement as an equation using $t(x)$: It takes 272 seconds to melt 1 gram of the compound at 400°C.
(b) Explain the following equations in words:
(i) $t(800) = 136$ (ii) $t^{-1}(68) = 1600$
(c) Above a certain temperature, doubling the temperature, x, halves the melting time. Express this fact with an equation involving $t(x)$.

69. **(a)** The Fibonacci sequence is a sequence of numbers that begins 1, 1, 2, 3, 5, ... Each term in the sequence is the sum of the two preceding terms. For example,

$$2 = 1+1, \quad 3 = 2+1, \quad 5 = 2+3, \ldots.$$

Based on this observation, complete the following table of values for $f(n)$, the n^{th} term in the Fibonacci sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1	1	2	3	5							

(b) The table of values in part (a) can be completed even though we don't have a formula for $f(n)$. Does the fact that we don't have a formula mean that $f(n)$ is not a function?
(c) Are you able to evaluate the following expressions using parts (a) and (b)? If so, do so; if not, explain why not.

$$f(0), \quad f(-1), \quad f(-2), \quad f(0.5).$$

70. Table 2.32 contains values of $g(t)$. Each function in parts (a)–(e) is a translation of $g(t)$. Find a possible formula for each of these functions in terms of g.

Table 2.32

t	-1	-0.5	0	0.5	1
$g(t)$	0.5	0.8	1.0	0.9	0.6

(a)

t	-1	-0.5	0	0.5	1
$a(t)$	1.0	1.3	1.50	1.4	1.1

(b)

t	-1	-0.5	0	0.5	1
$b(t)$	1.0	0.9	0.6	0.1	-0.4

(c)

t	-1	-0.5	0	0.5	1
$c(t)$	0.7	0.6	0.3	-0.2	-0.7

(d)

t	-1	-0.5	0	.5	1
$d(t)$	0	0.5	0.8	1.0	0.9

(e)

t	-1	-0.5	0	0.5	1
$e(t)$	1.2	1.7	2.0	2.2	2.1

71. A psychologist conducts an experiment to determine the effect of sleep loss on job performance. Let $p = f(t)$ be the number of minutes it takes the average person to complete a particular task if he or she has lost t minutes of sleep, where $t = 0$ represents exactly 8 hours of sleep. For instance, $f(60)$ is the amount of time it takes the average person to complete the task after sleeping for only 7 hours. Let $p_0 = f(0)$ and let t_1, t_2, and t_3 be positive constants. Explain what the following statements tell you about sleep loss and job performance.

(a) $f(30) = p_0 + 5$
(b) $f(t_1) = 2p_0$
(c) $f(2t_1) = 1.5f(t_1)$
(d) $f(t_2 + 60) = f(t_2 + 30) + 10$

72. Give a formula for a function whose domain is all non-negative values of x except $x = 3$.

73. Give a formula for a function that is undefined for $x = 8$ and for $x < 4$, but is defined everywhere else.

74. Many printing presses are designed with large plates that print a fixed number of pages as a unit. Each unit is called a signature. A particular press prints signatures of 16 pages each. Suppose $C(p)$ is the cost of printing a book of p pages, assuming each signature printed costs $0.14.

(a) What is the cost of printing a book of 128 pages? 129 pages? p pages?
(b) What are the domain and range of C?
(c) Graph $C(p)$ for $0 \leq p \leq 128$.

75. Table 2.33 shows the population, P, in millions, of Ireland[18] at various times between 1780 and 1910, with t in years since 1780.

(a) When was the population increasing? Decreasing?
(b) For each successive time interval, construct a table showing the average rate of change of the population.
(c) From the table you constructed in part (b), when is the graph of the population concave up? Concave down?
(d) When was the average rate of change of the population the greatest? The least? How is this related to part (c)? What does this mean in human terms?
(e) Graph the data in Table 2.33 and join the points by a curve to show the trend in the data. From this graph identify where the curve is increasing, decreasing, concave up and concave down. Compare your answers to those you got in parts (a) and (c). Identify the region you found in part (d).
(f) Something catastrophic happened in Ireland between 1780 and 1910. When? What happened in Ireland at that time to cause this catastrophe?

Table 2.33 *The population of Ireland from 1780 to 1910, where $t = 0$ corresponds to 1780*

t	0	20	40	60	70	90	110	130
P	4.0	5.2	6.7	8.3	6.9	5.4	4.7	4.4

STRENGTHEN YOUR UNDERSTANDING

Are the statements in Problems 1–46 true or false? Give an explanation for your answer.

1. If $f(t) = 3t^2 - 4$ then $f(2) = 0$.

2. If $f(x) = x^2 - 9x + 10$ then $f(b) = b^2 - 9b + 10$.

3. If $f(x) = x^2$ then $f(x+h) = x^2 + h^2$.

4. If $q = \dfrac{1}{\sqrt{z^2+5}}$ then the values of z that make $q = \frac{1}{3}$ are $z = \pm 2$.

5. If $W = \dfrac{t+4}{t-4}$ then when $t = 8$, $W = 1$.

6. If $f(t) = t^2 + 64$ then $f(0) = 64$.

7. If $f(x) = 0$ then $x = 0$.

8. If $f(x) = x^2 + 2x + 7$ then $f(-x) = f(x)$.

9. If $g(x) = \dfrac{3}{\sqrt{x^2+4}}$ then $g(x)$ can never be zero.

10. If $h(p) = -6p + 9$ then $h(3) + h(4) = h(7)$.

11. The domain of a function is the set of input values.

12. If a function is being used to model a real-world situation, the domain and range are often determined by the constraints of the situation being modeled.

13. The domain of $f(x) = \dfrac{4}{x-3}$ consists of all real numbers x, $x \neq 0$.

14. If $f(x) = \sqrt{2-x}$, the domain of f consists of all real numbers $x \geq 2$.

15. The range of $f(x) = \dfrac{1}{x}$ is all real numbers.

16. The range of $y = 4 - \dfrac{1}{x}$ is $0 < y < 4$.

17. If $f(x) = \frac{2}{5}x + 6$ and its domain is $15 \leq x \leq 20$ then the range of f is $12 \leq x \leq 14$.

[18]Adapted from D. N. Burghes and A. D. Wood, *Mathematical Models in the Social, Management and Life Science,* p. 104 (Ellis Horwood, 1980).

18. The domain of $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ is all real numbers.

19. The graph of the absolute value function $y = |x|$ has a V shape.

20. The domain of $f(x) = |x|$ is all real numbers.

21. If $f(x) = |x|$ and $g(x) = |-x|$ then for all x, $f(x) = g(x)$.

22. If $f(x) = |x|$ and $g(x) = -|x|$ then for all x, $f(x) = g(x)$.

23. If $y = \dfrac{x}{|x|}$ then $y = 1$ for $x \neq 0$.

24. If $f(x) = \begin{cases} 3 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 4 \\ 7 & \text{if } x > 4 \end{cases}$, then $f(3) = 0$.

25. Let $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 4. \\ -x & \text{if } x > 4 \end{cases}$ If $f(x) = 4$ then $x = 2$.

26. If $g(x) = f(x) + 3$ then the graph of $g(x)$ is a vertical shift of the graph of f.

27. If $g(t) = f(t-2)$ then the graph of $g(t)$ can be obtained by shifting the graph of f two units to the left.

28. Figure 2.50 suggests that $g(x) = f(x+2) + 1$.

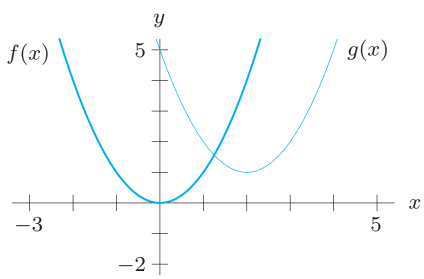


Figure 2.50

29. Figure 2.51 could be the graph of $f(x) = |x-1| - 2$.

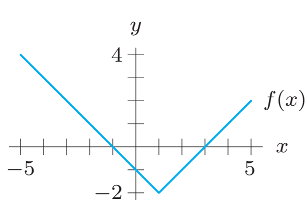


Figure 2.51

30. If $f(3) = 5$ and f is invertible, then $f^{-1}(3) = 1/5$.

31. If $h(7) = 4$ and h is invertible, then $h^{-1}(4) = 7$.

32. If $f(x) = \frac{3}{4}x - 6$ then $f^{-1}(8) = 0$.

33. If $R = f(S) = \frac{2}{3}S + 8$ then $S = f^{-1}(R) = \frac{3}{2}(R-8)$.

34. In general $f^{-1}(x) = (f(x))^{-1}$.

35. If $f(x) = \dfrac{x}{x+1}$ then $f(t^{-1}) = \dfrac{1/t}{1/t+1}$.

36. The units of the output of a function are the same as the units of output of its inverse.

37. The functions $f(x) = 2x + 1$ and $g(x) = \frac{1}{2}x - 1$ are inverses.

38. If $q = f(x)$ is the quantity of rice in tons required to feed x million people for a year and $p = g(q)$ is the cost, in dollars, of q tons of rice, then $g(f(x))$ is the dollar cost of feeding x million people for a year.

39. If $f(t) = t+2$ and $g(t) = 3t$, then $g(f(t)) = 3(t+2) = 3t + 6$.

40. A fireball has radius $r = f(t)$ meters t seconds after an explosion. The volume of the ball is $V = g(r)$ meter^3 when it has radius r meters. Then the units of measurement of $g(f(t))$ are meter^3/sec.

41. If the graph of a function is concave up, then the average rate of change of a function over an interval of length 1 increases as the interval moves from left to right.

42. The function f in the table could be concave up.

x	−2	0	2	4
$f(x)$	5	6	8	12

43. The function g in the table could be concave down.

t	−1	1	3	5
$g(t)$	9	8	6	3

44. A straight line is concave up.

45. A function can be both decreasing and concave down.

46. If a function is concave up, it must be increasing.