LECTURES ON TWISTED K-THEORY AND ORIENTIFOLDS

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This is an improved version of informal supplementary notes for lectures delivered in June, 2012 at the Erwin Schrödinger International Institute for Mathematical Physics as part of a graduate workshop in the program *K-Theory and Quantum Fields*. This is not a record of the actual lectures, but rather is in large part supplementary reading and exercises. These notes will also be the basis of a formal paper on some of this material, but that will necessarily contain less exposition, so we are making these informal notes available. We do not make any attempt to assemble even a representative list of references in these lecture notes; we do so in the published papers. There were four lectures, but these notes are divided into three lectures as is more natural.

The goal of the lectures is to give an overview of some aspects of ongoing joint work with Jacques Distler and Greg Moore about topological aspects of superstring theory. The inclusion of "orientifolds" has especially subtle topological features, and it is these that we focus on. There are two papers[1,2] about this work so far and several more to follow. These theories are known in the physics literature as "Type II"; the "Type I" theories are a special case of the orientifold construction, so are included here. The "heterotic" string is not part of this discussion, nor is "M-theory", "F-theory", or other variants of string theory with supersymmetry.

Lecture 1 is a purely mathematical discussion of twistings of K-theory, with a little about twisted K-theory. The simplest twisting of a cohomology theory is by degree shift, which is not always considered a twisting. The second simplest is by a double cover, and this sort of twisting exists for any cohomology theory. These two types of twistings are connected by a nontrivial k-invariant, and this already justifies including the degree as part of the twisting. For K-theory there is also a twisting by a "gerbe", and it is this one which is usually discussed in isolation. In fact, there is a whole tower of higher twistings, but they do not enter these lectures. These three "lowest" twistings of K-theory are all connected by nontrivial k-invariants, and we describe a geometric model which encodes them. It is based on the Donovan-Karoubi paper as well as on work with Mike Hopkins and Constantin Teleman. We describe the classifying spectrum for these twistings and some maps relating the real and complex cases.

Lecture 2 is an overview of the fields in the Type II superstring. In fact, we spend much of the lecture on the oriented bosonic string to illustrate the important ideas of orbifolds and orientifolds. Both are constructed by "gauging a symmetry". We interpret fields in field theory as simplicial sheaves, and in that context the natural quotient construction by a symmetry group is precisely

Date: June 15, 2012.

[1] J. Distler, D. S. Freed, G. W. Moore, *Orientifold précis*, in "Mathematical Foundations of Quantum Field Theory and Perturbative String Theory", Proceedings of Symposia in Pure Mathematics, **83** (2011), 159–172, (`arXiv:0906.0795`).

[2] J. Distler, D. S. Freed, G. W. Moore, *Spin structures and superstrings*, in Perspectives in Mathematics and Physics: Essays dedicated to Isadore Singer's 85th birthday, Surveys in Differential Geometry, **15** (2010), (`arXiv:1007.4581`).

the physicists' gauging. We emphasize the "B-field", which is different in the two string theories. Also noteworthy is the description of spin structures in the superstring; there is a more detailed discussion in [2]. We also include a general discussion of the notion of a field. Our use of the term is broader than usual: it includes topological structures such as orientations, for example.

Lecture 3 is a sketch of the anomaly cancellation on the worldsheet of the Type II superstring (with orientifold). There is an anomaly in the functional integral over the fermionic fields, as usual: it is the pfaffian of a Dirac operator, which is a section of a Pfaffian line bundle. In this case that bundle is flat and the line bundle has finite order. So the anomaly is a subtle torsion effect. The B-field amplitude is also—somewhat surprisingly—anomalous. The theorem states that these two anomalies cancel. The key ingredient in the cancellation is the twisted spin structure on spacetime. We remark that even in *oriented* Type II superstring (no orientifold) there is no anomaly in the B-field amplitude, and the trivialization of the Pfaffian line bundle uses the spin structure on spacetime, though we do not know an explicit construction. In fact, what is lacking here, as in other anomaly arguments, is a *categorified* index theorem which would construct a trivialization of the anomaly line bundle, not just prove that one exists, which is what we do here. The entire story depends very heavily on our choice of Dirac quantization condition for the B-field.

Contents

Lecture 1: Models of twistings

In this lecture we give concrete models for twistings of K-theory and its cousins (KO-theory, KR-theory). The model is based on joint work[3] with Michael Hopkins and Constantin Teleman and also on the original paper of Donovan and Karoubi.[4] Isomorphism classes of the twistings we need for the B-field in superstring theory are classified by a spectrum with *three* nonzero homotopy groups. The geometric model we use involves bundles of invertible complex $\mathbb{Z}/2\mathbb{Z}$-graded algebras, bimodules, and intertwiners. It has as special cases many other models which have appeared in the literature (including [3], which only uses two homotopy groups). We also give a concrete model for geometric representatives of twisted K-theory classes. There is a brief discussion of *differential* twistings as well. Differential twistings and differential twisted K-theory classes appear[5] in our description of superstring theory, but in subsequent lectures we work in a model-independent manner as nothing we do depends on a particular model, though we do use our model for some computations. There is also a notion of a hermitian structure on a twisting, analogous to a hermitian structure on a line bundle, but we do not describe it here. We end the lecture with several important constructions and formulas. The arguments we use go back and forth between the explicit models and general algebraic topology.

Introduction

(1.1) *Twisted real cohomology.* Twisted versions of cohomology—and the geometric objects which represent cohomology classes—arise in many situations. For example, on manifolds without an orientation there is cohomology twisted by the orientation bundle. In topology it appears in the statement of Poincaré duality. In geometry real cohomology is represented by differential forms, and forms of the top degree twisted by the orientation bundle are *densities*, which are the objects on a manifold which can be integrated. Concretely, if X is a smooth manifold untwisted real cohomology is computed by the *de Rham complex*

$$\Omega^\bullet(X):\ \Omega^0(X)\xrightarrow{d}\Omega^1(X)\xrightarrow{d}\cdots \tag{1.2}$$

Let $L\to X$ be a real line bundle with a flat covariant derivative. Then we can consider differential forms with values in L, which computes twisted cohomology:

$$\Omega^\bullet(X;L):\ \Omega^0(X;L)\xrightarrow{d}\Omega^1(X;L)\xrightarrow{d}\cdots \tag{1.3}$$

Features to notice: (i) $\Omega^\bullet(X)$ has a ring structure, but $\Omega^\bullet(X;L)$ does not; (ii) $\Omega^\bullet(X;L)$ is a rank 1 module over $\Omega^\bullet(X)$; (iii) the twisting L has a nontrivial automorphism—multiplication by -1—and it induces a nontrivial map on $\Omega^\bullet(X;L)$ and on the twisted cohomology.

[3] D. S. Freed, M. J. Hopkins, C. Teleman, *Loop groups and twisted K-theory I*, J. Topology, **4** (2011), 737–798 (`arXiv:0711.1906 [math.AT]`).

[4] They do not consider covers by locally equivalent groupoids, so do not realize all twistings with finite dimensional invertible algebra bundles, only those whose isomorphism class is torsion. In other approaches this is obviated by generalizing to infinite dimensional algebras.

[5] As we do not discuss Ramond-Ramond fields except in passing, there is very little about (differential) twisted K-theory; rather, we focus on the twistings.

We remark that vector bundles of rank greater than 1 also twist real cohomology, but the kinds of twistings of K-theory we discuss are rank 1.

(1.4) *Twisted vector spaces.* K-theory is made from a more primitive source than ordinary cohomology: linear algebra. The objects of interest in linear algebra over a point are vector spaces; for linear algebra over a space X they are vector bundles. Twistings are made by finding a "free rank 1 module" for the notion of a vector space. (For complex K-theory the ground field is $\mathbb{C}$; for real KO-theory the ground field is $\mathbb{R}$.) As a first example, suppose A is an algebra over the complex numbers, such as the algebra of $n \times n$ matrices. Then in lieu of a complex vector space we can consider an A-module. Over a space X we would then have a complex vector bundle $A \to X$ whose fibers are complex algebras, and in lieu of vector bundles $E \to X$ we consider vector bundles which are fiberwise A-modules: there is an action map $A \otimes E \to E$ which satisfies the usual associativity property $a_1(a_2 \cdot e) = (a_1a_2) \cdot e$. We consider only finite rank algebras, but elaborate on this idea to obtain more general notions of twisting.

More formally: Vector spaces are the objects of a symmetric monoidal category **Vect** and twisted vector spaces are the objects of a module category for **Vect**.

Question: What property of the algebra A tells that the collection of A-modules is free of rank 1 over **Vect**? Answer: A is *invertible* in a certain precise sense, and this translates to the classical notion that A is a *central simple algebra*.

For ordinary vector spaces over $\mathbb{C}$ this leads to nothing new: all invertible algebras are equivalent to the ground field $\mathbb{C}$ and so rank 1 modules over $\mathbf{Vect}^{\mathbb{C}}$ are equivalent to $\mathbf{Vect}^{\mathbb{C}}$. The situation is more interesting over $\mathbb{R}$ where up to equivalence there is a nontrivial possibility: the algebra $\mathbb{H}$ of quaternions. We work in the richer $\mathbb{Z}/2\mathbb{Z}$-*graded*, or *super*, world. This is a manifestation of the basic notions of group completion and stability in K-theory, which in the old literature is expressed in terms of formal differences $E^0 - E^1$ of vector spaces. Formally, we use the symmetric monoidal category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces $E = E^0 \oplus E^1$ with the usual tensor product and the symmetry

$$\begin{aligned} E \otimes E' &\longrightarrow E' \otimes E \\ e \otimes e' &\longmapsto (-1)^{|e||e'|} e' \otimes e, \end{aligned} \tag{1.5}$$

where $|e| \in \{0, 1\}$ denotes the parity of the homogeneous element $e \in E$. For simplicity we still use '**Vect**' to denote this category. Now there are 2 equivalence classes of complex invertible super algebras and 8 equivalence classes of real invertible super algebras. They are all represented by *Clifford algebras.*

(1.6) *Clifford algebras.* Let (V, Q) be a finite dimensional (ungraded) real vector space V with a nondegenerate quadratic form Q (equivalently, a nondegenerate symmetric bilinear form). The *Clifford algebra* $\mathrm{C}\ell(V, Q)$ is the free associative real algebra with identity 1 such that $v^2 = Q(v) \cdot 1$. It is naturally $\mathbb{Z}/2\mathbb{Z}$-graded. For $n \in \mathbb{Z}^{\neq 0}$ set $\mathrm{C}\ell_n = \mathrm{C}\ell(\mathbb{R}^n, Q_n)$, where Q_n is the standard quadratic form

$$Q_n(\xi^1, \dots, \xi^n) = \pm\big((\xi^1)^2 + \cdots + (\xi^n)^2\big) \tag{1.7}$$

with the sign chosen according to the sign of n. By convention $\mathrm{C}\ell_0 = \mathbb{R}$. There are also complex Clifford algebras $\mathrm{C}\ell^{\mathbb{C}}(V,Q) = \mathrm{C}\ell(V,Q) \otimes \mathbb{C}$. The Clifford algebra is invertible in the super sense, and super modules over a Clifford algebra are a free rank 1 module over super vector spaces. So this is one possibility for a twisted notion of vector space to make twisted K-theory.

Examples of twistings and twisted K-theory

(1.8) *Real vector bundles.* Let X be a nice topological space (locally contractible, paracompact, completely regular), for example, a smooth manifold. Let $V \to X$ be a real vector bundle of finite rank, and assume for simplicity that V is endowed with a positive definite quadratic form Q. Let $\mathrm{C}\ell(V,Q) \to X$ denote the associated bundle of *Clifford algebras*. It leads to a twisted notion of vector bundle: a $\mathrm{C}\ell(V,Q)$-twisted vector bundle is a real $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $E \to X$ which fiberwise is a left module over $\mathrm{C}\ell(V,Q)$. We can also consider complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundles, in which case we may as well use the complexification $\mathrm{C}\ell^{\mathbb{C}}(V,Q)$. When X is a Riemannian manifold this applies to the tangent bundle $V = TX$.

Let $\mathbf{Vect}(X)$ denote the category of super vector bundles over X and $\mathbf{Vect}^{\mathrm{C}\ell(V,Q)}(X)$ the category of $\mathrm{C}\ell(V,Q)$-twisted vector bundles. The bundle $\mathrm{C}\ell(V,Q) \to X$ is a left $\mathrm{C}\ell(V,Q)$-module by pointwise multiplication—a special twisted vector bundle called the *Euler class* of $V \to X$. It may happen that $\mathrm{C}\ell(V,Q) \to X$ is "equivalent" to a bundle of algebras with constant fiber a fixed Clifford algebra; we define the proper notion of equivalence below. Over the reals we need that the bundle $V \to X$ carry a *spin structure*; over the complexes $V \to X$ must carry a *spin^c-structure*. We revisit this in (**1.92**).

(1.9) *Twisted equivariant vector bundles.* Now suppose G is a compact Lie group and X a nice G-space. There is a symmetric monoidal category $\mathbf{Vect}_G(X)$ of equivariant super vector bundles $E \to X$. One way to define a twisted notion is to specify a *central extension* of G. Over the reals we take the center to be $\{\pm 1\}$, the multiplicative group of real numbers of unit norm; over the complexes the center is the circle group $\mathbb{T}$, the multiplicative group of complex numbers of unit norm. Let

$$1 \longrightarrow \{\pm 1\} \longrightarrow G^{\tau} \longrightarrow G \longrightarrow 1 \tag{1.10}$$

be a central extension in the real case. We use 'τ' to denote the central extension (1.10). Then a τ-twisted super bundle $E \to X$ carries an action of G^{τ} which covers the action of G on X. Furthermore, we require that the central element $-1 \in \{\pm 1\}$, which covers the identity map of X, act on each fiber as scalar multiplication by -1. Example: Let $X = \mathbb{T}$ with the action of $G = \mathbb{Z}/2\mathbb{Z}$ by a half-turn $x \mapsto -x$ for $x \in \mathbb{T}$. Then the Möbius line bundle $E \to X$ (whose grading is purely even) carries an action of the central extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{1.11}$$

and so is a twisted equivariant bundle.

For another kind of twisting we specify a homomorphism

$$\epsilon\colon G \longrightarrow \mathbb{Z}/2\mathbb{Z} \tag{1.12}$$

An ϵ-twisted super bundle $E = E^0 \oplus E^1 \to X$ carries an action of G in which $g \in G$ is an even transformation if $\epsilon(g) = 0$ and is an odd transformation if $\epsilon(g) = 1$. Recall that a transformation is even if it preserves the grading $E^0 \oplus E^1$ and odd if it reverses it. A group equipped with a homomorphism (1.12) is called a *$\mathbb{Z}/2\mathbb{Z}$-graded group.*

We have three kinds of twisting in the equivariant case specified by a triple (A, ϵ, τ) and it is natural to combine them. We use the Koszul sign rule (1.5) strictly. Notice that we can take $X = \mathrm{pt}$ to obtain twisted notions of the category $\mathbf{Vect}_G$ of $\mathbb{Z}/2\mathbb{Z}$-graded representations of G.

(1.13) *Examples.* Let $G = \mathbb{Z}/2\mathbb{Z}$ be nontrivially graded by the identity map $\epsilon\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Take $E = \mathbb{R}^{1|1}$ the $\mathbb{Z}/2\mathbb{Z}$-graded real vector space whose even and odd subspaces are each the trivial real line $\mathbb{R}$. Define

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.14}$$

Notice that each matrix represents an odd endomorphism of E and they commute in the super sense of the sign rule (1.5). (This last point is crucial: check it in detail!) We interpret it in two ways. First: γ generates the Clifford algebra $A = C\ell_1$ and x is the action of the generator of the nontrivial central extension (1.11) of $G = \mathbb{Z}/2\mathbb{Z}$. So E is a real representation of $\mathbb{Z}/2\mathbb{Z}$ twisted in all 3 ways: it is a module over $C\ell_1$, the group $\mathbb{Z}/2\mathbb{Z}$ is $\mathbb{Z}/2\mathbb{Z}$-graded, and there is a nontrivial central extension. Second interpretation: x generates the Clifford algebra $A = C\ell_{-1}$ and γ is the action of the generator of $G = \mathbb{Z}/2\mathbb{Z}$. In this interpretation the representation is a module over $C\ell_{-1}$ and $\mathbb{Z}/2\mathbb{Z}$ is nontrivially $\mathbb{Z}/2\mathbb{Z}$-graded, but there is no central extension. We revisit these twisted KO-classes in **(1.142)**. We remark that the first is the KO-theory Euler class of the sign representation of the group $G = \mathbb{Z}/2\mathbb{Z}$.

Exercise 1.15. Use the graded tensor product to define powers of both the module and the twisting. What is the product of the elements described by the two distinct interpretations? What is the square of the element in the first interpretation? (Very interesting point: the answer to the latter is in some sense the Bott element $u \in K^2(\mathrm{pt})$.)

(1.16) *A complex example with two morals.* The previous examples were over the reals. To twist over $\mathbb{C}$ in the equivariant situation we want a central extension by $\mathbb{T}$. If G is a finite dimensional Lie group every example factors through a central extension by a cyclic subgroup of $\mathbb{T}$. For example, the central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Spin}_n^c \longrightarrow SO_n \longrightarrow 1 \tag{1.17}$$

of the special orthogonal group factors through the extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_n \longrightarrow SO_n \longrightarrow 1 \tag{1.18}$$

To obtain something nontrivial we pass to an infinite dimensional example.

Let G be a compact Lie group and fix a principal G-bundle $P \to S^1$. Let $\mathcal{A}$ denote the affine space of connections on P and $\mathcal{G}$ the group of gauge transformations, i.e., the group of automorphisms of $P \to S^1$ which cover id_{S^1}. If $P = S^1 \times G$ is the trivial bundle then $\mathcal{G} = LG$ is canonically the *loop group* of maps $LG = \mathrm{Map}(S^1, G)$. It admits nontrivial central extensions by $\mathbb{T}$, and so defines a twisted notion of equivariant complex super vector bundle over $\mathcal{A}$. This becomes much more interesting when we connect the $\mathcal{G}$-space $\mathcal{A}$ with finite dimensional geometry. Fix a basepoint on P; its projection to S^1 is a basepoint on the base. Then there is a homomorphism $\mathcal{G} \to G$ which evaluates a gauge transformation on the fiber containing the basepoint. The holonomy is an equivariant map $\mathcal{A} \to G$, where the group G acts on the space G by conjugation. Below we will interpret this as a *local equivalence of groupoids*

$$\mathcal{A}/\!/\mathcal{G} \longrightarrow G/\!/G. \tag{1.19}$$

This is a global construction of canonical twistings of the equivariant complex K-theory group $K_G(G)$.

Moral 1: To define interesting twistings we must be allowed to pass to locally equivalent groupoids.

We will not pursue this here, but a positive energy representation of the central extension of the loop group determines an equivariant family of Fredholm operators parametrized by X. This takes the place of the finite rank equivariant super vector bundles of the previous examples. See [6] for details.

Moral 2: To define interesting "bundles" we should allow infinite rank bundles with a Fredholm operator and not restrict ourselves to finite rank bundles (with the zero operator).

(1.20) *Other motivation.* The model we present covers all of these examples and many more. Twisted notions of vector spaces and vector bundles arise in many other circumstances. Here are two more.

Let

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1 \tag{1.21}$$

be an extension of (topological) groups and let X be a space of isomorphism classes of representations of G' which is stable under the action of G''. In other words, if $\rho\colon G' \to \mathrm{Aut}(W)$ is a representation and $g \in G$ then there is a new representation $\tilde{\rho}(g') = \rho(gg'g^{-1})$, $g' \in G'$. Its isomorphism class only depends on the image of g in G''. We ask that if the isomorphism class of ρ lies in X, then so too does the isomorphism class of $\tilde{\rho}$ for all $g \in G$. In this situation a suitably finite representation of G determines a vector bundle $E \to X$ whose fibers are the multiplicity spaces of the restriction as a representation of G'. The group G'' does not in general lift to act on E, but

[6] D. S. Freed, M. J. Hopkins, C. Teleman, *Loop groups and twisted K-theory II*, (`arXiv:math.AT/0511232`).

rather there is a twisting of the groupoid $X/\!/G''$ which is defined and which acts. This is worked out in [3] and also in [7].

The second situation is a general story in quantum mechanics. The space of pure states in a quantum mechanical system is the projective space $\mathbb{P}\mathcal{H}$ of a complex separable Hilbert space $\mathcal{H}$, and it carries a natural symmetric function $p\colon \mathbb{P}\mathcal{H}\times\mathbb{P}\mathcal{H}\to[0,1]$ which encodes "transition probabilities". A fundamental theorem of Wigner asserts that every symmetry of $(\mathbb{P}\mathcal{H},p)$ lifts to either a unitary or antiunitary transformation of $\mathcal{H}$. If we denote the group of these transformations of $\mathcal{H}$ as $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$, then Wigner's theorem is encoded in the group extension

$$1\longrightarrow\mathbb{T}\longrightarrow\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})\longrightarrow\mathrm{Aut}(\mathbb{P}\mathcal{H},p)\longrightarrow 1 \tag{1.22}$$

The subgroup $\mathbb{T}$ of scalar transformations acts trivially on projective space, but it is not central as scalars do not commute with antiunitary transformations. This is a twisted central extension, and our first example of a twisting which is relevant to *KR-theory*.

Invertible super algebras, bimodules, and intertwiners

We begin by defining the twistings we use for the K-theory of a point, which is to say the twisted notion of a super vector space.

(1.23) *Preliminaries.* Let k be a field, which in our application will always be $\mathbb{R}$ or $\mathbb{C}$. Let $A=A^0\oplus A^1$ be a super algebra.. A homogeneous element z in its *center* satisfies $za=(-1)^{|z||a|}az$ for all homogeneous $a\in A$. The center is itself a super algebra, which is of course commutative (in the $\mathbb{Z}/2\mathbb{Z}$-graded sense). The *opposite* super algebra A^{op} to a super algebra A is the same underlying vector space with product $a_1\cdot a_2=(-1)^{|a_1||a_2|}a_2a_1$ on homogeneous elements. All algebras are assumed unital. Tensor products of super algebras are taken in the graded sense. Undecorated tensor products are over the ground field. Unless otherwise stated a module is a left module. An ideal $I\subset A$ in a super algebra is *graded* if $I=(I\cap A^0)\oplus(I\cap A^1)$.

Example 1.24. Let $S=S^0\oplus S^1$ be a finite dimensional super vector space over k. Then $\operatorname{End}S$ is a central simple super algebra. Endomorphisms which preserve the grading on S are even, those which reverse it are odd. A super algebra isomorphic to $\operatorname{End}S$ is called a *super matrix algebra*.

Exercise 1.25. Show that the opposite of the Clifford algebra $\mathrm{C}\ell(V,Q)$ is $\mathrm{C}\ell(V,-Q)$. In particular, $\mathrm{C}\ell_n$ and $\mathrm{C}\ell_{-n}$ are opposites.

Definition 1.26. A finite dimensional super algebra A over a field k is *central simple* if its center is k and its only graded 2-sided ideals are 0 and A.[8]

Central simple algebras in the super case were investigated by Wall and Deligne. We summarize some of their main results by recasting them as Theorem 1.28 below.

[7] D. S. Freed, M. J. Hopkins, C. Teleman, *Loop groups and twisted K-theory III*, Annals of Math., Volume 174 (2011), pp. 947–1007, (arXiv:math.AT/0312155).

[8] We also assume there exist $a_1,a_2\in A$ with $a_1a_2\neq 0$ to exclude one-dimensional algebras with the zero product map.

(1.27) *2-category of algebras-bimodules-intertwiners.* If A_0, A_1 are super algebras, then an (A_1, A_0)-*bimodule* is a $\mathbb{Z}/2\mathbb{Z}$-graded left $A_1 \otimes A_0^{op}$ module, a super vector space which is a simultaneous left module for A_1 and right module for A_0. If B, B' are (A_1, A_0)-bimodules, then an *intertwiner* $f\colon B \to B'$ is a linear map which commutes with the actions of A_0 and A_1. If $\alpha\colon A_0 \to A_1$ is a homomorphism of super algebras, then A_1 is an (A_1, A_0)-bimodule for which (a_1, a_0) acts on $a \in A_1$ to yield $a_1 \cdot a \cdot \alpha(a_0)$. In this way bimodules generalize homomorphisms of super algebras.

There is a 2-category $Alg = Alg_k$ whose objects are super k-algebras, 1-morphisms are bimodules, and 2-morphisms are intertwiners. Composition of 1-morphisms is via tensor product: if B is an (A_1, A_0)-bimodule and B' an (A_2, A_1)-bimodule, then $B' \otimes_{A_1} B$ is an (A_2, A_0)-bimodule. For any super algebra A the super vector space A is an (A, A)-bimodule which represents the identity map in Alg. Also, Alg is a symmetric monoidal category via tensor product (over k) of super algebras, bimodules, and intertwiners. Our interest is in the sub-2-category $Alg^\times = Alg_k^\times$ of *invertible* objects and morphisms in $Alg = Alg_k$.

An object $A \in Alg$ is invertible if it has an inverse under tensor product. Thus there exists $A' \in Alg$ and bimodules $B\colon k \to A \otimes A'$ and $B'\colon A \otimes A' \to k$ which are inverse isomorphisms. Since we are in a 2-category, this means there exist intertwiners $f\colon B' \otimes_{A \otimes A'} B \to k$ and $g\colon B \otimes B' \to A \otimes A'$ which are isomorphisms: f is an isomorphism of k-vector spaces and g is an isomorphism of $(A \otimes A', A \otimes A')$-bimodules.

Theorem 1.28. *A super algebra $A \in Alg$ is invertible if and only if A is (finite dimensional) central simple.*

Isomorphic algebras in $Alg_k^\times$ are called *Morita equivalent* and an invertible bimodule which gives the isomorphism defines a Morita equivalence. The group of isomorphism classes of objects in $Alg_k^\times$ is called the *super Brauer group* of the field k.

Exercise 1.29. Show that any super matrix algebra is Morita equivalent to the trivial algebra k. Prove the converse as well.

Theorem 1.30. *The super Brauer group of $\mathbb{C}$ is cyclic of order 2 and the super Brauer group of $\mathbb{R}$ is cyclic of order 8. In each case it is generated by the Clifford algebra $C(L, Q)$, where L is a line and Q a positive definite quadratic form.*

In particular, a complex central simple super algebra is either a super matrix algebra or a matrix algebra tensor the graded algebra $\mathbb{C} \oplus \mathbb{C}u^1$ with $(u^1)^2 = 1$. In the latter case the matrix algebra can be taken to be even.

For 1-morphisms in Alg invertibility amounts to the following: an (A_1, A_0)-bimodule B is invertible if there exists an (A_0, A_1)-bimodule B' and isomorphisms $B' \otimes_{A_1} B \xrightarrow{\cong} A_0$ and $B \otimes_{A_0} B' \xrightarrow{\cong} A_1$ as (A_0, A_0)- (respectively (A_1, A_1)-) bimodules. An invertible (k, k)-bimodule is a $\mathbb{Z}/2\mathbb{Z}$-graded line, a 1-dimensional k-vector space which is either even or odd. For any central simple super algebra A the 1-groupoid of invertible (A, A)-bimodules is canonically the groupoid of $\mathbb{Z}/2\mathbb{Z}$-graded lines: a $\mathbb{Z}/2\mathbb{Z}$-graded line L corresponds to the (A, A)-bimodule $A \otimes L$.

(1.31) *Morita equivalence and modules.* Suppose A_0, A_1 are invertible super algebras which are Morita equivalent, and let B an (A_1, A_0)-bimodule and B' an (A_0, A_1)-bimodule be invertible bimodules which implement the Morita equivalence. Let A-**Mod** denote the category of left A-modules. Then the functors

$$\begin{aligned} A_0\text{-}\mathbf{Mod} &\longrightarrow A_1\text{-}\mathbf{Mod} \\ E &\longmapsto B\otimes_{A_0} E \end{aligned} \tag{1.32}$$

and

$$\begin{aligned} A_1\text{-}\mathbf{Mod} &\longrightarrow A_0\text{-}\mathbf{Mod} \\ E &\longmapsto B'\otimes_{A_1} E \end{aligned} \tag{1.33}$$

are inverse equivalences. In other words, Morita equivalent super algebras define equivalent notions of twisted vector spaces.

The algebraic topology of $Alg^\times$

(1.34) *Groupoids and Picard groupoids.* Recall that a category $\mathcal{C}$ has a classifying space $B\mathcal{C}$. In this construction morphisms in the category become paths in the classifying space, and so are invertible up to homotopy: the homotopy inverse of a parametrized path is retreat along the same trajectory. A *groupoid* $\mathcal{C}$ is a category in which all morphisms are already invertible, so the classifying space construction does not invert any more. In this case the classifying space $B\mathcal{C}$ does not have any higher homotopy groups, only nonzero π_0,π_1. In the special case that the groupoid is a group π—that is, there is a single object $*$ and the $\pi=\mathrm{Aut}(*)$—then the classifying space has the single nonzero homotopy group $\pi_1=\pi$. In other words, the classifying space $B\pi$ is the Eilenberg-MacLane space $K(\pi,1)$. Returning to a general groupoid $\mathcal{C}$, if in addition $\mathcal{C}$ has a symmetric monoidal structure such that every object has an inverse, then $\mathcal{C}$ is called a *Picard groupoid*. In this case π_0,π_1 are abelian groups and the classifying space $B\mathcal{C}$ is an infinite loop space. Infinite loop spaces correspond to *spectra*, and as for spaces, spectra have a Postnikov decomposition. A Picard groupoid has only two nonzero homotopy groups, so the Postnikov tower is quite simple:

$$\begin{tikzcd} \Sigma H\pi_1 \arrow[r] & B\mathcal{C} \arrow[d] \\ & H\pi_0 \arrow[r, "k"] & \Sigma^2 H\pi_1 \end{tikzcd} \tag{1.35}$$

Here $H\pi$ is the Eilenberg-MacLane spectrum which has the single nonzero homotopy group π in degree 0. Then $\Sigma H\pi$ is the shift which has the nonzero homotopy group in degree 1, the de-looping of $H\pi$. (In the special case $\mathcal{C}$ has one object with automorphism group the abelian group π, then $B\mathcal{C}$ is the 0-space of the spectrum $\Sigma H\pi$.) The bottom arrow in (1.35) is called the k*-invariant* of the spectrum and it tells how the spectrum is glued together from the Eilenberg-MacLane building blocks. The k-invariant is a stable cohomology operation $H^q(-;\pi_0)\to H^{q+2}(-;\pi_1)$, which is equivalent to a homomorphism $\pi_0\otimes\mathbb{Z}/2\mathbb{Z}\to\pi_1$. Example: If $\pi_0=\pi_1=\mathbb{Z}/2\mathbb{Z}$, then there is a single nonzero stable operation $Sq^2\colon H^q(-;\mathbb{Z}/2\mathbb{Z}) \to H^{q+2}(-;\mathbb{Z}/2\mathbb{Z})$, the Steenrod square. The k-invariant can be computed directly from the Picard groupoid $\mathcal{C}$ as follows. First, observe that if $1 \in \mathcal{C}$ is the identity object, then for any $x \in \mathcal{C}$ there is an isomorphism $\mathrm{Aut}(1) \cong \mathrm{Aut}(1 \otimes x) \cong \mathrm{Aut}(x)$ which maps $f \mapsto f \otimes \mathrm{id}_x$. Then the homomorphism $\pi_0 \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_1$ is the image of the symmetry

$$\sigma_x\colon x \otimes x \longrightarrow x \otimes x \tag{1.36}$$

under the isomorphism $\mathrm{Aut}(x \otimes x) \cong \mathrm{Aut}(1)$.

Exercise 1.37. Check that this does indeed define a homomorphism $\pi_0 \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_1$.

Twistings, in particular $Alg^\times$, form a *Picard 2-groupoid*. This is a 2-category, which for our purposes means a category $\mathcal{C}$ in which the morphism spaces $\mathcal{C}(x, y)$ are also categories. An ordinary morphism $x \to y$ is called a *1-morphism* and a morphism in $\mathcal{C}(x, y)$ is called a *2-morphism*. The classifying space $B\mathcal{C}$ of a Picard 2-groupoid is an infinite loop space whose only nonzero homotopy groups are π_0, π_1, π_2. There is a filtration whose associated graded consists of two Picard 1-groupoids: the bottom one has the same objects as $\mathcal{C}$ and replaces $\mathcal{C}(x, y)$ by $\pi_0\mathcal{C}(x, y)$; the top one is the 1-groupoid $\mathrm{Aut}(1)$. These two groupoids do not in general determine $\mathcal{C}$. (In general, there are extensions when passing from an associated graded to the original filtered object.) However, in the situation here they are enough to do so; see Lemma 1.53.

(1.38) *The 2-groupoid $Alg^\times$.* We compute the homotopy groups π_0, π_1, π_2 for $Alg^\times$ over the fields $k = \mathbb{C}, \mathbb{R}$. First, π_0 is the super Brauer group of equivalence classes of invertible super algebras, the computation of which is recalled in Theorem 1.30. Next, π_1, π_2 are equal to π_0, π_1 of the 1-groupoid $\mathrm{Aut}(1)$ whose objects are $\mathbb{Z}/2\mathbb{Z}$-graded lines. There are two isomorphism classes of objects—the trivial line k and the odd line Π (generated by an odd element π)—and $\Pi^{\otimes 2} \cong k$. The tensor unit in $\mathrm{Aut}(1)$ is the trivial line k and its automorphism group is $k^\times$. So we obtain the following table of nonzero homotopy groups:

(1.39)

	$Alg^\times_{\mathbb{C}}$	$Alg^\times_{\mathbb{R}}$
π_2	$\mathbb{C}^\times$	$\mathbb{R}^\times$
π_1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
π_0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$

Here $\mathbb{C}^\times$ and $\mathbb{R}^\times$ have the *discrete* topology. There is a notion of a hermitian structure on algebras and bimodules, and then a corresponding 2-groupoid of unitary algebras-bimodules-intertwiners whose π_2 is $\mathbb{T}$ and $\mathbb{Z}/2\mathbb{Z}$ in the complex and real cases, respectively.

We are interested in a homotopy classification in which we use the *continuous* topology on intertwiners. By analogy consider the group $\mathbb{C}^\times$ and maps into it from a nice topological space X. If $\mathbb{C}^\times$ is discrete there is no notion of equivalence, and the set of continuous maps $X \to \mathbb{C}^\times$ is $H^0(X;\mathbb{C}^\times)$. If we use the continuous topology on $\mathbb{C}^\times$, then homotopy classes form the cohomology group $H^1(X;\mathbb{Z})$. The Bockstein map $H^0(X;\mathbb{C}^\times) \to H^1(X;\mathbb{Z})$ connects the two interpretations.

Similarly, for the topological classification of twistings we use the continuous topology on intertwiners, and so obtain the following table of nonzero homotopy groups, where the initial 'c' reminds of the continuous topology:

(1.40)

	$cAlg^\times_{\mathbb{C}}$	$cAlg^\times_{\mathbb{R}}$
π_3	$\mathbb{Z}$	0
π_2	0	$\mathbb{Z}/2\mathbb{Z}$
π_1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
π_0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$

Remark 1.41*.* The inclusion of π_0 amounts to including the degree in K-theory as part of the twisting. This is a natural idea and even in ordinary cohomology an alternative way to encode degree. It was built into our discussion from the beginning in **(1.4)**. From this point of view the periodicity of K-theory is built-in and arises much as in the classic paper of Atiyah-Bott-Shapiro.

(1.42) *k-invariants.* Next, we compute the k-invariants between the consecutive groups in (1.40). We focus on the complex case; the arguments in the real case are similar. The k-invariant on the bottom is a map $H\mathbb{Z}/2\mathbb{Z} \to \Sigma^2 H\mathbb{Z}/2\mathbb{Z}$, and as remarked in **(1.34)** the only nonzero possibility is Sq^2. To compute whether this occurs we take a representative of the nonzero element in π_0, which is the Clifford algebra $A = C\ell_1^{\mathbb{C}}$ with a single odd generator e. The symmetry $A\otimes A \to A\otimes A$ is the involution which fixes $1\otimes 1$, changes the sign of $e\otimes e$, and exchanges $1\otimes e \leftrightarrow e\otimes 1$. As stated in **(1.27)** it is represented by the vector space $A\otimes A$ as an $(A\otimes A, A\otimes A)$-bimodule using the involution to modify the right multiplication.

Exercise 1.43. Define an isomorphism of this bimodule with the bimodule $A\otimes A\otimes \Pi$ in which the left and right $A\otimes A$ actions are by multiplication on $A\otimes A$ tensor the identity on Π. This shows that the k-invariant is nonzero.

The other k-invariant is the standard one in the category $\mathrm{Aut}(1)$ of $\mathbb{Z}/2\mathbb{Z}$-graded lines. It is the nonzero map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$ (see (1.39)) as can be seen from the fact that the symmetry $\Pi\otimes\Pi \to \Pi\otimes\Pi$ is multiplication by -1. The corresponding k-invariant in (1.40) is the stable cohomology operation

$$\beta_{\mathbb{Z}} \circ Sq^2 \colon H^q(-;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{q+3}(-;\mathbb{Z}), \tag{1.44}$$

where $\beta_{\mathbb{Z}}$ is the integer Bockstein map.

The spectrum R and $cAlg^\times_{\mathbb{C}}$

Our goal here is to recognize the classifying space of $cAlg^\times_{\mathbb{C}}$, which is the 0-space of a spectrum, in terms of a more well-known spectrum, namely the spectrum KO. This will enable us to make topological arguments when we come to superstrings. We use a very small truncation of KO. We do not have a conceptual explanation of its appearance, except to say that the homotopy groups in the first column of (1.40) can only be fit together in one nontrivial way, and that occurs in the KO-spectrum. On the other hand, this description as a truncation of KO is crucial in Lecture 3 because of the Atiyah-Singer index theorem, which relates KO-theory to Dirac operators.

(1.45) *A Postnikov truncation of connective ko.* Let ko denote the connective KO-theory spectrum. It is more primitive than the periodic KO-spectrum: for example, its 0-space is the classifying space of the symmetric monoidal category $\mathbf{Vect}_{\mathbb{R}}$ of finite dimensional real vector spaces, a fact we use extensively at the end of this lecture. Its homotopy groups starting with π_1 are the Bott song $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$, ... Furthermore, ko is a commutative ring spectrum (an E^∞ ring spectrum), and we will use the ring structure extensively. There is a truncation procedure which produces a commutative ring spectrum from a Postnikov truncation. We introduce a central object in these lectures, the commutative ring spectrum

$$R := \pi_{\leqslant 4} ko = ko\langle 0 \cdots 4\rangle. \tag{1.46}$$

Its homotopy groups (and k-invariants) are those of the bottom part of ko:

(1.47)

	R	R^{-1}
π_4	$\mathbb{Z}$	0
π_3	0	$\mathbb{Z}$
π_2	$\mathbb{Z}/2\mathbb{Z}$	0
π_1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
π_0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

In the second column[9] we list the homotopy groups of the connective cover of $\Sigma^{-1}R$. Note that $\pi_{-1}\Sigma^{-1}R \cong \mathbb{Z}$, but it is not detected on spaces or groupoids, so taking the connective cover does not lose information in our application.

The Postnikov tower for R^{-1} is

$$\begin{tikzcd} & \Sigma^3 H\mathbb{Z} \arrow[r] & R^{-1} \arrow[d] & \\ & \Sigma H\mathbb{Z}/2\mathbb{Z} \arrow[r, "i"] & T \arrow[r, "j"] \arrow[d] & \Sigma^4 H\mathbb{Z} \\ & & H\mathbb{Z}/2\mathbb{Z} \arrow[r, "k"] & \Sigma^2 H\mathbb{Z}/2\mathbb{Z} \end{tikzcd} \tag{1.48}$$

for some spectrum T. (In fact, T is the spectrum which represents the Picard groupoid of $\mathbb{Z}/2\mathbb{Z}$-graded real lines, a fact we use in Lecture 3.) We have included the classifying maps for the various

[9]Our nomenclature 'R^{-1}' may not be optimal; nonetheless the two possible meanings of '$R^{-1}(X)$' for a space or groupoid X do coincide.

stages, which are the arrows j, k; their homotopy classes are the k-invariants. The k-invariants can be computed from the corresponding k-invariants of ko, and we take them as known. In particular, we have

$$\begin{aligned} k &\sim Sq^2 \\ j \circ i &\sim \beta_{\mathbb{Z}} \circ Sq^2 \end{aligned} \tag{1.49}$$

where as above $\beta_{\mathbb{Z}}$ is the integer Bockstein.

It follows from the Postnikov tower that for any space X there is a short exact sequence of abelian groups

$$0 \longrightarrow H^3(X;\mathbb{Z}) \longrightarrow R^{-1}(X) \xrightarrow{(t,a)} H^0(X;\mathbb{Z}/2\mathbb{Z}) \times H^1(X;\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \tag{1.50}$$

It is functorially split as an exact sequence of *sets*, but not as an exact sequence of *abelian groups*. Furthermore, the nonzero map k means that the product structure in the quotient is correct on the level of abelian groups, but not on the level of cohomology theories.

(1.51) *Comparing $cAlg_{\mathbb{C}}^{\times}$ and R^{-1}.* Tables (1.40) and (1.47) show that the classifying spectra of $cAlg_{\mathbb{C}}^{\times}$ and R^{-1} have the same homotopy groups. We claim more.

Theorem 1.52. *The classifying spectrum of $cAlg_{\mathbb{C}}^{\times}$ and the spectrum R^{-1} are isomorphic.*

This theorem would follow if we produce a map between them which induces an isomorphism on homotopy groups. Absent that, we check that the k-invariants agree. The k-invariants of the associated graded to the Postnikov filtration were computed in (**1.42**), and they agree with the homotopy classes of the maps k and $j \circ i$ in (1.48), as listed in (1.49). What we need to know is that the actual k-invariants, which correspond to the map j, agree. We are lucky(?) in this case: the extension problem of lifting from the associated graded is trivial.

Lemma 1.53. *Let $i\colon \Sigma H\mathbb{Z}/2\mathbb{Z} \to T$ denote the map in the Postnikov tower (1.48). Then any map $T \to \Sigma^4 H\mathbb{Z}$ is determined (up to homotopy) by its composition with i.*

Proof. Fibrations of spectra are also cofibrations, and from (1.48) the cofiber of i is $H\mathbb{Z}/2\mathbb{Z}$. Thus there is an exact sequence of abelian groups

$$[H\mathbb{Z}/2\mathbb{Z}, \Sigma^4 H\mathbb{Z}] \longrightarrow [T, \Sigma^4 H\mathbb{Z}] \xrightarrow{i^*} [\Sigma H\mathbb{Z}/2\mathbb{Z}, \Sigma^4 H\mathbb{Z}], \tag{1.54}$$

where $[S_1, S_2]$ denotes the group of homotopy classes of maps S_1, S_2 of spectra. The first group vanishes, as one can see by computing $H^{q+4}\big(K(\mathbb{Z}/2\mathbb{Z}, q); \mathbb{Z}/2\mathbb{Z}\big)$ iteratively for small values of q using the Serre spectral sequence. ∎

Bundles of invertible algebras

(1.55) *Review of groupoids.* This is a lightening review; we refer to [3] for a more detailed discussion. To avoid pathologies, as stated in **(1.8)** all topological spaces are assumed locally contractible, paracompact, and completely regular. A *topological groupoid* X consists of a pair of *spaces* X_0, X_1 which form the objects and morphisms (arrows) of a category in which all morphisms are invertible. The structure maps $p_0, p_1\colon X_1 \to X_0$ give the target and source of a morphism; identity morphisms are identified as the image of a map $X_0 \to X_1$; there is an associative composition map $X_1 \times_{X_0} X_1 \to X_1$; and there is an inversion $X_1 \to X_1$. All structure maps are assumed continuous. Let X_n denote the space whose elements are n composable morphisms. There are maps $p_i\colon X_n \to X_{n-1}$, $i = 0, \dots, n$ whose value on $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) \in X_n$ replaces the pair f_i, f_{i+1} with the composition $f_{i+1}f_i$ and omits the node x_i. (The map p_0 omits $x_0 \xrightarrow{f_1}$ and the map p_n omits $\xrightarrow{f_n} x_n$.) For example, composition of two arrows is the map $p_1\colon X_2 \to X_1$. A map of topological groupoids $F\colon X \to Y$ is a pair of continuous maps $F_i\colon X_i \to Y_i$, $i = 0, 1$, which commute with the structure maps. It is a *local equivalence* if it is an equivalence of the underlying categories and has continuous local sections. Topological groupoids related by a chain of local equivalences are termed *weakly equivalent* and represent the same underlying *topological stack*. (There is an alternative, invariant approach to topological and smooth stacks.) Let X and Y be topological groupoids. A *morphism* $X \to Y$ of the underlying topological stacks is a local equivalence $\tilde{X} \to X$ and a map of groupoids $\tilde{X} \to Y$. If G is a topological group acting on a space S, then there is a *global quotient groupoid* $X = S/\!/G$ in which $X_0 = S$ and $X_1 = S \times G$. A topological groupoid X is a *local quotient groupoid* if each point in X has a neighborhood which is weakly equivalent to a global quotient $S/\!/G$, where S is a Hausdorff space and G a compact Lie group.

A *Lie groupoid* X is one for which X_0, X_1 are smooth manifolds, all structure maps are smooth, and the maps $p_0, p_1\colon X_1 \to X_0$ are submersions. A Lie groupoid X is a *local quotient Lie groupoid* if each point has a neighborhood weakly equivalent to $S/\!/G$, where S is a finite dimensional smooth manifold and G a compact Lie group acting smoothly. A Lie groupoid X is *étale* if the target and source maps $p_0, p_1\colon X_1 \to X_0$ are local diffeomorphisms. It is *proper* if $p_0 \times p_1\colon X_1 \to X_0 \times X_0$ is a proper map. If X is proper and étale, then the underlying topological stack is called an *orbifold* or *smooth Deligne-Mumford stack* and the representing groupoid an *orbifold groupoid*. In particular, the stabilizers are finite groups. We remark that smooth DM stacks may be presented by Lie groupoids which are not étale—for example, if $P \to M$ is a principal G-bundle over a smooth manifold, then $P/\!/G$ is locally equivalent to M, a Deligne-Mumford stack which is a manifold. Orbifolds have a more concrete differential-geometric description as "V-manifolds" in the work of Satake, Kawasaki, Thurston and others.

Definition 1.56. A *vector bundle* $E \to X$ over a topological groupoid X is a pair $E = (E_0, \psi)$ consisting of a vector bundle $E_0 \to X_0$ and an isomorphism $\psi\colon p_1^*E_0 \to p_0^*E_0$ on X_1 which satisfies the cocycle constraint

$$\psi_{f_2 \circ f_1} = \psi_{f_2} \circ \psi_{f_1}. \tag{1.57}$$

for $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in X_2$.

The notation is that the isomorphism ψ at $(x_0 \xrightarrow{f} x_1) \in X_1$ is $\psi_f \colon (E_0)_{x_0} \to (E_0)_{x_1}$. This data determines a groupoid $E = \Big(E_0 \overset{\tilde{p_1}}{\underset{\tilde{p_0}}{\leftleftarrows}} E_1 \Big)$ where E_1 is the pullback $p_1^*E_0$ and $\tilde{p}_0 \colon E_1 \to E_0$ is the composition $p_1^*E_0 \xrightarrow{\psi} p_0^*E_0 \to E_0$.

(1.58) *Bundles of invertible algebras over topological groupoids.* We give a sequence of precise definitions here for the assiduous reader. They have their own internal music—somewhat like a passacaglia—and once the reader gets the pattern it should be no problem to supply all details.

Definition 1.59. Let X be a topological groupoid. An *invertible algebra bundle* (A, B, λ) over X is

(i) a fiber bundle $A \to X_0$ of central simple super algebras;
(ii) a super vector bundle $B \to X_1$ which is an invertible (p_0^*A, p_1^*A)-bimodule, that is, for $(x_0 \xrightarrow{f} x_1) \in X_1$ an invertible (A_{x_1}, A_{x_0})-bimodule B_f;
(iii) and an isomorphism of $(p_1^*p_0^*A, p_2^*p_1^*A)$-bimodules $\lambda \colon p_0^*B \otimes_{p_0^*p_2^*A} p_2^*B \to p_1^*B$, that is, for $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in X_2$ an invertible (A_{x_2}, A_{x_0})-intertwiner

$$\lambda_{f_2,f_1} \colon B_{f_2} \otimes_{A_{x_1}} B_{f_1} \to B_{f_2f_1}, \tag{1.60}$$

such that for $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3) \in X_3$ the diagram

$$\begin{tikzcd} B_{f_3} \otimes_{A_{x_2}} B_{f_2} \otimes_{A_{x_1}} B_{f_1} \arrow[r, "\mathrm{id} \otimes \lambda_{f_2,f_1}"] \arrow[d, "\lambda_{f_3,f_2} \otimes \mathrm{id}"'] & B_{f_3} \otimes_{A_{x_2}} B_{f_2f_1} \arrow[d, "\lambda_{f_3,f_2f_1}"] \\ B_{f_3f_2} \otimes_{A_{x_1}} B_{f_1} \arrow[r, "\lambda_{f_3,f_2,f_1}"] & B_{f_3f_2f_1} \end{tikzcd} \tag{1.61}$$

commutes.

The collection of invertible algebra bundles over a fixed topological groupoid X forms a Picard 2-groupoid $Alg^\times(X)$. We use the notations $Alg^\times_{\mathbb{C}}(X)$ and $Alg^\times_{\mathbb{R}}(X)$ when we need to make the ground field explicit.

Definition 1.62. Let $\mathcal{A} = (A, B, \lambda)$ and $\mathcal{A}' = (A', B', \lambda')$ be invertible algebra bundles over a topological groupoid X.

(i) A *1-morphism* $\mathcal{A} \to \mathcal{A}'$ is a pair (C, μ) consisting of
  (a) A fiber bundle $C \to X_0$ of invertible (A', A)-bimodules,
  (b) For $(x_0 \xrightarrow{f} x_1) \in X_1$ an isomorphism

$$\mu_f \colon C_{x_1} \otimes_{A_{x_1}} B_f \longrightarrow B'_f \otimes_{A'_{x_0}} C_{x_0} \tag{1.63}$$

such that for $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in X_2$ the diagram

(1.64)
$$\begin{tikzcd} C_{x_2} \otimes_{A_{x_2}} B_{f_2} \otimes_{A_{x_1}} B_{f_1} \arrow[r, "\mu_{f_2} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \lambda_{f_2,f_1}"'] & B'_{f_2} \otimes_{A'_{x_1}} C_{x_1} \otimes_{A_{x_1}} B_{f_1} \arrow[r, "\mathrm{id} \otimes \mu_{f_1}"] & B'_{f_2} \otimes_{A'_{x_1}} B'_{f_1} \otimes_{A'_{x_0}} C_{x_0} \arrow[d, "\lambda'_{f_2 f_1} \otimes \mathrm{id}"] \\ C_{x_2} \otimes_{A_{x_2}} B_{f_2 f_1} \arrow[rr, "\mu_{f_2 f_1}"] & & B'_{f_2 f_1} \otimes_{A'_{x_0}} C_{x_0} \end{tikzcd}$$

commutes.

(ii) A 2-morphism $(C, \mu) \to (C', \mu')$ is an isomorphism

$$\nu \colon C \longrightarrow C' \tag{1.65}$$

of (A', A)-bimodules on X_0 such that for $(x_0 \xrightarrow{f} x_1) \in X_1$ the diagram

$$\begin{tikzcd} C_{x_1} \otimes_{A_{x_1}} B_f \arrow[r, "\mu_f"] \arrow[d, "\mu \otimes \mathrm{id}"'] & B'_f \otimes_{A'_{x_0}} C_{x_0} \arrow[d, "\mathrm{id} \otimes \nu"] \\ C'_{x_1} \otimes_{A_{x_1}} B_f \arrow[r, "\mu'_f"] & B'_f \otimes_{A'_{x_0}} C'_{x_0} \end{tikzcd} \tag{1.66}$$

commutes.

There is a symmetric monoidal structure on $Alg^{\times}(X)$.

Definition 1.67. The *tensor product* of invertible algebra bundles $\mathcal{A} = (A, B, \lambda)$ and $\mathcal{A}' = (A', B', \lambda')$ over a topological groupoid X is the invertible algebra bundle $(A \otimes A', B \otimes B', \lambda \otimes \lambda')$, where the intertwiner $(\lambda \otimes \lambda')_{f_2,f_1}$ associated to $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2)$ uses the symmetry of graded tensor products:

$$\begin{aligned} (B_{f_2} \otimes B'_{f_2}) \otimes_{A_x \otimes A'_x} (B_{f_1} \otimes B'_{f_1}) &\xrightarrow{(1.5)} (B_{f_2} \otimes_{A_{x_1}} B_{f_1}) \otimes (B'_{f_2} \otimes_{A'_{x_1}} B'_{f_1}) \\ &\xrightarrow{\lambda \otimes \lambda'} B_{f_2 f_1} \otimes B'_{f_2 f_1} \end{aligned} \tag{1.68}$$

The symmetry $\mathcal{A} \otimes \mathcal{A}' \to \mathcal{A}' \otimes \mathcal{A}$ is constructed from the symmetry (1.5).

(1.69) *Special cases.* Specializations of these definitions appear in the literature, as we now illustrate.

Example 1.70. Here is the original construction of Donovan and Karoubi in the late 1960s. Suppose the topological groupoid X is a space, so that $X_1 = X_0$ consists only of identity arrows. Then an invertible algebra bundle over X is a fiber bundle of central simple super algebras. They were introduced by Donovan and Karoubi to twist K-theory and, in particular, to construct a generalized Thom isomorphism. Subsequently, fiber bundles of infinite dimensional C^*-algebras—Dixmier-Douady bundles—were used by Rosenberg as more general twistings. Below (Definition 1.78) we allow a space X to be replaced by a weakly equivalent groupoid, and so we realize all isomorphism classes of infinite dimensional bundles of C^*-algebras in our finite dimensional model.

Example 1.71. Consider the special case in which $(A, B, \lambda) \in Alg^\times(X)$ has A the trivial bundle with fiber the ground field $\mathbb{R}$ or $\mathbb{C}$. Then the bundle $B \to X_1$ is a line bundle and λ is an isomorphism of line bundles; cf. the paragraph preceding (**1.31**). This is the model explained in much greater detail in [3].

Example 1.72. Consider a further special case in which A is trivial and B is the constant line bundle with fiber k (the ground field) or Π (the trivial odd line). This is equivalent to a function $\epsilon\colon X_1 \to \mathbb{Z}/2\mathbb{Z}$ which encodes the grading, and the existence of λ is in this case a condition, not data: it asserts that ϵ is a cocycle. In fact, ϵ determines a double cover of the groupoid X; see (**1.94**).

We remark that double covers twist *any* cohomology theory.

The next examples, which are commonly used in the literature, do not have $\mathbb{Z}/2\mathbb{Z}$-gradings so only model the spectrum $\Sigma^3 H\mathbb{Z}$, not R^{-1}. In this case we use the term 'gerbe' in place of 'invertible algebra bundle'. We will see in the next lecture that gerbes are relevant to the bosonic string, whereas all of the homotopy groups of R^{-1} are used in the superstring.

Example 1.73. The next example is another special case of Example 1.71. Here we let B be possibly nonconstant, but require that it be a purely even line bundle.

Example 1.74. Suppose X is a space and $Y \to X$ a surjective submersion. We form the groupoid $\tilde{Y}$ defined by iterated fiber products $\tilde{Y}_0 := Y$, $\tilde{Y}_1 := Y \times_X Y$, $\tilde{Y}_2 := Y \times_X Y \times_X Y$, etc. Following Murray, we say a *bundle gerbe* is a line bundle over $\tilde{Y}_1$ with a product on $\tilde{Y}_2$ which satisfies an associativity constraint on $\tilde{Y}_3$. This is an object in $Alg^\times(\tilde{Y})$ of the type in Example 1.71.

Example 1.75. As a special case of Example 1.74 suppose X is a space and $\{U_i\}_{i \in I}$ an open cover. Then the disjoint union $Y := \bigsqcup_i U_i$ surjects onto X and a bundle gerbe reduces to a widely used model for gerbes consisting of a line bundle on each intersection $U_{i_0} \cap U_{i_1}$ and an isomorphism on each triple intersection $U_{i_0} \cap U_{i_1} \cap U_{i_2}$ which satisfies a cocycle condition on quadruple intersections.

Example 1.76. In another direction if G is a topological group and $X = BG$ the groupoid with X_0 a point and $X_1 = G$ then an invertible algebra bundle of the type in Example 1.71 is equivalent to a $\mathbb{Z}/2\mathbb{Z}$-graded central extension of G by $k^\times$ where $k = \mathbb{R}$ or $k = \mathbb{C}$.

Exercise 1.77. Define the *opposite* or *inverse* of an invertible algebra bundle. It is canonical if you use the opposite algebra, inverse line bundle, etc. Notice the special case of Example 1.73. What is the effect on the equivalence class in $R^{-1}(X)$ (or $H^3(X; \mathbb{Z})$ in the special case)?

Twistings of K-theory

Following §2.3 of [3], which is motivated by the first moral of (**1.16**), we make the following definition.

Definition 1.78. Let X be a topological groupoid. A *twisting* on X is a pair $\tau = (P, \mathcal{A})$ consisting of a local equivalence $P \to X$ and an invertible algebra bundle $\mathcal{A} \in Alg^\times(P)$. It is called a *$K$-twisting* in the complex case and a *KO-twisting* in the real case.

Remark 1.79. It is important that we have the flexibility to "replace" X with a locally equivalent groupoid P, as foreshadowed in **(1.16)**. For example, if $X = S^3$ is the 3-sphere then every invertible algebra bundle over S^3 is equivalent to a constant invertible algebra bundle in which all fibers are identified. However, if we replace S^3 by the groupoid formed as in Example 1.75 from the open cover consisting of two sets $U_i = S^3 \backslash \{x_i\}$, $i = 0, 1$, for distinct points $x_0, x_1 \in S^3$, then there are nontrivial complex invertible algebra bundles of the type in Example 1.71 for which the line bundle on $U_0 \cap U_1$ is topologically nontrivial. These represent nonzero elements of $H^3(S^3; \mathbb{Z})$.

(1.80) *The Picard 2-groupoid of twistings.* The twistings in Definition 1.78 are objects in Picard 2-groupoids $\mathfrak{Twist}_K(X)$ and $\mathfrak{Twist}_{KO}(X)$. For example, a 1-morphism $\tau = (P, \mathcal{A}) \to \tau' = (P', \mathcal{A}')$ is a 1-morphism $p^*\pi^*\tau \to p^*\pi'^*\tau'$ in $Alg^\times(P'')$, where in the diagram

(1.81)
$$\begin{tikzcd} & P'' \arrow[d, "p"] & \\ & P \times_X P' \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ P & & P' \end{tikzcd}$$

the arrow p is a local equivalence. This idea is developed in [3] for the subcategory of twistings in Example 1.71, and the discussion goes through without change in the general case. One result is that $\mathfrak{Twist}_K(X)$ and $\mathfrak{Twist}_{KO}(X)$ are unchanged (up to equivalence) under a local equivalence $X' \to X$ of groupoids, so only depend on the stack which underlies X.

The topological classification of twistings is asserted in the following theorem. We will not give a proof here, but refer to Corollary 2.25 in [3] for the proof in case we do not include the degree as part of the twisting.

Theorem 1.82. *Let X be a local quotient groupoid. Then there is an isomorphism*

$$\pi_0 \mathfrak{Twist}_K(X) \cong R^{-1}(X). \tag{1.83}$$

We have not yet explained what $R^{-1}(X)$ means for a groupoid, only for a space. We do so now.

(1.84) *Cohomology of local quotient groupoids.* Let X be a local quotient groupoid. There is an associated topological space $|X|$—the *geometric realization* or *classifying space* of X—for which X_n is the space of n-simplices. If $X = S/\!/G$ is a global quotient, then $|X|$ is homotopy equivalent to the usual Borel construction. Define the cohomology of X with coefficients in an abelian group A as the ordinary topological cohomology of the geometric realization:

$$H^\bullet(X; A) := H^\bullet(|X|; A). \tag{1.85}$$

For a global quotient $H^\bullet(S/\!/G; A) \cong H^\bullet_G(S; A)$ is the usual Borel equivariant cohomology. The same works for general cohomology theories, and so $R^{-1}(X) = R^{-1}(|X|)$ is the abelian group of homotopy classes of maps of $|X|$ into the 0-space of the spectrum $\Sigma^{-1}R$. If X is a global quotient S/G of a nice space S by the action of a compact Lie group G, then $R^{-1}(X)$ is the Borel equivariant cohomology of X.

Twisted vector bundles and Fredholm operators

There is no point to twisting K-theory if we do not have twisted K-theory; so far we only have the twisting. We first give the definition of a twisted vector bundle for a fixed invertible algebra bundle; it is a modification of Definition 1.56.

Definition 1.86. Let X be a topological groupoid and $\mathcal{A} = (A, B, \lambda)$ an invertible algebra bundle. Then an $\mathcal{A}$-*twisted vector bundle* over X is a pair $E = (E_0, \psi)$ consisting of an A-module $E_0 \to X_0$ and an isomorphism $\psi\colon B \otimes_{p_1^*A} p_1^*E_0 \to p_0^*E_0$ of p_0^*A-modules over X_1. It is required to satisfy a cocycle relation on X_2 analogous to (1.57).

Exercise: Write the exact cocycle relation.

Remark 1.87. Implicit in Definition 1.86 is that the bundle $E_0 \to X_0$ has finite rank. But such bundles are *not* sufficient to represent all twisted K-theory classes, even if we allow general twistings as in Definition 1.78. Here is a sketch argument in case X is a space. Given an invertible algebra bundle choose a good open cover (Example 1.75) over which we can trivialize the bundle of algebras. For simplicity—and because it's enough to make the point—suppose these algebras are Morita trivial. Then on the cover we can choose an equivalent invertible algebra bundle for which the bundle of algebras is trivial, so a twisting of the form in Example 1.71. Now if E is a twisted vector bundle for that invertible algebra bundle, and its rank is k, then the determinant bundle $\operatorname{Det} E$ is a trivialization of the k^{th} power of the invertible algebra bundle. In other words, the twisting is necessarily torsion of order k.

(1.88) *Fredholm operators.* We therefore introduce a more general model which represent elements of K-theory to realize all twisted classes. Suppose $E = E^0 \oplus E^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space. A bounded linear map $t\colon E^0 \to E^1$ is *Fredholm* if it has closed range and finite dimensional kernel and cokernel. We consider bounded linear operators $T\colon E \to E$ which are *odd* and *skew-adjoint*, so have the form $\left(\begin{smallmatrix} 0 & -t^* \\ t & 0 \end{smallmatrix}\right)$. Then T is Fredholm iff t is. If T is Fredholm, then $\operatorname{Ker} T$ is a finite dimensional super vector space which is equivalent to the pair (E, T). But in a continuous family of odd skew-adjoint Fredholm operators the kernels do not form a vector bundle as the rank jumps, though the (graded) dimension—called the *index*—is locally constant.

Let **Fred** be the category of super Hilbert spaces and odd skew-adjoint Fredholms. It, rather than **Vect**, gives a model of K-theory which generalizes to the twisted case over a space X. We first generalize it to a twisted model over a point, as in **(1.4)**. Thus if A is an invertible super algebra we let $\mathbf{Fred}^A$ denote the category of super Hilbert spaces which are left A-modules and which carry an odd skew-adjoint Fredholm operator T which (graded) commutes with the A-action. We remark that we allow the super Hilbert space E to be finite dimensional, and there are many interesting classes (e.g. the Thom class) which have a nice finite dimensional model. The finite dimensional model is essentially the "difference construction" in the early K-theory literature.

Remark 1.89. There is a tricky sign in the proper definition of "skew-adjoint" in the super world—the matrix we wrote above is not really odd skew-adjoint. There is a way around that sign to a more standard convention and in any case we will not need to worry about this level of detail.

(1.90) *Continuous families of Fredholms.* To extend Definition 1.86 to twisted families of Fredholm operators, there is an important technical point: what is a fiber bundle whose fiber is a (super) Hilbert space? The issue is the operator topology with which we measure continuity. This is dealt with nicely in an appendix to the paper of Atiyah-Segal on twisted K-theory. The strong operator topology, which is essentially the compact-open topology, is the appropriate one to use here, not the norm topology. Not only does this work theoretically, but important examples are continuous in the compact-open topology and not in the norm topology. One then needs an adapted model for Fredholm operators, which ends up being a homotopy equivalent slight thickening of that space of operators. We defer to all of these references for details. And after all of the preceding detailed definitions, we leave it to the reader to complete the Exercise: Write carefully the generalization of Definition 1.86 to twisted families of Fredholm operators, including the correct cocycle relation.

(1.91) *Twisted K-theory.* Finally, given a twisting (Definition 1.78) we must define twisted K-theory groups. Again we defer to [3]. We want in the end to obtain a twisted cohomology theory with all of the exact sequences etc., so we need to do more than just define K-theory groups separately for each pair (X, τ) of a topological groupoid X and a twisting τ. Rather we define a bundle of spectra over X and use it to define K-theory groups. The idea is that for a fixed super Hilbert space there is a spectrum defined by odd skew-adjoint Fredholms, using the standard Clifford algebras $\mathrm{C}\ell_n$ to make the shift operators. This was developed in a classic paper of Atiyah-Singer. Given a twisting τ there exist "locally universal" τ-twisted Hilbert bundle, and we use twisted odd skew-adjoint Fredholms to form the bundle of spectra we need. The K-theory groups we define are in the end canonically independent of the choice of the twisted Hilbert bundle. This means that we can represent K-theory classes by any τ-twisted Hilbert bundle (even finite rank) equipped with an odd skew-adjoint Fredholm operator.

(1.92) *Thom class revisited: an extended exercise.* We already gave several examples of twisted K-theory classes at the beginning of this lecture. The reader should revisit them now. Together we will only revisit **(1.8)**, and we invite the reader to work out the details of the following as practice in the formalism.

Let X be a topological groupoid and $V \to X$ a real vector bundle of rank n with a positive definite quadratic form Q. Show that $\mathrm{C}\ell(V, Q) \to X$ is an invertible algebra bundle which is in fact a bundle of algebras. It is one model for the twisting $\tau^{KO}(V)$ determined by V. Here is another. Define a locally equivalent groupoid $P \to X$ which is the principal O_n-bundle of orthonormal frames. The group O_n has an extension

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Pin}_n^- \longrightarrow O_n \longrightarrow 1 \tag{1.93}$$

where Pin_n^- lies in the Clifford algebra $\mathrm{C}\ell_{-n}$. Use this extension to define an invertible algebra bundle over P of the type in Example 1.71. Construct an isomorphism with the twisting given by $\mathrm{C}\ell(V, Q) \to X$. This latter model will play an important role in Lecture 3.

Pull these twistings back by the map $\pi \colon V \to X$ to obtain twistings over V. For the first we pull back the opposite Clifford algebras $\pi^*\mathrm{C}\ell(V, -Q) \to V$, and consider the vector bundle $\pi^*\mathrm{C}\ell(V, -Q) \to V$ as a *right* $\mathrm{C}\ell(V, -Q)$-module. Define the family of odd skew-adjoint Fredholms (in finite rank every operator is Fredholm) whose value at $\xi \in V$ is *left* multiplication by $\xi \in V \subset \mathrm{C}\ell(V, -Q)$. This graded commutes with the *left* action by $\mathrm{C}\ell(V, +Q)$, defined from the right action of $\mathrm{C}\ell(V, -Q)$ using Exercise 1.25. Note that multiplication by ξ is invertible off of the 0-section of π. The K-theory (or KO-theory) class so defined is the *Thom class*. It lies in $KO^{\pi^*Cl(V,Q)}(V)_c$, where the subscript 'c' denotes 'compact support'. Its restriction to the 0-section is the *Euler class* defined in (**1.8**).

Now make a model of the Thom class using the equivalent twisting based on the pin group. You'll need to observe that the pullback $\pi^*V \to P$ of $V \to X$ over $P \to X$ is canonically equivalent to the vector bundle with constant fiber $\mathbb{R}^n$. (Hint: A point of P is an isometry $\mathbb{R}^n \to V_x$ for some x.) For this we use the vector bundle over π^*V with constant fiber $\mathrm{C}\ell_{-n}$, left Clifford multiplication by $\xi \in \mathbb{R}^n$ and the right $\mathrm{C}\ell_{-n}$-action by multiplication.

Define a spin structure in two equivalent ways. As usual it is a reduction of the O_n-bundle $P \to X$ to a Spin_n-bundle. Show that this trivializes the twisting you defined above in the sense that it is equivalent to a twisting which shifts the degree by n. Use the trivialization to define an untwisted Thom class. Use the spin structure to define a vector bundle $C \to X$ which is an invertible $\big(\mathrm{C}\ell_n, \mathrm{C}\ell(V, Q)\big)$-bimodule. Compare with Definition 1.62 to see that we obtain an isomorphism of the twisting $\mathrm{C}\ell(V, Q) \to X$ with the twisting defined by the constant algebra $\mathrm{C}\ell_n$, so a degree shift by $+n$.

Twistings of KR-theory

Suppose Y is a space with an involution $\sigma\colon Y \to Y$. In this situation it is natural to consider complex vector bundles $E \to Y$ with a lift $\tilde{\sigma}\colon \sigma^*E \to E$ of the involution to a linear isomorphism on the bundle whose square is the identity. These are simply vector bundles over the groupoid quotient $X = Y/\!/\sigma$. But it is also possible to consider lifts which are complex *antilinear*, or equivalently a linear isomorphism $\sigma^*\overline{E} \to E$ from the pullback of the complex conjugate bundle $\overline{E}$ to E. This is another form of "twisting" of K-theory. We remark that in the previous case twisted K-theory is a module over K-theory; for this kind of twisting we obtain a module over KO-theory. This is Atiyah's KR-theory. Our focus on twistings leads us to regard the KR-theory of Y as a twisted form of K-theory on the quotient X. From that point of view Y with its involution is replaced by X with its double cover $Y \to X$. (Recall that double covers have already been used in Example 1.72 to define more usual twists of K-theory.)

(1.94) *Double covers of topological groupoids.* A *double cover* of a local quotient groupoid X is a morphism $X \to B\mathbb{Z}/2\mathbb{Z}$, where the groupoid $B\mathbb{Z}/2\mathbb{Z} = */\!/(\mathbb{Z}/2\mathbb{Z})$ is the global quotient of a point by the trivial involution. More concretely, we have the following.

Definition 1.95. Let X be a local quotient groupoid. A *double cover* of X is a pair (Y, ϕ) consisting of a local equivalence $Y \to X$ and a continuous homomorphism $\phi\colon Y_1 \to \mathbb{Z}/2\mathbb{Z}$.

The homomorphism property means that for $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in Y_2$ we have $\phi(f_2 \circ f_1) = \phi(f_2) + \phi(f_1)$. The *trivial double cover* is the map $X \to B\mathbb{Z}/2\mathbb{Z}$ which takes every element in X_1 to the identity arrow in $B\mathbb{Z}/2\mathbb{Z}$. Given two double covers of X we can find a single local equivalence $P \to X$ and homomorphisms $\phi_1, \phi_2\colon P_1 \to \mathbb{Z}/2\mathbb{Z}$ which describe them. The double covers are isomorphic if and only if $\phi_1 - \phi_2$ evaluated on $(x_0 \xrightarrow{f} x_1)$ is $\chi(x_1) - \chi(x_2)$ for some function $\chi\colon P_0 \to \mathbb{Z}/2\mathbb{Z}$. See §2.2 in [3] for more details.

If $(\tilde{X}, \phi)$ is a double cover of X we construct a new groupoid X_w as the pullback

$$\begin{tikzcd} X_w \arrow[r, dashed] \arrow[d, dashed] & \mathrm{pt} \arrow[d] \\ X \arrow[r] & B\mathbb{Z}/2\mathbb{Z} \end{tikzcd} \tag{1.96}$$

Explicitly, $(X_w)_0 = X_0 \times \mathbb{Z}/2\mathbb{Z}$ and each arrow $(x_0 \xrightarrow{f} x_1) \in X_1$ gives rise in X_w to two arrows $\bigl((x_0, i_0) \to (x_1, i_1)\bigr) \in (X_w)_1$, where $i_0 = i_1$ in $\mathbb{Z}/2\mathbb{Z}$ if $\phi(f) = 0$ and $i_0 \neq i_1$ if $\phi(f) = 1$.

A double cover of X is classified up to isomorphism by a class $w \in H^1(X; \mathbb{Z}/2\mathbb{Z})$, where we recall **(1.84)** for the definition of cohomology of a local quotient groupoid. In the presentation above the function $\phi\colon X_1 \to \mathbb{Z}/2\mathbb{Z}$ on 1-simplices determines w. We often employ 'w' in the notation for a double cover thus: $X_w \to X$. We also use 'w' to denote the double cover itself.

(1.97) *w-twisting a twisting.* Let V be a complex super vector space. Set

$$ {}^{\phi}V = \begin{cases} V, & \phi = 0; \\ \overline{V}, & \phi = 1, \end{cases} \tag{1.98}$$

where $\overline{V}$ is the complex conjugate super vector space. The same notation applies to super algebras and bimodules.

Now suppose that $X_w \to X$ is a double cover of topological groupoids, as in Definition 1.95. This is specified by a homomorphism $\phi\colon X_1 \to \mathbb{Z}/2\mathbb{Z}$, possibly after replacing X by a weakly equivalent groupoid (which we still denote 'X'). We use the double cover to twist the notion of invertible algebra bundles introduced in **(1.58)**.

Definition 1.99. Let $X_w \to X$ be a double cover of topological groupoids. A *w-twisted complex invertible algebra bundle* $\mathcal{A} = (A, B, \lambda)$ over X is given as in Definition 1.59 with the following modifications. In (ii), B_f is an $({}^{\phi(f)}A_{x_1}, A_{x_0})$-bimodule. In (iii), the intertwiner is a map

$$\lambda_{f_2,f_1}\colon {}^{\phi(f_1)}B_{f_2} \otimes_{A_{x_1}} B_{f_1} \longrightarrow B_{f_2 f_1}. \tag{1.100}$$

Finally, in the w-twisted case diagram (1.61) reads

$$\begin{tikzcd} {}^{\phi(f_2f_1)}B_{f_3} \otimes_{A_{x_2}} {}^{\phi(f_1)}B_{f_2} \otimes_{A_{x_1}} B_{f_1} \arrow[r, "\mathrm{id}\otimes\lambda_{f_2,f_1}"] \arrow[d, "\lambda_{f_3,f_2}\otimes \mathrm{id}"'] & {}^{\phi(f_2f_1)}B_{f_3} \otimes_{A_{x_2}} B_{f_2f_1} \arrow[d, "\lambda_{f_3,f_2f_1}"] \\ {}^{\phi(f_1)}B_{f_3f_2} \otimes_{A_{x_1}} B_{f_1} \arrow[r, "\lambda_{f_3,f_2,f_1}"] & B_{f_3f_2f_1} \end{tikzcd} \tag{1.101}$$

By way of explanation, if $\phi(f_1) = 1$ then in the tensor product $\overline{B_{f_2}} \otimes_{A_{x_1}} B_{f_1}$ in (1.100) we have $\bar{b}_2 \otimes \bar{a}b_1 = \bar{b}_2\bar{a} \otimes b_1$ for all $a \in A_{x_1}$, $b_1 \in B_{f_1}$, and $b_2 \in B_{f_2}$.

Remark 1.102. The following special case is illuminating. Suppose X_w is a space with involution σ. Suppose $A \to X_w$ is a complex invertible algebra bundle and the involution σ is lifted to an isomorphism of algebras $\tilde{\sigma}\colon \sigma^*A \to \overline{A}$ such that $\overline{\tilde{\sigma}}\tilde{\sigma}$ is the identity. Then $\tilde{\sigma}$ determines a w-twisted complex invertible algebra bundle over the quotient groupoid $X := X_w/\!/(\mathbb{Z}/2\mathbb{Z})$. (The homomorphism $\phi\colon X_1 \to \mathbb{Z}/2\mathbb{Z}$ maps non-identity arrows in X to $1 \in \mathbb{Z}/2\mathbb{Z}$.)

We leave the necessary modifications of other definitions in the w-twisted case to the reader. There is a 2-groupoid $Alg^{\times}_{\mathbb{C}}(X)^w$ of w-twisted invertible algebra bundles. As in Definition 1.78 we can also form a 2-groupoid $\mathfrak{Twist}_K(X)^w$ of "twisted twistings" on X; it is more familiar as a 2-groupoid of twistings of KR-theory on X_w. There is a w-twisted version of Theorem 1.82.

Theorem 1.103. *Let X be a local quotient groupoid and $X_w \to X$ a double cover. Then there is an isomorphism*

$$\pi_0\mathfrak{Twist}_K(X)^w \cong R^{w-1}(X), \tag{1.104}$$

where $R^w(X)$ is the R-cohomology of X twisted by the double cover $X_w \to X$.

Recall that double covers twist every cohomology theory, so in particular R, and that degree shifts are also twists. Twistings form a Picard (multi-)groupoid, so can be added using its group law.

Differential twistings

(1.105) *General discussion of differential cohomology.* Let h be a generalized cohomology theory. Then $h \otimes \mathbb{R}$ is (noncanonically) isomorphic to ordinary Eilenberg-MacLane cohomology $H\mathcal{R}$ with coefficients in the $\mathbb{Z}$-graded vector space $\mathcal{R}^\bullet = h^\bullet(pt) \otimes \mathbb{R}$. (We ignore possible finiteness issues which are satisfied in our applications.) On a smooth manifold X we can compute $H(X;\mathcal{R})^\bullet$ using smooth differential forms with values in $\mathcal{R}$. (The $\mathbb{Z}$-grading adds the differential form degree and the degree in $\mathcal{R}$.) Furthermore, there is a "marriage" of h and these differential forms into a *differential cohomology theory* $\check{h}$. This is an active area of development, and we freely use variations (such as equivariant differential theories) which are perhaps not fully in place.

Rather than delve into any details, we just paint a picture[10] of $\check{h}^q(X)$ for an integer q and a smooth manifold X. The main point is that $\check{h}^q(X)$ is a (usually infinite-dimensional) abelian Lie group. An abelian Lie group only has nonzero homotopy groups π_0, π_1, and these are computed by topological cohomology:

$$\begin{aligned} \pi_0\, \check{h}^q(X) &\cong h^q(X) \\ \pi_1\, \check{h}^q(X) &\cong h^{q-1}(X)/\text{torsion} \end{aligned} \tag{1.106}$$

[10]This is the correct picture for the examples $h = H\mathbb{Z}, R, KO, K$ that enter these lectures. We could also take $h = H\mathbb{R}$, for example, in which case the topology we would introduce on $\check{h}^\bullet(X)$ would not satisfy (1.106) or other aspects of our picture.

The identity component of $\check{h}^q(X)$ carries a free action of the finite dimensional torus $h^{q-1}(X)\otimes\mathbb{R}/\mathbb{Z}$, which sits inside as a subgroup, and the quotient is the space of exact differential forms $d\Omega(X;\mathcal{R})^{q-1}$. The same torus acts freely on every other component, and the quotient is naturally an affine translate of $d\Omega(X;\mathcal{R})^{q-1}$ in the space of closed forms in $\Omega(X;\mathcal{R})^q$. The map to differential forms is a kind of curvature:

$$\mathrm{curv}\colon \check{h}^q(X) \longrightarrow \Omega(X;\mathcal{R})^q_{\text{closed}}. \tag{1.107}$$

The image of (1.107) is a union of affine translates of the subspace $\Omega(X;\mathcal{R})^q_{\text{exact}} \subset \Omega(X;\mathcal{R})^q_{\text{closed}}$, and the image of this union of subspaces under the de Rham map

$$\Omega(X;\mathcal{R})^q_{\text{closed}} \longrightarrow H(X;\mathcal{R})^q \tag{1.108}$$

is a full lattice, which is the image of $h^q(X) \to H(X;\mathcal{R})^q$. The fiber of (1.107) over a differential form $\omega \in \Omega(X;\mathcal{R})^q_{\text{closed}}$ is a torsor for the group $h^{q-1}(X;\mathbb{R}/\mathbb{Z})$ of *flat* elements in $\check{h}^q(X)$. Note that the flat elements $h^{q-1}(X;\mathbb{R}/\mathbb{Z})$ form an abelian Lie group whose identity component is the torus $h^{q-1}(X)\otimes\mathbb{R}/\mathbb{Z}$.

(1.109) *Differential R-theory.* Following this general discussion there are differential cohomology groups $\check{R}^\bullet(X)$ attached to a manifold X. In our discussion of superstring theory we encounter $\check{R}^{-1}(X)$ and, for a double cover $X_w \to X$ a twisted version $\check{R}^{w-1}(X)$. There is a concrete differential geometric model for these differential classes which we now sketch.

Let X be a smooth manifold and $P \to X$ a locally equivalent *Lie groupoid* (see (**1.55**)). Let $\mathcal{A} = (A, B, \lambda)$ be an invertible algebra bundle over P, as in Definition 1.59. First, there is contractible space of "hermitian structures" on $\mathcal{A}$. It consists of a "positive $*$-structure" on the bundle of algebras $A \to X_0$ and a compatible positive hermitian structure on the bimodule $B \to X_1$. We will not spell out these notions here, but mention that in case A is trivial (Example 1.71), then it carries a canonical positive $*$-structure and a positive hermitian structure on the $\mathbb{Z}/2\mathbb{Z}$-graded line bundle $B \to X_1$ is the usual notion of a positive definite hermitian metric. We fix such a hermitian structure. Without it we would have complex differential forms in what follows, instead of the real forms we need.

The differential geometric datum is then a pair (B, ∇) consisting of a real 2-form[11] $B \in \Omega^2(P_0)$ and a covariant derivative ∇ on $B \to P_1$ which is compatible with the bimodule structure and the hermitian structure. Since the endomorphisms compatible with those structures are sections of a real line bundle over P_1, the curvature F_∇ is a real 2-form on P_1. We demand that (B, ∇) satisfies the constraint

$$p_1^*B - p_0^*B = \frac{i}{2\pi}F_\nabla \tag{1.110}$$

on P_1. There is a global closed 3-form $H = dB$, and since by (1.110) it satisfies $p_1^*H = p_0^*H$ it drops to a global 3-form on the smooth manifold X. This is the *curvature* of the differential twisting.

[11]Sorry for the notation clash, but this is the 'B' of 'B-field'.

A differential twisting has an equivalence class in $\check{R}^{-1}(X)$.

If $X_w \to X$ is a double cover, specified by a homomorphism $\phi\colon P_1 \to \mathbb{Z}/2\mathbb{Z}$ as in Definition 1.95, then a differential KR-twisting is the same sort of data on a w-twisted invertible algebra bundle over P (Definition 1.99), but now (1.110) is modified to:

$$(-1)^{\phi} p_1^*B - p_0^*B = \frac{i}{2\pi} F_{\nabla}. \tag{1.111}$$

The curvature is now a twisted differential form $H \in \Omega^{w+3}(X)$.

A differential KR-twisting has an equivalence class in $\check{R}^{w-1}(X)$.

(1.112) *Reminder about ordinary orientations.* Let V be a real vector space. The set of orientations of V is a canonical $\mathbb{Z}/2\mathbb{Z}$-torsor attached to V, and may be defined as

$$\mathfrak{o}(V) := \pi_0\bigl(\operatorname{Det} V \backslash \{0\}\bigr), \tag{1.113}$$

where $\operatorname{Det} V$ is the determinant line of V, its highest exterior power. An *orientation* of V is a choice of element of $\mathfrak{o}(V)$.

Let M be a smooth manifold. There is a canonical *orientation double cover* $\widehat{M} \to M$ whose fiber at $m \in M$ is $\mathfrak{o}(T_mM)$. It represents the first Stiefel-Whitney class $w_1(M)$. An *orientation* of M is a section of $\widehat{M} \to M$. If $\mathfrak{o}$ is an orientation, then the *opposite orientation* $-\mathfrak{o}$ is the section obtained by applying the deck transformation to $\mathfrak{o}$.

(1.114) *Integration in differential cohomology theories.* Let $\pi\colon X \to Y$ be a fiber bundle of smooth manifolds with fiber compact manifolds of dimension n. Let h be a multiplicative cohomology theory[12] (based on a commutative ring spectrum). Then there is a notion of h-orientation, and if π is h-oriented there is an "integration" map

$$\pi_*\colon h^q(X) \longrightarrow h^{q-n}(Y). \tag{1.115}$$

An h-orientation induces an ordinary orientation and so an integration of differential forms

$$\pi_*\colon \Omega^q(X;\mathcal{R}) \longrightarrow \Omega^{q-n}(Y;\mathcal{R}). \tag{1.116}$$

There is a notion of a *differential h-orientation*, introduced in work of Hopkins-Singer, which is used to define an integration map

$$\pi_*\colon \check{h}^q(X) \longrightarrow \check{h}^{q-n}(Y) \tag{1.117}$$

in the differential theory which is "compatible" with (1.115) and (1.116); see (1.120) below.

[12] There is a version of this discussion for a module theory over a ring theory.

We mention three cases of (1.117). For $h = H\mathbb{Z}$ ordinary Eilenberg-MacLane cohomology an $H\mathbb{Z}$-orientation is a usual orientation and there is no extra data necessary to define a differential $H\mathbb{Z}$-orientation. For $h = R$ an R-orientation is a spin structure, just as it is for ko-theory. Again there is no additional data necessary to integrate in differential R-theory. Finally, for $h = KO$ a KO-orientation is again a spin structure, only now there is additional data in a differential KO-orientation. The most efficient is a *Riemannian structure* on $\pi\colon X \to Y$, which is a pair (g, H) consisting of a metric g on the relative tangent bundle $T(X/Y) \to X$ and a horizontal distribution, which is a splitting of the short exact sequence

$$0 \longrightarrow T(X/Y) \longrightarrow TX \longrightarrow \pi^*TY \longrightarrow 0 \tag{1.118}$$

of vector bundles over X. This pair determines a Levi-Civita covariant derivative on $T(X/Y) \to X$, so a curvature and a closed differential form $\hat{A}(g,H) \in \Omega^\bullet(X)$. It enters into the compatibility between the integrations (1.117) and (1.116). Namely, if

$$\omega\colon \widecheck{KO}^q(X) \to \Omega(X;\mathcal{R})^q \tag{1.119}$$

is the curvature map, then

$$\omega(\pi_*\check{x}) = \pi_*\big(\hat{A}(g,H) \wedge \omega(\check{x})\big), \qquad \check{x} \in \widecheck{KO}^q(X). \tag{1.120}$$

There is a similar compatibility for integration in $\widecheck{H\mathbb{Z}}$ and $\check{R}$.

The classifying spectrum of $cAlg_{\mathbb{R}}^\times$ and the transfer map

We conclude this lecture with more about the algebraic topology of twistings. Some of what we sketch here will be useful—even crucial—in the application to superstring theory. Theorem 1.52 locates $Alg_{\mathbb{C}}^\times$ in algebraic topology, as a certain truncation of ko. We now investigate the Picard 2-groupoid $cAlg_{\mathbb{R}}^\times$ of real invertible algebras. Here there is not an "off-the-shelf" spectrum to compare to, so we simply give its classifying spectrum a name: 'r'. (Despite the choice of letter, this is *not* a ring spectrum.) Nonetheless, r does receive a map from ko via the Clifford algebra construction, and is in fact a ko-module (which factors down to an R-module). Real algebras are the "fixed points" of complex algebras under complex conjugation, and this leads to another realization in algebraic topology, also connected to ko and its truncation R. Of course, the involution by complex conjugation is used in defining w-twisted twistings of K-theory; see Definition 1.99. We also define a quadratic map—a "transfer map"—from complex algebras to real algebras. This is only interesting when considered equivariantly for complex conjugation. We prove a formula for this quadratic map which is crucial in proving the anomaly cancellation in Lecture 3. We remark that this transfer map is the effect on twistings of a norm map on twisted K-theory classes, and that norm map is part of the definition of Ramond-Ramond currents, which are not part of these lectures. We also revisit the Thom twistings attached to a real vector bundle and give formulas for them in algebraic topology.

I thank Matt Ando and David Gepner for valuable discussions about some of this material.

(1.121) *The spectrum r.* Let r denote the classifying spectrum of the Picard 2-groupoid $Alg_{\mathbb{R}}^{\times}$. We computed its homotopy groups in (1.40). The computation of the k-invariants is similar to the computation **(1.42)** in the complex case. The k-invariants are nonzero, but we won't need them for any computations, so we don't pursue the matter further. Analogous to Theorem 1.82 we have the following.

Theorem 1.122. *Let X be a local quotient groupoid. Then there is an isomorphism*

$$\pi_0 \mathfrak{Twist}_{KO}(X) \cong r^0(X). \tag{1.123}$$

(1.124) *The Clifford map.* The Clifford algebra of a real vector space is the map

$$c\colon \mathbf{Vect}_{\mathbb{R}} \longrightarrow Alg_{\mathbb{R}}^{\times}. \tag{1.125}$$

which assigns to a real vector space V the Clifford algebra $C\ell(V,Q)$ for any positive definite quadratic form Q on V. The set of positive definite inner products on V is contractible, so to make the construction we can form a category equivalent to $\mathbf{Vect}_{\mathbb{R}}$ whose objects are real vector spaces with positive definite inner product. Direct sum provides a monoidal structure on $\mathbf{Vect}_{\mathbb{R}}$ and there is an obvious symmetry. The map c is a homomorphism of symmetric monoidal Picard 2-groupoids:

$$C\ell(V_1 \oplus V_2, Q_1 \oplus Q_2) \cong C\ell(V_1,Q_1) \otimes C\ell(V_2,Q_2). \tag{1.126}$$

Therefore it induces a map of the classifying spectra. As mentioned earlier, connective ko is the classifying spectrum of $\mathbf{Vect}_{\mathbb{R}}$, so we obtain

$$c\colon ko \longrightarrow r. \tag{1.127}$$

We claim that c factors to a map

$$\begin{tikzcd} ko \arrow[r] \arrow[d, "c"'] & R \arrow[dl, "\bar{c}"] \\ r & \end{tikzcd} \tag{1.128}$$

This follows easily since the restriction of c to the homotopy fiber of $ko \to R$ is null: there is no possible nonzero map on homotopy groups. We easily compute the effect of $\bar{c}$ on homotopy groups:

(1.129)

	R	$\longrightarrow$	r
π_4	$\mathbb{Z}$	$\longrightarrow$	0
π_3	0	$\longrightarrow$	0
π_2	$\mathbb{Z}/2\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/2\mathbb{Z}$
π_1	$\mathbb{Z}/2\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/2\mathbb{Z}$
π_0	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/8\mathbb{Z}$

The map on π_0 is surjective and the maps on π_1, π_2 are isomorphisms.

(1.130) *Complexification.* There is an obvious complexification map

$$\otimes_{\mathbb{C}} \colon Alg_{\mathbb{R}}^{\times} \longrightarrow Alg_{\mathbb{C}}^{\times} \tag{1.131}$$

and it induces a map of spectra

$$\otimes_{\mathbb{C}} \colon r \longrightarrow R^{-1}. \tag{1.132}$$

The composite $ko \xrightarrow{c} r \xrightarrow{\otimes_{\mathbb{C}}} R^{-1}$ attaches the complex Clifford algebra to a real vector space.

Remark 1.133. We do not believe this factors through k, the connective K-theory spectrum which is the classifying spectrum of $\mathbf{Vect}_{\mathbb{C}}$, since the space of nondegenerate quadratic forms on a complex vector space is not contractible.

(1.134) *The universal double cover.* Define (see also (**1.94**))

$$B\mathbb{Z}/2\mathbb{Z} = \mathrm{pt}/\!/(\mathbb{Z}/2\mathbb{Z}), \tag{1.135}$$

the groupoid with a single object and automorphism group $\mathbb{Z}/2\mathbb{Z}$. There is a canonical double cover

$$\pi_0 \colon \mathrm{pt} \longrightarrow B\mathbb{Z}/2\mathbb{Z}. \tag{1.136}$$

It is the universal double cover in the sense that any double cover, say $\pi\colon X_w \to X$, is classified by a unique map

$$\begin{tikzcd} X_w \arrow[r] \arrow[d, "\pi"] & \mathrm{pt} \arrow[d, "\pi_0"] \\ X \arrow[r] & B\mathbb{Z}/2\mathbb{Z} \end{tikzcd} \tag{1.137}$$

Let w_0 denote the double cover (1.136).

(1.138) *Complex conjugation.* Complex conjugation is the antilinear map $\mathbb{C} \to \mathbb{C}$ which takes

$$z \longmapsto \bar{z}. \tag{1.139}$$

The fixed points are $\mathbb{R} \subset \mathbb{C}$. What we need is a categorical version. For example, there is an involution $\mathbf{Vect}_{\mathbb{C}} \to \mathbf{Vect}_{\mathbb{C}}$ which takes a complex vector space W to the complex conjugate vector space $\overline{W}$ and a linear map to the complex conjugate linear map. (It contains the previous complex conjugation by restricting to $\mathrm{End}(\mathbb{C}) = \mathbf{Vect}_{\mathbb{C}}(\mathbb{C},\mathbb{C})$.) Whereas (1.139) is an involution on the set $\mathbb{C}$—that is a condition on a map—to say that complex conjugation is an involution on the category $\mathbf{Vect}_{\mathbb{C}}$ is to give extra data which then satisfies a condition. We do not spell that out here.

Similarly, we can say that the "fixed points" of that involution on $\mathbf{Vect}_{\mathbb{C}}$ is the category $\mathbf{Vect}_{\mathbb{R}}$, but here 'fixed points' must be understood as 'homotopy fixed points'. So to give a fixed point is to give a pair (W, J) consisting of $W \in \mathbf{Vect}_{\mathbb{C}}$ and an isomorphism $J\colon W \to \overline{W}$ which satisfies a condition relative to the involution which amounts to $\overline{J}J = \mathrm{id}_W$.

Our interest here is in complex conjugation on the Picard 2-groupoid $Alg^{\times}_{\mathbb{C}}$. It maps an invertible complex algebra A to its complex conjugate algebra $\overline{A}$, and similarly for invertible bimodules and intertwiners. The homotopy fixed point category is $Alg^{\times}_{\mathbb{R}}$.

In terms of algebraic topology the involution of complex conjugation defines a fibering

$$E \longrightarrow B\mathbb{Z}/2\mathbb{Z} \tag{1.140}$$

with fiber R^{-1}, and the spectrum r is the spectrum $\Gamma(B\mathbb{Z}/2\mathbb{Z}, E)$ of global sections of (1.140). Now (1.140) is the twisting of R-cohomology from the universal double cover (1.136), and so this gives a new interpretation of the cohomology theory r in terms of a twisted version of R:

$$r^{\bullet}(X) \cong R^{w_0+\bullet-1}(X \times B\mathbb{Z}/2\mathbb{Z}). \tag{1.141}$$

Since twisted R-cohomology is an R-module, we deduce that r is an R-module, and by pullback a ko-module as well.

Note that the pullback of the bundle (1.140) of spectra under (1.136) is simply the spectrum R^{-1}, so pullback by π_0 is a model for complexification (1.132).

(1.142) *Two universal twisted KO-theory classes.* We introduce

$$\theta \in R^{w_0-1}(B\mathbb{Z}/2\mathbb{Z}). \tag{1.143}$$

It, and the class χ to be introduced shortly, was already defined in **(1.13)** in terms of our concrete model. Namely, the twisting w_0 of $B\mathbb{Z}/2\mathbb{Z}$ is defined by the nontrivial grading of the group $\mathbb{Z}/2\mathbb{Z}$: the identity homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Twisted ko-classes are then represented by real super vector spaces with an action of $\mathbb{Z}/2\mathbb{Z}$ compatible with the grading. The matrix γ in (1.14) is odd and squares to the identity, so defines such an action on the vector space $\mathbb{R}^{1|1}$. The degree shift -1 is implemented by a $C\ell_{-1}$-module structure which (graded) commutes with the group action, and the generator of the Clifford action is the matrix x in (1.14). This defines a class in $ko^{w_0-1}(B\mathbb{Z}/2\mathbb{Z})$, and we let (1.143) be its image under the map $ko \to R$.

Similarly, we define the class

$$\chi \in R^{-w_0+1}(B\mathbb{Z}/2\mathbb{Z}). \tag{1.144}$$

The twisting $-w_0$ of $B\mathbb{Z}/2\mathbb{Z}$ is defined by the nontrivial $\mathbb{Z}/2\mathbb{Z}$-graded central extension of $\mathbb{Z}/2\mathbb{Z}$, where the grading is as before and the central extension is

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{1.145}$$

We simply switch the role of the matrices in (1.14): now x is the action of the generator of $\mathbb{Z}/2\mathbb{Z}$ and γ the action of the generator of $C\ell_1$. Again this is a class in twisted ko and we define (1.144) as its image under $ko \to R$.

We need two facts about θ and χ. First, the product $\theta\chi$ is a class in $R^0(B\mathbb{Z}/2\mathbb{Z})$, which is the image of a class in $ko^0(B\mathbb{Z}/2\mathbb{Z})$, the spectrum associated to (untwisted) representations of the group $\mathbb{Z}/2\mathbb{Z}$. We claim that

$$\theta\chi = \rho, \tag{1.146}$$

where ρ is the image of the regular representation of $\mathbb{Z}/2\mathbb{Z}$. Second, the complexification of θ is a class

$$\eta \in R^{-1}(\mathrm{pt}) \tag{1.147}$$

which is represented by the Clifford module $\mathbb{R}^{1|1}$ for the Clifford algebra $C\ell_{-1}$. Again, the Clifford module represents in the first instance a class in $ko^{-1}(\mathrm{pt})$ and we use the map $ko \to R$.

Exercise 1.148. Verify these two facts using the explicit models. For the first you'll need to use a Morita trivialization of the Clifford algebra $C\ell_{1,1}$. For the second you'll need to use that complexification is the map π_0^*, as explained at the end of (**1.138**).

(1.149) *A commutative diagram.* The following shows a relationship between two ways of passing from a class in ko to a class in r.

Lemma 1.150. *The diagram*

$$\begin{tikzcd} ko \arrow[r] \arrow[d, "c"'] & R \arrow[d, "\theta"] \arrow[dl, "\bar{c}"'] \\ r \arrow[r, "(1.141)"] & \Gamma(B\mathbb{Z}/2\mathbb{Z}, E) \end{tikzcd} \tag{1.151}$$

commutes.

Proof. Since all spectra in (1.151) are ko-modules, it suffices to verify the diagram for the multiplicative unit $1 \in ko^0(\mathrm{pt})$. I do not see a proof using the explicit models, since we don't have a model which sees constructs complex invertible super algebras (R^{-1}) from Cl_{-1}-modules (ko^{-1}). Absent that we use the following topological computation, proved in §3 of [2]. We can then define θ as the generator of this cyclic group which makes the diagram commute. □

Theorem 1.152. *The group $R^{w_0-1}(B\mathbb{Z}/2\mathbb{Z}) \cong R^{w_0-2}(B\mathbb{Z}/2\mathbb{Z};\mathbb{R}/\mathbb{Z})$ is cyclic of order 8. Furthermore, the pullback of a generator θ under $\pi_0\colon \mathrm{pt} \to B\mathbb{Z}/2\mathbb{Z}$ is η.*

This group has an interpretation as the group of degrees in KO-theory, and the $\mathbb{Z}/8\mathbb{Z}$ is that of Bott periodicity. The last statement in Theorem 1.152 was observed around (1.147).

(1.153) *A quadratic map on twistings.* According to Theorem 1.103 the group of isomorphism classes of twistings of KR-theory on X_w, also known as w-twisted twistings, is $R^{w-1}(X)$. For any orbifold X the group of isomorphism classes of twistings of KO-theory on X, as a special case of the same theorem, is $R^{w_0-1}(X \times B\mathbb{Z}/2\mathbb{Z})$, where the double cover (1.136) is pulled back via projection onto the second factor. Define

$$\Re\colon R^{w-1}(X) \longrightarrow R^{w_0-1}(X \times B\mathbb{Z}/2\mathbb{Z}) \tag{1.154}$$

as pushforward under the map

$$p\colon X \longrightarrow X \times B\mathbb{Z}/2\mathbb{Z}, \tag{1.155}$$

where p is the identity onto the first factor and the bottom arrow in (1.137) on the second. Note that (1.137) provides an isomorphism $p^*w_0 \xrightarrow{\cong} w$, and this is used to define the pushforward (1.154). This pushforward may be regarded as a *transfer map* in equivariant cohomology.

$\Re$ maps KR-twistings to KO-twistings. It has a geometric definition in the model of Lecture 1. Roughly, if β is a KR-twisting, then $\Re(\beta)$ is a real lift of $\bar\beta + \beta$. In terms of an invertible algebra bundle (A, B, λ), we form $(\overline{A} \otimes A, \overline{B} \otimes B, \overline{\lambda} \otimes \lambda)$. The "real lift" is accomplished via a Morita equivalence, which is canonical if we introduce a hermitian structure.

According to (1.141) we can also view $\Re$ as a map

$$\Re\colon R^{w-1}(X) \longrightarrow r^0(X). \tag{1.156}$$

The following lemma plays a crucial role at the end of Lecture 3.

Lemma 1.157. *The map $\Re\colon R^{w-1}(X) \to r^0(X)$ is multiplication by $q^*\chi \in R^{-w+1}(X)$ followed by $\bar c\colon R^0(X) \to r^0(X)$.*

Here $\chi \in R^{-w_0+1}(B\mathbb{Z}/2\mathbb{Z})$ is the class defined in (1.144).

Proof. We must show that the two maps (1.154) and multiplication by $q^*\chi$ are equal on the category of spaces X equipped with a double cover. There is an equivalent statement about $\mathbb{Z}/2\mathbb{Z}$-equivariant cohomology on the double cover. Now to prove that the two maps are equal it suffices[13] to check two universal cases: (i) $X = B\mathbb{Z}/2\mathbb{Z}$ with the nontrivial double cover π_0 in (1.136), and (ii) the lift to $X = \mathrm{pt}$. In the universal cases it suffices to check on a generator, since both are R-module maps. For (i) the generator is $\theta \in R^{w_0-1}(B\mathbb{Z}/2\mathbb{Z})$, as defined in **(1.142)**. Now for $X = B\mathbb{Z}/2\mathbb{Z}$ the map (1.155) is the diagonal $\Delta\colon B\mathbb{Z}/2\mathbb{Z} \to B\mathbb{Z}/2\mathbb{Z} \times B\mathbb{Z}/2\mathbb{Z}$ and (1.154) is the pushforward

$$\Delta_*\colon R^{w_0-1}(B\mathbb{Z}/2\mathbb{Z}) \longrightarrow R^{w_0-1}(B\mathbb{Z}/2\mathbb{Z} \times B\mathbb{Z}/2\mathbb{Z}), \tag{1.158}$$

[13]This follows from a general theorem in equivariant homotopy theory. We used the nonequivariant version of this theorem in the proof of Lemma 1.150: two maps of spectra are homotopic if they induce the same map on homotopy groups. In the equivariant case we need to check this on all fixed point spectra, and we need to be careful to use the correct notion of fixed points.

where the twisting w_0 in the codomain is pulled back from the second factor. Using (1.141) we identify this as the pushforward

$$(\pi_0)_*\colon r^0(\mathrm{pt}) \longrightarrow r^0(B\mathbb{Z}/2\mathbb{Z}), \tag{1.159}$$

and this pushforward on the ko-module r is induced by the pushforward

$$(\pi_0)_*\colon ko^0(\mathrm{pt}) \longrightarrow ko^0(B\mathbb{Z}/2\mathbb{Z}), \tag{1.160}$$

on ko. The generator $1 \in ko^0(\mathrm{pt})$ is represented by the trivial real line $\mathbb{R}$, and its image under $(\pi_0)_*$ is represented by the regular representation ρ of $\mathbb{Z}/2\mathbb{Z}$; this is the standard description as the induced representation of the trivial representation. Now the desired equality follows from (1.146) and tracing back to (1.158).

For (ii) we prove the equality of $\Re\colon R^{-1}(\mathrm{pt}) \to r^0(\mathrm{pt})$ and multiplication by the lift of χ to $R^1(\mathrm{pt})$. Since $R^1(\mathrm{pt}) = 0$, the latter map is the zero map. As for the former, we check that the action on homotopy groups (see (1.40)) is trivial. Note the map takes an complex invertible algebra A to $A \otimes \overline{A}$, which is Morita trivial. This proves the map is zero on π_0. For π_1 we observe that if L is a complex super line, then $L \otimes \overline{L}$ is even, so the induced map on π_1 is zero. The maps on π_2, π_3 are obviously zero. ☐

(1.161) *Thom twistings revisited.* The canonical twisting $\tau^h(V)$ in the cohomology theory h associated to a real vector bundle $V \to X$ is described in **(2.44)**, where its role in integration is emphasized. Particular models for K-theory are given in **(1.92)**. Here we simply state formulas for the isomorphism classes of the various twistings. They all have flat differential lifts which we don't put into the notation in this section.

Suppose $X_w \to X$ is a double cover with classifying map

$$\begin{tikzcd} X_w \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ X \arrow[r, "q"] & B\mathbb{Z}/2\mathbb{Z} \end{tikzcd} \tag{1.162}$$

Then for the K-theory and KR-theory Thom twistings we have

$$\begin{aligned} \big[\tau^K(V)\big] &= \eta\,[V] && \in R^{-1}(X) \\ \big[\tau^{KR}(V)\big] &= q^*(\theta)\,[V] && \in R^{w-1}(X) \end{aligned} \tag{1.163}$$

As a special case of the second equation we have

$$\big[\tau^{KO}(V)\big] = \theta\,[V] \quad \in R^{w_0-1}(X \times B\mathbb{Z}/2\mathbb{Z}), \tag{1.164}$$

which, by Lemma 1.150 is equivalent to

$$\big[\tau^{KO}(V)\big] = c(V) \quad \in r^0(X). \tag{1.165}$$

Exercise 1.166. Prove (1.163)–(1.165). Here are a few hints. It is easiest to begin with (1.165) and observe that the real Clifford bundle, which is the map c, is a model for the Thom twisting; see (**1.92**). Then (1.164) follows from (1.151). The K-theory twisting is gotten by complexification, for which we use the second part of Exercise 1.148.

Lecture 2: Fields and superstrings

In the remaining lectures we turn to geometric structures in superstring theory. This lecture contains definitions; the next contains a theorem. We execute the traditional three steps in applied mathematics: (i) model a system external to core mathematics—in this case a physical system—in mathematical terms; (ii) prove theorems about the mathematical model; and (iii) draw conclusions about the external system from the mathematical theorems. Today's lecture is part of Step (i); tomorrow's is an example of Step (ii). We do not discuss the physics of Step (iii) here, except to say that the theorem in Lecture 3 is a consistency check on the physical system. There are other physical consequences of our work which we do not discuss here. One attraction of this particular application of mathematics, as with many others, is that it suggests problems and developments in core mathematics. Here what it suggests are ideas in a rich mix of homotopy theory, differential geometry, and global analysis. For example, the modern developments in twisted K-theory were directly inspired by this physical system.

More specifically, we work with a "semi-classical" model for strings in terms of fields, which are classical objects that belong to differential geometry, though this example presses us to bring in homotopy theory as well. There are a few key "quantum" ideas which enter also—Dirac's quantization of charge, the fermionic functional integral—and they shape our considerations. The main point of this lecture is to pin down the topological aspects of fields in superstring theory. In fact, here we only describe the definition of the "fields", for the most part not the "action" with the notable exception of the B-field amplitude on the worldsheet. The theorems we prove from these definitions, including the one in Lecture 3, provide evidence that our mathematical model for the fields is "correct". Usual considerations in physics pin down the local field content; the topological subtleties involve more refined physical and mathematical considerations.

Superstring theory is studied in its usual perturbative formulation as a 2-dimensional field theory on "string worldsheets". In this theory spacetime is an external 10-dimensional manifold X. There is a low energy approximation which is a 10-dimensional field theory formulated in terms of fields on X. The topological nature of the fields is the same across the two theories, and it is striking how tightly constrained the system is and how well the same data works in the 2-dimensional and 10-dimensional theories. The "orientifold" construction provides the best testing ground and the deepest agreement between the two theories.

We begin with a general discussion of the notion of a field, leading to a definition in terms of simplicial sheaves. A key construction in this lecture is what physicists call "gauging a symmetry". In the context of simplicial sheaves this is the natural quotient construction, at least for finite groups of symmetries. This gives the field content of the gauged theory; the action of the gauged theory requires additional data. We only discuss one term in the action—the "B-field amplitude" in the 2-dimensional theory—and the lecture ends with a puzzle: There is no natural extension of this action to orientifolds. The resolution of the mystery is in Lecture 3.

General discussion of fields

Physicists speak in terms of "scalar" fields, "gauge fields", etc. Here we give a mathematical framework in which to think about the notion of a field. We take the notion more broadly—and more abstractly—than is usual. For example, we consider topological structures such as orientations and spin structures to be fields. One motivation for this discussion is our description below in (**2.39**) of the orientifold construction as a gauging of the orientation-reversal symmetry. The reader can safely skip it and get more quickly to the less structural discussions in subsequent sections.

(2.1) *Examples of fields.* Let's begin by listing some types of fields by telling what they are on a smooth manifold M:

(i) a *scalar field* with values in a fixed manifold Z is a map $M \to Z$, and if $Z = \mathbb{R}$ is is called a *real scalar field*;
(ii) a *metric* ('gravitational field' in physics-speak) is a Riemannian[14] metric on M;
(iii) an *orientation* is...well...an orientation on M; similarly for a *spin structure*;
(iv) given a complex spinor representation of Spin_n, a *spinor field* on an n-dimensional manifold with a Riemannian metric and spin structure is a section of the bundle associated to the given representation;
(v) for a fixed Lie group G a G*-connection* ('gauge field' in physics-speak) is a pair (P, Θ) of a principal bundle $P \to M$ and a connection Θ on P;
(vi) the term 'B*-field*' has many meanings; in Type II superstring theory it is a twisting of K-theory on M (Definition 1.78).

(2.2) *Categories of manifolds.* We want the notion of a field $\mathcal{F}$ to be valid not just on a single manifold M, but rather on a collection of manifolds, and fields must pullback under a collection of maps between the manifolds. In other words, we will define a *category* **Man** of manifolds and a target category $\mathcal{C}$ such that a field is a *homomorphism*[15]

$$\mathcal{F} \colon \mathbf{Man}^{op} \longrightarrow \mathcal{C}. \tag{2.3}$$

For example, a real scalar field is the homomorphism $\mathcal{F} \colon \mathbf{Man}^{op} \to \mathbf{Set}$ which assigns the set $\mathcal{F}(M) = C^\infty(M)$ to each smooth manifold M. We would then say a real scalar field on M is a particular element in $\mathcal{F}(M)$.

How should we define **Man**? For scalar fields (i) we can take **Man** to be the category of all smooth manifolds and all smooth maps between them. However, this is too big for (ii): metrics do not pull back under arbitrary maps. To accommodate metrics, then, we can still take the category of all smooth manifolds, but restrict to immersions. Now (iii) forces us to rule out immersions which are not local diffeomorphisms, so we are reduced to the latter. The spinor field (iv) depends on a particular spinor representation, so requires us to further fix a dimension n and consider only n-manifolds.

Definition 2.4. For each integer $n \geqslant 0$ define $\mathbf{Man}_n$ as the category whose objects are smooth n-manifolds and morphisms are local diffeomorphisms.

[14]We work in the "Wick rotated" framework; we could instead use Lorentz metrics.
[15]a (contravariant) *functor* in traditional terminology

(2.5) *The codomain.* For many of the examples in (**2.2**) we can take the codomain $\mathcal{C}$ in (2.3) to be **Set**: on a fixed manifold M scalar fields, metrics, orientations, and spinor fields form a set. But spin structures and G-connections have automorphisms, and we need a structure which tracks them. In these examples we can take $\mathcal{C} = \mathbf{Gpd}$ to be the category of groupoids. The morphisms of groupoid-valued fields are called "gauge transformations". But for (vi) we need to go further: twistings of K-theory on M form a 2-groupoid—see (**1.80**). Though it is not the subject of these lectures, we remark that the "C-field" in M-theory takes values in a 3-groupoid. To accommodate all of these examples, and to have a flexible mathematical framework with plenty of foundational work in the literature, it is convenient to take $\mathcal{C} = \mathbf{Set}_\Delta$, the category of *simplicial sets.* A general field in an n-dimensional field theory, then, is a homomorphism

$$\mathcal{F}\colon \mathbf{Man}_n^{op} \longrightarrow \mathbf{Set}_\Delta. \tag{2.6}$$

This is sometimes termed a *simplicial presheaf* on the category $\mathbf{Man}_n$. Fields in physics satisfy a locality property, encoded in mathematics by the *sheaf* property, which is crucial, but which we do not spell out here.

Definition 2.7. A *field*, or *collection of fields*, in an n-dimensional field theory is a simplicial sheaf $\mathcal{F}\colon \mathbf{Man}_n^{op} \to \mathbf{Set}_\Delta$.

Remark 2.8. In a given field theory we define $\mathcal{F}$ to be the collection of *all* fields in the theory. Notice that if a theory has, say, a scalar field and a metric, then $\mathcal{F} = \mathcal{F}_{\text{scalar}} \times \mathcal{F}_{\text{metric}}$ is a product. But if the theory has a metric, spin structure, and spinor field, then $\mathcal{F}$ is not the product of three factors since a spinor field cannot be defined without first having a metric and spin structure. So in general $\mathcal{F}$ is an iterated fibration of the individual fields. Furthermore, the fields may be divided into *background* and *dynamical* fields (also called *fluctuating fields* in a quantum theory). For example, in the theory of a scalar field on a Riemannian manifold we might consider the metric as background and the scalar field as dynamical. Fields pertinent to the intrinsic geometry of the manifold—metrics, orientations, spin structures—are background fields in non-gravitational theories but are dynamical in theories of gravity. String theory *is* a theory of gravity. Thus, for example, in the superstring one sums over the spin structures on the worldsheet.

Quantization of charge

(2.9) *Classical electromagnetism.* We briefly recall the setup for Maxwell's theory of the electromagnetic field. We work on a four-dimensional spacetime of the form $M^4 = \mathbb{E}^1 \times N^3$, where (N^3, g_N) is a Riemannian manifold. We endow M with the Lorentz metric $dt^2 - g_N$, where t is a (time) coordinate on $\mathbb{E}^1$ and the speed of light is set to unity. Minkowski spacetime is the case $N = \mathbb{E}^3$.

Classical electromagnetism involves four time-dependent fields:

$$\begin{aligned} E(t) &\in \Omega^1(N) && \text{electric field} \\ B(t) &\in \Omega^2(N) && \text{magnetic field} \\ \rho_E(t) &\in \Omega^3_c(N) && \text{electric charge density} \\ J_E(t) &\in \Omega^2_c(N) && \text{electric current} \end{aligned} \tag{2.10}$$

Here Ω_c denotes differential forms of compact support. Traditional texts identify E, B, J_E with vector fields and ρ_E with a function.[16] But the differential form language is more convenient and leads to a better geometric picture. The classical Maxwell equations are

$$\begin{aligned} dB &= 0 & \frac{\partial B}{\partial t} + dE &= 0 \\ d *_N E &= \rho_E & *_N \frac{\partial E}{\partial t} - d *_N B &= J_E \end{aligned} \tag{2.11}$$

We reformulate these equations using differential forms on M with its Lorentz metric and corresponding Hodge $*$ operator as follows. Set

$$\begin{aligned} F &= B - dt \wedge E && \in \Omega^2(M) \\ j_E &= \rho_E + dt \wedge J_E && \in \Omega^3(M). \end{aligned} \tag{2.12}$$

The *electric current* j_E has compact *spatial* support. Maxwell's equations (2.11) are equivalent to the pair of equations

$$\begin{aligned} dF &= 0 \\ d * F &= j_E. \end{aligned} \tag{2.13}$$

As a consequence of the second equation we have

$$dj_E = 0. \tag{2.14}$$

The de Rham cohomology class of ρ_E in $H^3_c(N;\mathbb{R})$ is the *electric charge*; (2.14) implies that it is independent of time.

(2.15) *Charges in quantum theory.* To write a quantum mechanical theory which incorporates electromagnetism—for example, the nonrelativistic Schrödinger equation for a charged particle moving in a background electromagnetic field—the gauge potential A, and not just the electromagnetic field $F = dA$, appears. This assertion has an experimental basis, due to Aharanov and Bohm. Furthermore, it is an empirical fact that nobody has written a quantum theory in terms of F alone. Accepting the necessity of the gauge potential, the quantization of charge is based

[16] This assumes that M is oriented. If not, then ρ_E, J_E are forms twisted by the orientation bundle. The vector field which corresponds to B is also twisted.

on: (i) the existence of a system in which magnetic current j_B and electric current j_E are both nonzero, and (ii) the particular coupling of A to the electric current in the quantum theory. The text in this discussion, beginning with **(2.9)**, is taken almost *verbatim* from the leisurely discussion in [17] to which I refer the reader for a continuation. The upshot is that in a quantum theory the electric and magnetic charges—the de Rham cohomology classes of j_E and j_B restricted to a time slice—are required to live in a full lattice in the de Rham cohomology vector space. There are further considerations which prompt a refinement of the lattice to an abelian group of charges, which may include torsion charges. Finally, that abelian group should depend locally on space, so it is reasonable to postulate that it is a cohomology group in some generalized cohomology theory.

For Maxwell electromagnetism **(2.9)** the "correct" cohomology theory is ordinary Eilenberg-MacLane cohomology $H\mathbb{Z}$. So the de Rham class of ρ_E in $H^3_c(N;\mathbb{R})$ lies in the image of $H^3_c(N;\mathbb{Z})$, and in fact there is a cohomology class $Q \in H^3(M;\mathbb{Z})$ (with compact spatial support) which is compatible with j_E in the sense that the de Rham class of j_E is the image of Q under $H^3(M;\mathbb{Z}) \to H^3(M;\mathbb{R})$. Recalling the discussion in **(1.105)** it is natural to assume a refinement $\check{j}_E \in \check{H}^3(M)$ which fits into the diagram:

(2.16)
$$\begin{tikzcd} \check{j}_E \arrow[r, mapsto] \arrow[d, mapsto] & j_E \arrow[d, mapsto] \\ Q \arrow[r, mapsto] & Q_R \end{tikzcd}$$

The left map gives the component in the differential cohomology group (see (1.106)), the top map is the curvature, the bottom map is $\otimes_{\mathbb{Z}}\mathbb{R}$, and the right map is the de Rham cohomology class. The right column is classical; the left column is the quantum refinement taking into account Dirac charge quantization. There is a diagram analogous to (2.16)—with all degrees reduced by 1—for the gauge field itself. The upper right corner of that diagram is the *field strength* F and the lower left corner is sometimes called the *flux*. The upper left corner is an object in $\check{H}^2(M)$, which can be taken to be a principal $\mathbb{T}$-bundle with connection. This is the usual geometric model for the Maxwell gauge field.

Perhaps better is to think directly in terms of the picture sketched in **(1.105)**. We then lift the classical current j_E, which is a closed differential form, to its fiber of the map (1.107). The fiber is a torsor for $H^2(M;\mathbb{R}/\mathbb{Z})$ and this tells the extra information in the quantum theory—beyond the restriction that the de Rham class of j_E lie in a full lattice.

(2.17) *Which cohomology theory?* The Dirac argument is taken to apply to all "generalized abelian gauge fields" in quantum field theories. The field is locally a differential form, but globally has integrality encoded in a generalized cohomology theory h, and the art is in finding the correct cohomology theory. Then there are diagrams (2.16) for both the current and gauge field. What physical and mathematical considerations go into the choice of generalized cohomology theory h and a particular degree in that theory? There are several possible:

(i) First and foremost, $h\otimes\mathbb{R}$ in the given degree must duplicate the known local field content.

[17] D. S. Freed, *K-theory in quantum field theory*, Current Developments in Mathematics 2001, International Press, Somerville, MA, pp. 41-87 (`arXiv:math-ph/0206031`).

(ii) There may be torsion charges not detected in the classical formulation with differential forms, and the theory may contain charged solitonic objects which exhibit the torsion charges.

(iii) Anomaly cancellation, a condition for the theory to be consistent, is sensitive to the choice of h. This occurs in the "Green-Schwarz mechanism", which occurs in several contexts. Here some form of K-theory is involved as an anomaly from the currents must cancel against an anomaly computed using the Atiyah-Singer index theorem.

(iv) There may be an equivariant version, e.g. on orientifolds, and then an appropriate equivariant version of h comes into play and must exist with the correct field content after tensoring with $\mathbb{R}$.

(v) There may be special geometric features in the system.

I refer to [18] for a more detailed discussion of charge quantization for generalized gauge fields and for many examples.

(2.18) *Generalized abelian gauge fields in superstring theory.* There are two generalized abelian gauge fields in the Type II superstring: the "B-field" and the "Ramond-Ramond field". We will argue that the former is a differential twisting of K-theory and the latter an object representing a twisted differential K-theory class. All of the considerations listed in **(2.17)** are relevant. Point (i) is *always* a consideration. In the Type I superstring, for example, there are solitonic objects with torsion charges, as first identified by Witten, who used (ii) to argue for the correctness of K-theory to quantize the Ramond-Ramond charges. Lecture 3 is an illustration of (iii) on the worldsheet. There are also anomaly cancellation arguments for the 10-dimensional spacetime theory, including the original argument of Green-Schwarz in the Type I theory. Consideration (iv) applies very neatly to Ramond-Ramond fields on orbifolds. There we use Atiyah-Segal equivariant K-theory, and the tensor product with $\mathbb{R}$ has contributions from twisted sectors (via the localization theorem in equivariant K-theory) which matches perturbative computations in string theory. Finally, (v) also applies: the coupling of the B-field and Ramond-Ramond field seen in terms of differential forms is manifest by using the quantum B-field (with integrality in R-theory) to twist K-theory and so the notion of a Ramond-Ramond field.

The oriented bosonic string

(2.19) *Two sets of fields.* Quantum field theory has a fixed dimension of spacetime and fields are local objects (Definition 2.7) on manifolds of the given dimension. String theory is confusing at first as there is, in addition to spacetime, a *worldsheet* of dimension 2. We consistently use the letter 'X' to denote spacetime and 'Σ' to denote the worldsheet. Both are smooth manifolds; Σ has dimension 2 and the dimension of X depends on the particular string theory. We discuss two cases. For the oriented bosonic string $\dim X = 26$ and for the superstring $\dim X = 10$. These dimensions arise out of standard nontopological considerations and we do not discuss them further here. We do remark that the number 10 (reduced mod 8) plays a crucial role in Lecture 3; the argument

[18] D. S. Freed, *Dirac charge quantization and generalized differential cohomology*, Surv. Differ. Geom. VII, 2000, 129–194 (`arXiv:hep-th/0011220`).

there would not work if the dimension of spacetime were 11. We always take Σ to be compact. In these lectures it will be closed in the sense that it is a manifold without boundary. Worldsheets with boundary are very important in string theory—they are "open strings"—but we will not have time to deal with them in these lectures.

For each of string theories there is a set of fields $\mathcal{F}_{26}$ or $\mathcal{F}_{10}$ on spacetime and a set of fields $\mathcal{F}_2[X]$ on the worldsheet. The order is important. A fixed choice of spacetime X and of fields on X—an element in $\mathcal{F}_{26\text{ or }10}(X)$—is used as external data to define $\mathcal{F}_2[X]$ and is indicated by '$[X]$' in the notation. This external data is analogous to both the choice of gauge group in a gauge theory and to a choice of coupling constants in any theory. On the other hand, $\mathcal{F}_{26\text{ or }10}$ does not depend on the worldsheet at all.

(2.20) $\mathcal{F}_{26}$. There are three fields on a spacetime in oriented bosonic string theory, and they are *independent* in the sense that $\mathcal{F}_{26}$ is the Cartesian product of three sheaves. In other words, each of the three fields may be defined without defining the other two. The fields in $\mathcal{F}_{26}(X)$ are (see **(2.1)**):

(i) a metric;
(ii) a real scalar field, called the "dilaton";
(iii) a "B-field" $\check{\beta}$, a "gerbe with connection", which on a manifold X has an equivalence class in $\check{H}^3(X)$.

For the purposes of these lectures we take a gerbe on X to have a geometric model given by $(P; L, \lambda)$, which is a restricted kind of twisting of K-theory. Here, as in Definition 1.78, P is a topological groupoid equipped with a local equivalence $P \to X$. (Some typical examples appear in Example 1.74 and Example 1.75.) The pair (L, λ) is a special complex invertible algebra bundle over P in which the bundle of algebras over P_0 is the trivial bundle with fiber $\mathbb{C}$ and the line bundle $L \to P_1$ is purely even; it was mentioned in Example 1.73. The connection data in this model is a pair (B, ∇), as described in **(1.109)**. This case is simpler than the general invertible algebra bundle described there as ∇ is a covariant derivative on a line bundle.

Remark 2.21. These three fields appear in the superstring as well, but then the B-field is there an arbitrary twisting of K-theory. The use of the phrase 'restricted kind of twisting of K-theory' to describe the oriented bosonic string B-field is pure convenience: there is no K-theory in the oriented bosonic string and the B-field doesn't twist anything. The metric and dilaton do not play any role in these lectures, which focus on topological aspects of strings.

(2.22) $\mathcal{F}_2[X]$. As mentioned in **(2.19)** to define the worldsheet fields $\mathcal{F}_2[X]$ we fix an *oriented string background*, which consists of a smooth 26-dimensional manifold X and a choice of fields in $\mathcal{F}_{26}(X)$. Let $\check{\beta}$ denote the chosen B-field. There are three worldsheet fields for the oriented bosonic string, and they are independent in the sense described in **(2.20)**. The fields are:

(i) an orientation $\mathfrak{o}$;
(ii) a metric;
(iii) a scalar field ϕ with values in X.

On a worldsheet Σ the scalar field is a smooth map $\phi\colon \Sigma \to X$. If X were a true spacetime with a Lorentz metric, ϕ would encode the spacetime motion of a string. Here ϕ is an analog in Riemannian field theory. We remark that one can consider bosonic string theory without an orientation, in which case there is no B-field.

Note that $\mathcal{F}_2[X]$ is *set*-valued.

(2.23) *The B-field amplitude in the oriented bosonic string.* For the most part we do not discuss the worldsheet action for the string and we do not discuss at all the spacetime action which approximates string theory in terms of low energy fields. The one exception is the B-field amplitude in the worldsheet action. For the oriented bosonic string it is straightforward. First, recall that in classical field theory on a manifold Y the action is a function $S\colon \mathcal{F}(Y)\to\mathbb{R}$ on the set of fields. (In case $\mathcal{F}(Y)$ is a simplicial set, as it is here because of the presence of the B-field, we require that S factor through a function on $\pi_0\mathcal{F}(Y)$.) In the functional integral formulation of quantum field theory, it is the exponential e^{iS} or e^{-S} which is relevant, depending on the signature of the metric. We call this the *exponentiated action*. It may happen that only this exponential is well-defined, and often in those cases the exponential is the same independent of the signature of the metric. In this case we can view S as a function into $\mathbb{R}/2\pi\mathbb{Z}$. (We will move the '$2\pi$' to $e^{2\pi iS}$, so take S to have values in $\mathbb{R}/\mathbb{Z}$.)

Now suppose $\phi\colon\Sigma\to X$ is a scalar field as in (**2.22**)(iii). Then $\phi^*\check{\beta}$ is a geometric object on Σ which represents a class in $\check{H}^3(\Sigma)$. We can use the orientation $\mathfrak{o}$ on Σ to define an integration map, as in (1.117) and the discussion at the beginning of the paragraph which follows it. Denote that integration as

$$\pi_* = \int_{(\Sigma,\mathfrak{o})}\colon \check{H}^3(\Sigma)\longrightarrow \check{H}^1(\mathrm{pt})\cong\mathbb{R}/\mathbb{Z}. \tag{2.24}$$

This last isomorphism follows from the second isomorphism in (1.106), but of course the reader will need to understand more about differential cohomology to truly understand it. In any case we define the $\mathbb{R}/\mathbb{Z}$-valued B-field amplitude as

$$S(\phi,\mathfrak{o}) = \int_{(\Sigma,\mathfrak{o})}\phi^*\check{\beta}. \tag{2.25}$$

Note that the integral only depends on the equivalence class of $\check{\beta}$ in $\check{H}^3(X)$.

There is no interesting topology in this expression, but the analogous expressions for orientifolds, superstrings, and superstring orientifolds are more interesting from that point of view.

Orbifolds in string theory and in geometry

(2.26) *Gauging a symmetry.* We explain the general idea in field theory of gauging, working in the general framework of Definition 2.7. We gauge the symmetry of a finite, or discrete, group Γ. (To gauge a Lie group of symmetries, replace Galois covers with connections.)

Let $\mathcal{F}\colon \mathbf{Man}_n^{op}\to\mathbf{Set}_\Delta$ be a collection of fields in an n-dimensional field theory. Let Γ be a finite group, and suppose that Γ acts of $\mathcal{F}$. Before indicating what 'Γ acts on $\mathcal{F}$' means, let's give an example. Let Y be a 26-dimensional smooth manifold equipped with a Γ-action. Use it as an external background to define the fields $\mathcal{F}_2[Y]$ in the oriented bosonic string, as in (**2.22**). (We need to also choose a metric, dilaton, and B-field on Y which are Γ-invariant, but they are not essential to this discussion.) Then for each 2-manifold Σ, the group Γ acts on the *set* of fields $\mathcal{F}_2[Y](\Sigma)$, and the action commutes with pullback by local diffeomorphisms (in fact all smooth maps) of 2-manifolds. The action on the orientation and metric on Σ is trivial; the action of $\gamma \in \Gamma$ on $\phi\colon \Sigma \to Y$ yields $f_\gamma \circ \phi\colon \Sigma \to Y$, where $f_\gamma\colon Y \to Y$ is the action of γ on Y.

The general definition should be clear, except that for groupoid- or multi-groupoid valued fields there is more data to specify. We do not give details here.

Given a Γ-action on $\mathcal{F}$, define the sheaf of *gauged fields*

$$\mathcal{F}_\Gamma(\Sigma) = \{(P,\Phi) : P \to \Sigma \text{ is a principal } \Gamma\text{-bundle, } \Phi \in \mathcal{F}(P) \text{ is } \Gamma\text{-invariant}\}. \tag{2.27}$$

A principal Γ-bundle is also called a Galois covering space with Galois group (group of deck transformations) equal to Γ. The Γ-invariance of Φ means

$$R_\gamma^*\Phi = \gamma \cdot \Phi, \qquad \text{for all } \gamma \in \Gamma, \tag{2.28}$$

where on the left we pull back by the map $R_\gamma\colon P \to P$ and on the right we use the Γ-action on $\mathcal{F}(P)$. Note that $\mathcal{F} \subset \mathcal{F}_\Gamma$ as the gauged fields with trivial Γ-bundle.

Remark 2.29. In this context $\mathcal{F}_\Gamma$ is the natural quotient of $\mathcal{F}$ by the symmetry Γ. To see this suppose first that Y is a manifold and Γ acts as a group of symmetries on Y. The natural quotient construction in the world of simplicial sets is the groupoid $Y/\!/\Gamma$, which as a simplicial set is $Y_0 = Y$, $Y_1 = Y \times \Gamma$, etc; see (**1.55**). Now Y corresponds to a sheaf $\mathcal{F}$ whose value on a (test) manifold Z is the set of smooth maps $Z \to Y$. The groupoid $Y/\!/\Gamma$ corresponds to a simplicial sheaf whose value on a (test) manifold M is the set of smooth maps $M \to Y/\!/\Gamma$, and this is a pair consisting of a principal Γ-bundle $P \to M$ and a Γ-equivariant map $P \to Y$.

Exercise 2.30. Work out $\mathcal{F}_\Gamma$ for the worldsheet example given above. You should see that ϕ is now a section of a fiber bundle over Σ with fiber Y. Also, check that for fixed $P \to \Sigma$ its automorphism group acts on the space of Γ-invariant fields in $\mathcal{F}(P)$. What does that say in the special case when P is the trivial bundle?

Principal Γ-bundles form a groupoid, and $\mathcal{F}_\Gamma$ contains the information of isomorphisms of bundles. So $\mathcal{F}_\Gamma(\Sigma)$ naturally breaks up as a union over the isomorphism classes of bundles $P \to \Sigma$. These isomorphism classes are called *twisted sectors*, except for the isomorphism class of the trivial bundle, which is called the *untwisted sector*.

Remark 2.31. We have only discussed the fields here, not the action. We can also do that in a formal framework—there is a bordism category of n-manifolds equipped with a field and an action is an invertible field theory on that bordism category, and now we can see what it means to extend the action—but we will not carry that out here. We just point out that to extend the action is to provide more data, whereas to construct the fields in the gauged theory there is no further data required.

(2.32) *The usual meaning of twisted sectors.* In the case $n = 2$ of the string worldsheet theory, consider a cylinder $\Sigma = [0,1] \times S^1$, which models the propagation of a single string. Then twisted sectors are labeled by isomorphism classes of Γ-bundles over the cylinder, which is the same as isomorphism classes over S^1, and these are labeled by conjugacy classes in Γ.

Exercise 2.33. In the situation of Exercise 2.30 suppose we fix a basepoint in $P \to S^1$. Define the holonomy as an element $\gamma \in \Gamma$. Show that Γ-invariant maps $\phi\colon P \to Y$ correspond to maps $\tilde\phi\colon \mathbb{R} \to Y$ such that $\tilde\phi(t+1) = f_\gamma \circ \tilde\phi(t)$. What happens as we change the basepoint (in the same fiber of $P \to S^1$?)

(2.34) *Orbifolds in geometry.* Suppose as above a finite group Γ acts on a smooth 26-manifold Y. Points of Y connected by elements of Γ represent the same points of spacetime—Γ is a gauge symmetry—so it is natural to take spacetime as the quotient $Y/\!/\Gamma$. We keep track of isotropy subgroups, due to non-identity elements $\gamma \in \Gamma$ and $y \in Y$ with $f_\gamma(y) = y$. Now an old construction in differential geometry of Satake, called the 'orbifold' by Thurston, does exactly that. Furthermore, we can admit as spacetimes orbifolds X which are not global quotients by finite groups, thus widening the collection of models introduced in the previous paragraph. Orbifolds are presented by a particular class of groupoids, as was explained in **(1.55)**: each point has a neighborhood weakly equivalent to $Y/\!/\Gamma$ for a smooth manifold Y and a finite group Γ. Of course, a special case is the global quotient $X = Y/\!/\Gamma$. A worldsheet is then a map $\phi\colon \Sigma \to X$ of orbifolds, and the infinite-dimensional orbifold of such maps is precisely $\mathcal{F}_2[X](\Sigma)$, once we add a metric and orientation.

The upshot of this paragraph is that for a scalar field with values in a Γ-manifold Y, the orbifold quotient implements the gauging **(2.26)**.

And the upshot for oriented bosonic string theory is that we allow spacetime X to be a smooth 26-dimensional orbifold (in the sense of Satake-Thurston). But then we must extend $\mathcal{F}_{26}$ in **(2.20)** to orbifolds. This is straightforward for the metric and dilaton. But for the B-field we need some discussion, as anticipated in **(2.17)**(iv). We turn to that now.

(2.35) *Equivariant cohomology.* There are many extensions of a given cohomology theory h to an equivariant cohomology theory for spaces Y with the action of a compact Lie group G. The simplest is the *Borel construction*. It attaches to (Y,G) the space $Y_G = EG \times_G Y$, where EG is a contractible space with a free G-action. Then one defines the Borel equivariant h-cohomology as $h_G(Y) := h(Y_G)$. This is not a new cohomology theory, but rather the nonequivariant theory applied to the Borel construction, a functor from G-spaces to spaces. That functor generalizes to orbifolds which are not necessarily global quotients—the functor is *geometric realization*—and so leads to a notion of "Borel cohomology" theories on orbifolds, which we described in greater generality in **(1.84)**. But usually h has other extensions to an equivariant theory. For example, the Atiyah-Segal geometric version of equivariant K-theory, defined in terms of equivariant vector bundles, is more delicate: Borel equivariant K-theory appears as a certain completion. The Atiyah-Segal theory is extended to orbifolds, in fact to "local quotient groupoids", in [3].

We use generalized *differential* cohomology on orbifolds without further comment. There are papers which develop it in the case of a global quotient, but as far as I know there is work to be done in the general case.

(2.36) *The B-field on orbifolds.* We posit the following generalization of **(2.20)**(iii) to allow for X a smooth orbifold.

(iii) the B-field $\breve{\beta}$ is a gerbe with connection on the orbifold X.

Recall that our geometric model for a gerbe with connection, as discussed in **(2.20)**, already makes sense for an orbifold (and indeed in much greater generality). We use the classification result Theorem 1.82, restricted to gerbes rather than more general twistings of K-theory, to conclude that the topological classification of B-fields is by the cohomology group $H^3(|X|;\mathbb{Z})$, where as in **(1.84)** $|X|$ is the geometric realization of X.

The B-field amplitude (2.25) is unchanged when spacetime X is allowed to be an orbifold; the worldsheet Σ is still a smooth manifold, not an orbifold, and there is nothing new to say to define the integral.

Orientifolds of the oriented bosonic string

(2.37) *A bigger version of worldsheet fields.* In the context of the general discussion of fields at the beginning of the lecture, culminating in Definition 2.7, observe that a constant (simplicial) sheaf is trivially a field. In that spirit we now include the spacetime fields, heretofore viewed as external to $\mathcal{F}_2[X]$, as part of the worldsheet fields. We do this to encode the action of orientation reversal in string theory, which acts simultaneously on worldsheet and spacetime fields. Thus for a fixed 26-dimensional orbifold[19] X define $\widetilde{\mathcal{F}_2}[X]$ to include the fields:

(i) an orientation $\mathfrak{o}$;
(ii) a metric;
(iii) a scalar field ϕ with values in X;
(iv) a metric on X;
(v) a real scalar field on X;
(vi) a gerbe with connection $\breve{\beta}$ on X.

The fields (iv)–(vi) are constant in the sense that $\widetilde{\mathcal{F}_2}[X]$ is a homomorphism $\mathbf{Man}_2 \to \mathbf{Set}_\Delta$. If $\Sigma \in \mathbf{Man}_2$ is a smooth 2-manifold, then the metric (ii) in $\widetilde{\mathcal{F}_2}[X](\Sigma)$ is a metric on Σ, so depends on Σ, as do the fields (i) and (iii).

(2.38) *The involution on $\widetilde{\mathcal{F}_2}$.* Let Y be an orbifold and $\sigma\colon Y \to Y$ be an involution. The corresponding *parity* involution on $\widetilde{\mathcal{F}_2}[Y]$ has the following action on the fields enumerated in **(2.37)**:

(i) $\mathfrak{o} \mapsto -\mathfrak{o}$ (the opposite orientation $\mathfrak{o}$ is defined in **(1.112)**);
(ii) the metric on the 2-manifold is fixed;
(iii) $\phi \mapsto \sigma \circ \phi$;
(iv) the metric on Y is pulled back by σ;
(v) the scalar field on Y is pulled back by σ;
(vi) the B-field $\breve{\beta}$ transforms as $\breve{\beta} \mapsto -\sigma^*\breve{\beta}$.

[19]The reader can first think through the case when X is a smooth manifold. Allowing X to be an orbifold combines the orbifold construction above with the orientifold construction.

The motivation for the minus sign in (vi) is the B-field amplitude (2.25), which is then preserved by the involution (since both $\mathfrak{o}$ and $\check{\beta}$ change sign). The construction of the opposite B-field in the model of Lecture 1 is hinted at in Exercise 1.77.

(2.39) *Gauging the involution.* Now apply the general gauging construction in **(2.26)** to **(2.38)**. So on $\Sigma \in \mathbf{Man}_2$ there is a new field $P \to \Sigma$ which is an arbitrary double cover. (We are gauging an action of the group $\mathbb{Z}/2\mathbb{Z}$, and a double cover is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle.) Then we need each of the fields in **(2.37)** on P and require them to be invariant under simultaneously pulling back by the deck transformation of $P \to \Sigma$ and executing the involution **(2.38)**.

The key observation is that if P is an oriented surface, any map $P \to \Sigma$ has a canonical lift $P \to \widehat{\Sigma}$ to the orientation double cover. Here, since $P \to \Sigma$ is itself a double cover, this map is an orientation-preserving diffeomorphism. In other words, the field $P \to \Sigma$ is not arbitrary but must be the orientation double cover and the orientation $\mathfrak{o}$ is the canonical orientation on $\widehat{\Sigma}$. So *the orientifold of the oriented bosonic string does not have an orientation field*: it is an unoriented string theory.

The metric (ii) in the gauged theory is simply a metric on Σ. More interesting is the gauged field ϕ, for which Exercise 2.30 is relevant. Namely, it is an equivariant map $\widehat{\Sigma} \to Y$. The metric (iv) and dilaton (v) descend to a metric and dilaton on the quotient $Y /\!/ \sigma$ of Y by the involution σ. The B-field $\check{\beta}$ also descends to the quotient, but it is twisted by the double cover $Y \to Y /\!/ \sigma$, due to the minus sign in the transformation law.

(2.40) *Bosonic orientifold background.* We recast the gauged fields **(2.39)** into new definitions of $\mathcal{F}_{26}$ and $\mathcal{F}_2$ which account for orientifolds. We put hats for the orientifold construction. Note that the original definitions in **(2.20)** and **(2.22)** are the special case when the orientifold double cover is trivial.

The fields in $\widehat{\mathcal{F}}_{26}(X)$ are:

(i) a double cover $X_w \to X$, called the *orientifold double cover*;
(ii) a metric;
(iii) a real scalar field, the *dilaton*;
(iv) a w-twisted gerbe with connection $\check{\beta}$, called the *B-field*.

The B-field in the bosonic orientifold is a special case of a w-twisted twisting of K-theory, as defined towards the end of Lecture 1 (as twistings of KR-theory). The equivalence class of $\check{\beta}$ lies in $\check{H}^{w+3}(X)$. The orientifold double cover is unramified; it is an ordinary double cover in the sense of orbifolds, as reviewed in **(1.94)**.

(2.41) *Bosonic worldsheet fields for the orientifold.* Fix a bosonic orientifold background, which means a smooth 26-manifold X and a set of fields in $\widehat{\mathcal{F}}(X)$. Then the worldsheet fields in that background form a sheaf $\widehat{\mathcal{F}}_2[X]\colon \mathbf{Man}_2 \to \mathbf{Set}$ whose fields on a 2-manifold Σ are:

(i) a metric on Σ;
(ii) a map $\phi\colon \Sigma \to X$;
(iii) an isomorphism $\nu\colon \phi^* w \to w_1(\Sigma)$.

Recall that $w_1(\Sigma)$ is represented by the orientation double cover, so concretely the field in (iii) is an isomorphism of double covers

$$\begin{tikzcd} \phi^*X_w \arrow[rr, "\nu"] \arrow[dr] & & \widehat{\Sigma} \arrow[dl] \\ & \Sigma & \end{tikzcd} \tag{2.42}$$

Exercise 2.43. Check that if the double cover $X_w \to X$ in the bosonic orientifold background is trivial, then (iii) reduces to the orientation $\mathfrak{o}$ in **(2.22)**.

(2.44) *Integration of densities.* Let M be a smooth manifold of dimension n. Then a *density* on M is an element of $\Omega^{w_1(M)+n}(M)$, an n-form twisted by the orientation double cover $\widehat{M} \to M$. One representation is as an anti-invariant element of $\Omega^n(\widehat{M})$, one which changes sign under pullback by the deck transformation. Densities can be integrated without any choice of orientation.

There is an analogous story for any cochain theory which represents ordinary cohomology. Thus, if say M is compact there is an integration map $H^{w_1(M)+n}(M) \to \mathbb{Z}$. Similar integration maps exist for fiber bundles. There are analogous integration maps for a general multiplicative cohomology theory h, but in that case w_1 is replaced by the obstruction to h-orientation. More precisely, a real vector bundle—here $TM \to M$—gives rise to an associated twisting $\tau^h(M)$ of $h(M)$, and the integration is a map $h^{\tau^h(M)+q}(M) \to h^q(\mathrm{pt})$.

These twisted integrations combine to give a twisted integration on differential cohomology.

(2.45) *The B-field amplitude in the bosonic orientifold.* Analogous to (2.24), but without the orientation, we have an integration

$$\int_\Sigma : \check{H}^{w_1(\Sigma)+3}(\Sigma) \longrightarrow \check{H}^1(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}. \tag{2.46}$$

Analogous to (2.25) the B-field amplitude is defined to be

$$S(\phi,\nu) = \int_\Sigma \nu\phi^*\check{\beta}, \tag{2.47}$$

where the $\nu\phi^*$ is the composition

$$\check{H}^{w+3}(X) \xrightarrow{\phi^*} \check{H}^{\phi^*w+3}(\Sigma) \xrightarrow{\nu} \check{H}^{w_1(\Sigma)+3}(\Sigma). \tag{2.48}$$

As with (2.25) the integral (2.47) depends only on the equivalence class of $\check{\beta}$ in $\check{H}^{w+3}(X)$.

Fields in the oriented Type II superstring

We deploy the term 'Type II superstring' when we include the orientifold construction and use 'oriented Type II superstring' when the orientifold data is absent (or viewed as present and trivial). Some of the structures we describe here are spelled out in greater detail in [2].

We begin with a technical point.

Remark 2.49. The worldsheet of the perturbative superstring is more properly treated as a supermanifold, which leads to a more integrated description of some of the fields. Then the moduli space of super surfaces with conformal structure is itself a supermanifold, and the functional integral of the perturbative superstring becomes an integral over that supermanifold. That complex supermanifold does not in general admit a holomorphic splitting, whereas it does admit a C^∞ splitting, as does any supermanifold. We implicitly use one in our description of the fields and in our treatment of the fermions in Lecture 3. We believe that the issues of supergeometry are irrelevant for our topological considerations.

(2.50) *Review of spin structures.* As a preliminary we quickly review spin structures. Recall that the intrinsic geometry of a smooth n-manifold M is encoded in its principal $GL_n\mathbb{R}$-bundle of frames $\mathcal{B}(M) \to M$. A point of $\mathcal{B}(M)$ is a linear isomorphism $\mathbb{R}^n \to T_mM$ for some $m \in M$. Choose a Riemannian metric on M, equivalently, a reduction to an O_n-bundle of frames $\mathcal{B}_O(M) \to M$. The spin group

$$\rho\colon \mathrm{Spin}_n \longrightarrow O_n \tag{2.51}$$

is the double cover of the index two subgroup $SO_n \subset O_n$. A *spin structure* on M is a principal Spin_n-bundle $\mathcal{B}_{\mathrm{Spin}} \to M$ together with an isomorphism of the associated O_n-bundle with $\mathcal{B}_O(M)$. It induces an orientation on M via the cover $\mathrm{Spin}_n \to SO_n$. The space of Riemannian metrics is contractible, so a spin structure is a topological choice and can alternatively be described in terms of a double cover of an index two subgroup of $GL_n\mathbb{R}$. An isomorphism of spin structures is a map $\mathcal{B}_{\mathrm{Spin}} \to \mathcal{B}'_{\mathrm{Spin}}$ such that the induced map on O_n-bundles commutes with the isomorphisms to $\mathcal{B}_O(M)$. The *opposite spin structure* to $\mathcal{B}_{\mathrm{Spin}} \to M$ is the complement of $\mathcal{B}_{\mathrm{Spin}}$ in the principal Pin_n^--bundle associated to the inclusion[20] $\mathrm{Spin}_n \hookrightarrow \mathrm{Pin}_n^-$. If M admits spin structures, then the collection of spin structures forms a groupoid whose set of equivalence classes $\mathcal{S}(M)$ is a torsor for $H^0(M;\mathbb{Z}/2\mathbb{Z})\times H^1(M;\mathbb{Z}/2\mathbb{Z})$; the action of a function $\delta\colon \pi_0M \to \mathbb{Z}/2\mathbb{Z}$ in $H^0(M;\mathbb{Z}/2\mathbb{Z})$ sends a spin structure to its opposite on components where $\delta = 1$ is the nonzero element. The automorphism group of any spin structure is isomorphic to $H^0(M;\mathbb{Z}/2\mathbb{Z})$; a function $\delta\colon \pi_0M \to \mathbb{Z}/2\mathbb{Z}$ acts by the central element of Spin_n on components where $\delta = 1$. The manifold M admits spin structures if and only if the Stiefel-Whitney classes $w_1(M), w_2(M)$ vanish.

(2.52) *Spacetime fields for the oriented Type II superstring.* We defer the orientifold construction but do allow spacetime X to be an orbifold. In the super case there are new fields. Most importantly, the B-field has a different geometric structure than in the oriented bosonic superstring (2.20). Spacetime in the superstring is 10-dimensional, as mentioned in (2.19). Let $\mathcal{F}^s_{10}\colon \mathbf{Man}_{10} \to \mathbf{Set}_\Delta$

[20] Recall that $\mathrm{Pin}_n^\pm$ sits in the Clifford algebra $\mathrm{C}\ell_n$. Either sign can be used to construct the opposite spin structure.

denote the sheaf of fields in the Type II superstring. As in (**2.34**) we allow the domain $\mathbf{Man}_{10}$ to be replaced by the category of 10-dimensional orbifolds and local diffeomorphisms. The fields in $\mathcal{F}^s_{10}(X)$ are:

(i) a metric on X;
(ii) a real scalar field on X;
(iii) a differential twisting $\check{\beta}$ of K-theory on X (see (**1.109**));
(iv) a spin structure κ on X;
(v) a Ramond-Ramond field;
(vi) fermionic fields.

The local information in the B-field (iv) is the same as in the B-field in the oriented bosonic string: it is a closed 3-form $H \in \Omega^3(X)$. But the extra global torsion nonnegative homotopy groups in the cohomology theory $\Sigma^{-1}R$, as opposed to the theory $\Sigma^3 H\mathbb{Z}$, carries relevant information about the superstring. Let β denote the (nondifferential) twisting which underlies $\check{\beta}$; it has an equivalence class in $R^{-1}(X)$. From the short exact sequence (1.50) we deduce classes

$$t = t(\check{\beta}) \in H^0(X;\mathbb{Z}/2\mathbb{Z}), \qquad a = a(\check{\beta}) \in H^1(X;\mathbb{Z}/2\mathbb{Z}) \tag{2.53}$$

which are topological invariants of the B-field. The class t is the *type* of the theory. In the usual nomenclature

$$\begin{aligned} t = 0 &\text{ is the Type IIB superstring; and} \\ t = 1 &\text{ is the Type IIA superstring.} \end{aligned} \tag{2.54}$$

(If X is not connected, then there is a type—A or B—on each component.) One interpretation of the class a is to define a second spin structure $\kappa + a$ on X. Then we consider κ as the "left-moving" spin structure and $\kappa + a$ as the "right-moving" spin structure. These correlate to two spin structures on the worldsheet, which we define below.

Remark 2.55. We could, therefore, organize the data differently for the Type II superstring. But that different organization would not generalize to the orientifold.

We do not discuss the Ramond-Ramond field in detail in these lectures, so we only make a few comments here. One salient point is that the Ramond-Ramond field is *self-dual*. This means its quantization is treated differently from that of the other fields in the theory. As with all self-dual fields we focus on the *current* rather than the gauge field; see (**2.9**) for a reminder about currents and gauge fields in the more familiar context of Maxwell electromagnetism. The Ramond-Ramond current represents an element of $\check{R}^{\check{\beta}}(X)$, twisted differential K-theory group on an orbifold. Here the relevant equivariant version of K-theory is the Atiyah-Segal theory based on equivariant vector bundles, or equivariant families of Fredholm operators.

Also, we do not treat the fermionic fields on spacetime in these lectures, though it is interesting to fit them (and the action) into our picture of the other fields.

Remark 2.56. When we use spacetime fields as fixed external data for the worldsheet theory, we set the Ramond-Ramond field and fermionic fields to zero.

(2.57) *Worldsheet fields for the oriented Type II superstring.* Fix a Type II spacetime background, by which we mean a smooth 10-dimensional orbifold X and a set of fields in $\mathcal{F}^s_{10}(X)$. As just remarked, we assume that the Ramond-Ramond field and fermionic fields vanish. The worldsheet fields $\mathcal{F}^s_2[X](\Sigma)$ on a 2-manifold Σ are:

(i) an orientation $\mathfrak{o}$;
(ii) a spin structure α_ℓ which refines the orientation $\mathfrak{o}$, and a spin structure α_r which refines the opposite orientation $-\mathfrak{o}$;
(iii) a metric;
(iv) a scalar field ϕ with values in X;
(v) spinor field ψ_ℓ, ψ_r on Σ with coefficients in $\phi^*(TX)$;
(vi) spinor fields χ_ℓ, χ_r on Σ with coefficients in $T^*\Sigma$.

The fields (i), (iii), and (iv) are as in the oriented bosonic string **(2.22)**. The fields (ii), (v), (vi) are new. We emphasize: the spin structures α_ℓ, α_r are independent of each other.

Recall that a spin structure is a trivialization of the first two Stiefel-Whitney classes w_1, w_2, which detect the bottom two stages of the Postnikov tower for the classifying space BO. An orientation is a trivialization of w_1, or the bottom stage of the Postnikov tower. It is in that sense that a spin structure can refine an orientation. See **(2.50)** for a more concrete description. Physicists speak in terms of left-movers (ℓ) and right-movers (r), nomenclature which derives from the wave equation on two-dimensional Minkowski spacetime. There is one spin structure for each orientation. When we come to orientifolds there is no global orientation, as we have already seen in **(2.39)**, but locally there are still two spin structures which refine opposite orientations.

The spinor fields ψ_ℓ, χ_ℓ are associated to the spin structure α_ℓ. The spinor field ψ_ℓ has positive chirality and χ_ℓ has negative chirality. The spinor fields ψ_r, χ_r are associated to the spin structure α_r. The spinor field ψ_r has positive chirality and χ_r has negative chirality. (These last chiralities are measured with respect to $-\mathfrak{o}$, the underlying orientation of α_r.)

(2.58) *The B-field amplitude in the oriented Type II superstring.* Let X be a 10-manifold—a superstring spacetime—and $\check{\beta}$ a B-field on X as defined in **(2.52)**. We define the oriented superstring B-field amplitude, which only depends on the equivalence class of $\check{\beta}$ in $\check{R}^{-1}(X)$. To do so we replace (2.24) with a pushforward in differential R-theory; see **(1.114)**. The main point is that the cohomology theory R is Spin-oriented, that is, there is a pushforward in topological R-theory on spin manifolds. It is the Postnikov truncation of the pushforward in ko-theory defined from the spin structure (which by the Atiyah-Singer index theorem has an interpretation as an index of a Dirac operator). In fact, because we are in sufficiently low dimensions we can identify it exactly with the pushforward in ko, a fact which is useful in the proof of the Theorem 2.66 below. The pushforward we need is

$$\int_{(\Sigma,\alpha_\ell)} : \check{R}^{-1}(\Sigma) \longrightarrow \check{R}^{-3}(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z} \tag{2.59}$$

in differential R-theory defined using the spin structure α_ℓ on Σ. Then the B-field amplitude is

$$S(\phi,\alpha_\ell) = \int_{(\Sigma,\alpha_\ell)} \phi^*\check{\beta}. \tag{2.60}$$

(2.61) *Special B-field amplitudes.* There is an isomorphism

$$\check{R}^{-1}(\mathrm{pt}) \cong R^{-1}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}. \tag{2.62}$$

Let $\check{\eta}$ denote the generator. It is a universal B-field on any spacetime X: pullback using $X \to \mathrm{pt}$. There is an explicit model for the topological class $\eta \in R^{-1}(\mathrm{pt})$ underlying the differential class $\check{\eta}$. (Because of the first isomorphism in (2.62) there is no extra information in the differential class.) Following Atiyah-Bott-Shapiro we use the model for $KO(\mathrm{pt})$ in terms of Clifford algebras, as described at the beginning of Lecture 1, and the fact that R is a Postnikov truncation of ko means we just need give a model for the generator of $KO^{-1}(\mathrm{pt})$. This is the real super vector space $\mathbb{R}^{1|1}$ with Clifford generator

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.63}$$

This already appeared in Lecture 1 around (1.147).

The B-field amplitude for this universal B-field is independent of ϕ and is an important function in the theory of spin 2-manifolds. We explain the statement here, cribbing from [2] and refer to that reference for proofs and more elaboration. Let $(\Sigma, \mathfrak{o})$ be a closed oriented surface and $\mathcal{S}(\Sigma, \mathfrak{o})$ the set of equivalence classes of spin structures which refine the given orientation. Note $\mathcal{S}(\Sigma, \mathfrak{o})$ is a torsor for $H^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. Let

$$q\colon \mathcal{S}(\Sigma, \mathfrak{o}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \tag{2.64}$$

be the affine quadratic function which distinguishes even and odd spin structures. It dates back to Riemann and is the Kervaire invariant in dimension two. The characteristic property of the quadratic function q is

$$q(\alpha+a_1+a_2)-q(\alpha+a_1)-q(\alpha+a_2)+q(\alpha) = a_1\cdot a_2, \qquad \alpha \in \mathcal{S}(\Sigma, \mathfrak{o}), \quad a_1, a_2 \in H^1(\Sigma; \mathbb{Z}/2\mathbb{Z}), \tag{2.65}$$

where $a_1 \cdot a_2 \in \mathbb{Z}/2\mathbb{Z}$ is the mod 2 intersection pairing.

Theorem 2.66. *Let $\check{\eta}$ be the nonzero universal B-field. For any superstring worldsheet $\phi\colon \Sigma \to X$, the B-field amplitude is $(-1)^{q(\alpha_\ell)}$.*

This demonstrates that the B-field amplitude (2.60) is sensitive to the worldsheet spin structure.

(2.67) *Physics interpretation of Theorem 2.66.* The perturbative superstring is a 2-dimensional supergravity theory. In nongravitational field theories the intrinsic geometry which is present in the theory is fixed. This includes the underlying manifold, topological structures on its tangent bundle, and a metric or conformal structure. In the language of Remark 2.8 these are background fields. But in a gravitational theory they are dynamical, which means they are integrated over in the path integral formulation. In particular, in the perturbative superstring one sums over the spin structures.

Now in path integrals over spaces of fields with many components there are often signs or phases which are attached to each component. They go by different names: "θ-angles", "discrete torsion", etc. But they are not arbitrary: they must be derived from local computations and obey all of the gluing laws that non-locally-constant quantities obey. In this case the spin structures α_ℓ, α_r can be used to distinguish components of field space, and so there is a possibility of phases entering. In fact, there *are* signs which enter into the usual formulation of the sum over spin structures, and the precise choice of those signs governs the distinction between Type IIB and Type IIA. In our approach these signs are embedded in the B-field amplitude because of our choice of the cohomology theory R^{-1} to quantize the B-field charges. Theorem 2.66 expresses the signs used to go from Type IIB to Type IIA, and the signs agree with those in the traditional approach.

The complete Type II superstring

Now we include the orientifold construction into the Type II superstring. Rather than repeat the gauging procedure described in **(2.39)**, we simply use the formulation in **(2.40)** and **(2.41)**, adapted to the Type II superstring. As mentioned earlier, we use 'Type II superstring' to include the orientifold, and use 'oriented Type II superstring' if it is absent.

(2.68) *More on spin structures.* We mentioned in **(2.44)** that a real vector bundle $V \to X$ determines a twisting $\tau^h(V)$ of any multiplicative cohomology theory h. It includes the rank of the vector bundle and is an ingredient in the general Thom isomorphism theorem. An h-orientation (see **(1.114)**) is a trivialization of $\tau^h(V - \operatorname{rank} V)$, where $\operatorname{rank} V\colon X \to \mathbb{Z}$ is the rank. Such a trivialization is an isomorphism $0 \to \tau^h(V - \operatorname{rank} V)$ in the (multi-)groupoid of h-twistings. For $h = KO$ this is a spin structure. A twisted notion of spin structure enters into **(2.69)** below.

(2.69) *General Type II background.* Let X be a 10-dimensional orbifold. The fields in $\widehat{\mathcal{F}^s}_{10}(X)$, the spacetime fields including the orientifold, are:

(i) a double cover $X_w \to X$, called the *orientifold double cover*;
(ii) a metric on X;
(iii) a real scalar field on X;
(iv) a w-twisted differential twisting $\check{\beta}$ of X (see **(1.109)**);
(v) a "twisted spin structure" $\kappa\colon \Re(\beta) \to \tau^{KO}(TX - 10)$;
(vi) a Ramond-Ramond field;
(vii) fermionic fields.

The equivalence class of $\check{\beta}$ lies in $\check{R}^{w-1}(X)$. The Ramond-Ramond current is now also twisted by w and represents a twisted differential KR-theory class on X_w. A concrete geometric model for the twisted spin structure (v) is given in the last section of [2]. There is a topological constraint forced by the existence of a twisted spin structure:

$$\begin{aligned} w_1(X) &= t(\check{\beta})w \\ w_2(X) &= a(\check{\beta})w + t(\check{\beta})w^2 \end{aligned} \tag{2.70}$$

where $t(\breve{\beta}), a(\breve{\beta})$ are defined in (2.53). These equations generalize the topological constraints $w_1(X) = 0,\ w_2(X) = 0$ imposed by an ordinary spin structure.

(2.71) *The Type I superstring.* There is a special case of the Type II superstring which is important in string theory: the Type I superstring. (The nomenclature derives from that in supergravity theories. For example the 'I' and 'II' reflect the amount of supersymmetry present in these theories.) In this case $X_w = Y$ is a 10-dimensional orbifold with trivial involution; the quotient is $X = Y \times B\mathbb{Z}/2\mathbb{Z}$, where $B\mathbb{Z}/2\mathbb{Z}$ is defined in (1.135). The B-field reduces to an object representing a class in $H^2(Y;\mathbb{R}/\mathbb{Z}) \subset \check{H}^3(Y) \to \check{R}^{-1}(Y)$; it is a good exercise to see why this is so. The twisted spin structure reduces to an ordinary spin structure on Y.

(2.72) *General Type II superstring worldsheet fields.* Fix a Type II background, which means a smooth 10-dimensional orbifold X and a set of fields in $\widehat{\mathcal{F}^s}(X)$. Then the worldsheet fields in that background form a sheaf $\widehat{\mathcal{F}^s}_2[X]\colon \mathbf{Man}_2 \to \mathbf{Set}$ whose fields on a 2-manifold Σ are:

(i) a spin structure α on the total space of the orientation double cover $\widehat{\pi}\colon \widehat{\Sigma} \to \Sigma$;
(ii) a metric on Σ;
(iii) a map $\phi\colon \Sigma \to X$;
(iv) an isomorphism $\nu\colon \phi^*w \to w_1(\Sigma)$;
(v) a positive chirality spinor field ψ on $\widehat{\Sigma}$ with coefficients in $\widehat{\pi}^*\phi^*(TX)$;
(vi) and a negative chirality spinor field χ on $\widehat{\Sigma}$ with coefficients in $T^*\widehat{\Sigma}$ (the gravitino).

The spin structure and the spinor fields are the same locally as in the oriented Type II superstring, but the absence of a global orientation makes the description in terms of the orientation double cover natural. As far as we know, this description of the spin structure does not appear in the string theory literature, even for Type I. In particular, we might have thought that the unoriented superstring would have a pin structure, but this is not the case. The local picture is as in the oriented case, and in that case the two worldsheet spin structures α_ℓ, α_r are independent; see **(2.57)**. Note that a pin structure would restrict α since then it would be isomorphic to its pullback by the deck transformation of $\widehat{\Sigma} \to \Sigma$, which would be analogous to requiring in the oriented case that α_r be the opposite spin structure to α_ℓ.

Remark 2.73. To illustrate, suppose that the superstring orientifold worldsheet Σ is diffeomorphic to a 2-dimensional torus. Even though Σ is orientable, the fields **(2.72)** do not include an orientation. The field α is equivalent to a pair of spin structures α', α'' on Σ with opposite underlying orientations. Up to isomorphism there are 4 choices for each of α', α'', so 16 possibilities in total. Of those 4 refine uniquely to pin$^-$ structures on Σ.

Exercise 2.74. Check that if the double cover $X_w \to X$ in the Type II background is trivial, then **(2.72)** reduces to **(2.57)**. The first step is Exercise 2.43. Then you'll need to reconcile the description of the spinor fields.

(2.75) *B-field amplitude in the general Type II superstring.* Now we come to a puzzle, which we won't resolve until the next lecture. How do we combine (2.59) and (2.46) to integrate the pullback $\phi^*\breve{\beta}$ of the B-field? Well, using the isomorphism ν the pullback $\nu\phi^*\breve{\beta}$ is computed by the composition (compare (2.48))

$$\check{R}^{w-1}(X) \xrightarrow{\phi^*} \check{R}^{\phi^*w-1}(\Sigma) \xrightarrow{\nu} \check{R}^{w_1(\Sigma)-1}(\Sigma). \tag{2.76}$$

But the cohomology theory R is oriented for spin manifolds, and w_1-twisted classes are not "densities" in the sense of (**2.44**); (w_1, w_2)-twisted classes are densities. We might try to use the spin structure α on the worldsheet, but it does not move us to densities. Conclusion: There is no obvious combination of the data which produces a quantity we can integrate in differential R-theory.

(2.77) *Pin structures.* While not directly relevant to the physics, we can consider the case when Σ has a pin$^-$ structure. (There are two types of pin structures: pin$^+$ and pin$^-$. Any 2-manifold admits pin$^-$ structures, but not every 2-manifold admits pin$^+$ structures.) Recall from (**1.92**) that the tangent bundle $T\Sigma \to \Sigma$ has an associated KO-twisting $\tau^{KO}(\Sigma) = \tau^R(\Sigma)$. The pin$^-$ structure provides an isomorphism of that twisting with the twisting defined by the orientation double cover: $\tau^R(\Sigma) \xrightarrow{\cong} w_1(\sigma)$. We sketch that in the exercise below. Thus the pin$^-$ structure determines a pushforward, or integration, of $w_1(\Sigma)$-twisted R-theory classes, so in particular a definition of the integral of $\nu\phi^*(\check{\beta})$; see (2.76).

As in (**2.61**) we can evaluate these amplitudes for a universal B-field $\check{\theta}$ which is the differential refinement of θ in Theorem 1.152. (Note that $\check{R}^{w_0-1}(B\mathbb{Z}/2\mathbb{Z}) \cong R^{w_0-1}(B\mathbb{Z}/2\mathbb{Z})$, so there is no additional information in the differential refinement.) In §4 of [2] we prove that the resulting integral is a $\mathbb{Z}/8\mathbb{Z}$-valued quadratic function on the $H^1(\Sigma; \mathbb{Z}/2\mathbb{Z})$-torsor of pin$^-$ structures. It is the Kervaire invariant of pin$^-$ surfaces, and it induces an isomorphism $\Omega_2^{\mathrm{Pin}^-} \to \mathbb{Z}/8\mathbb{Z}$, where the domain is the bordism group of pin$^-$ surfaces. This provides a (twisted) R-theory, or KO-theory, interpretation of this Kervaire invariant.

Lecture 3: Worldsheet anomalies

In this lecture we work exclusively with the worldsheet theory. Thus fix a smooth 10-dimensional orbifold X and a set of background fields in $\widehat{\mathcal{F}^s}_{10}(X)$, as listed in (**2.69**). The only fields which play a role are the orientifold double cover $X_w \to X$, the B-field $\breve{\beta}$, and the twisted spin structure $\kappa\colon \Re(\beta) \to \tau^{KO}(TX - 10)$. The worldsheet fields $\widehat{\mathcal{F}^s}_2[X]$ are listed in (**2.72**). Let Σ be a closed 2-manifold. We use all of the worldsheet fields on Σ..

There are two quantities in the *effective* exponentiated action on which we focus. The first is obtained by integrating out the two spinor fields ψ, χ. The fermionic path integral over these fields may be treated formally, and the result is the pfaffian of a Dirac operator on $\widehat{\Sigma}$. It is not a number, but rather an element of a line,[21] the *Pfaffian line* L_{Pfaff}. The second quantity is the exponential of the B-field amplitude, which was *not* defined in (**2.75**). We will define it in this lecture as an element of another line, which we call the *B-field line* L_B, which depends on the B-field. What we would like to assert is that the data in the theory gives a trivialization of the tensor product of the Pfaffian line and the B-field line. Furthermore, it is the twisted spin structure κ, pulled back via ϕ, which gives the trivialization. We do not have a direct construction of a geometric trivialization, so instead prove the weaker statement that a trivialization exists. The stronger statement would be an example of a "categorified index theorem"; the statement we prove is a geometric index theorem whose proof leans heavily on results of Atiyah-Patodi-Singer.

The reader may rightly object that any two complex lines are isomorphic—no argument from the author there—so to get a meaningful statement we work in families of surfaces over a variable base S. Then $L_{\text{Pfaff}} \to S$, $L_B \to S$ are flat hermitian line bundles (in fact of order 2, though that is not directly evident for L_{Pfaff}), so their isomorphism classes are elements of $H^1(S;\mathbb{R}/\mathbb{Z})$. Our main result is

Theorem 3.1. *The flat line bundle $L_{\text{Pfaff}} \otimes L_B \to S$ is trivializable.*

The line bundle $L_{\text{Pfaff}} \to S$ is defined analytically (see §3 of [22]), whereas $L_B \to S$ is defined purely topologically. Note that the isomorphism class of a flat line bundle is determined[23] by the holonomies around all loops in S, so Theorem 3.1 is an equality between sets of numbers. To compute these numbers we take the base of the family of surfaces to be the circle S^1, and so the total space is a 3-manifold: $N^3 \to S^1$. Then the holonomy of L_{Pfaff} around the base S^1 is computed by an Atiyah-Patodi-Singer η-invariant. We use a geometric index theorem—a version of the "flat index theorem" in the 3rd APS paper—to relate it to the holonomy of L_B.

We can say this all a bit more nicely in the differential theory, though not every statement has been proved. (The main theorem in the PhD thesis of Kevin Klonoff locates the η-invariant in differential K-theory, but does not prove the refinement in KO-theory in dimension 3 that we need here.) To do so we need a differential version of the the real[24] index theorem for families,

[21] The Pfaffian line and the B-field line below should be regarded as $\mathbb{Z}/2\mathbb{Z}$-graded, though the gradings are even for both.

[22] D. S. Freed, *On determinant line bundles*, in "Mathematical Aspects of String Theory" (ed. S. T. Yau), World Scientific Publishing, 1987.

[23] This is true for any line bundle with connection; the flatness means the holonomy depends only on the homology class of the loop.

[24] There are differential version of the index theorem in complex K-theory, for example in D. S. Freed, J. Lott, *An index theorem in differential K-theory*, Geometry & Topology, **14** (2010), pp. 903–966, (`arXiv:0907.3508`).

the topological version of which is in the fifth of the classic Atiyah-Singer series of papers. That theorem asserts that an analytic index is computed by a pushforward in KO-theory. We use a truncation of the analytic index—the Pfaffian line bundle—and correspondingly a truncation—the cohomology theory R—of KO-theory. It is very pretty that these truncations match. This works nicely because we are dimension 2. In higher dimensions we could also truncate KO-theory to compute the Pfaffian line bundle, but the truncation would keep more homotopy groups and would not be as geometric as in this low dimension. This relation to the index theorem, and the fact that the truncation computes the Pfaffian line bundle, is one of the many pieces of supporting evidence for our choice (**2.52**) of Dirac quantization condition on the B-field. Again, we need the differential version of all this, but only in truncated form for the Pfaffian line bundle.

In this lecture we sketch a proof of Theorem 3.1. *Warning: Some details in these notes are not correct and hopefully a corrected version will appear as a paper in due time.*

Digression: a categorified index theorem

There is a much easier analog of Theorem 3.1 in supersymmetric quantum mechanics, where it is an index theorem on 1-manifolds rather than on 2-manifolds. The pfaffian in this case is computed by an even simpler truncation of KO-theory: mod 2 cohomology! (We get there by adding the simplifying hypothesis that spacetime X in that supersymmetric quantum mechanics theory is oriented.) The topological theorem is that the Pfaffian line bundle is computed by transgressing $w_2(X)$. The categorified version is that a trivialization of $w_2(X)$—a spin structure on X—induces a trivialization of the Pfaffian line bundle. In more detail: the index formula in this case has the shape

$$L_{\mathrm{Pfaff}} \cong \int_C \phi^* w_2(X). \tag{3.2}$$

If we interpret $w_2(X)$ as a cohomology class, then the right hand side computes the isomorphism class of L_{Pfaff}. But we claim that the formula makes sense on the level of geometric objects and their isomorphisms: the formula (3.2) actually computes the Pfaffian line bundle, and we can integrate a trivialization $0 \xrightarrow{\cong} w_2(X)$ to a trivialization $0 \xrightarrow{\cong} L_{\mathrm{Pfaff}}$.

A version of this claim is explained in §5.2 of [2], though not precisely as an integration formula (3.2). Nonetheless, it is proved that L_{Pfaff} has a canonical trivialization if X is spin. The proof there is a bit of a poof, and the ambitious reader will enjoy writing out more details; the argument is very explicit and geometric.

A twisted R-class on Σ

In this section we use the spin structure α on $\widehat{\Sigma}$ to define a class

$$\delta \in R^{\tau_0}(\Sigma), \tag{3.3}$$

where

$$\tau_0 = \tau^R(\Sigma) - 2 \tag{3.4}$$

is the reduced R-twisting defined by the tangent bundle $T\Sigma \to \Sigma$. Since R is a truncation of KO, we have $\tau^R = \tau^{KO}$. Explicit models for $\tau^{KO}(\Sigma)$ are given in (**1.92**). We will also give a lift of δ to a *flat* differential class

$$\check{\delta} \in \check{R}^{\tau_0}(\Sigma). \tag{3.5}$$

(3.6) *An easy construction.* Recall from (**2.68**) that we can regard the spin structure α on the total space of $\widehat{\pi}\colon \widehat{\Sigma} \to \Sigma$ as an isomorphism

$$0 \xrightarrow{\alpha} \tau^{KO}(T\widehat{\Sigma} - 2) \cong \widehat{\pi}^*\tau^{KO}(T\Sigma - 2) = \widehat{\pi}^*\tau_0. \tag{3.7}$$

Now integrate (push forward) (3.7) over the double cover $\widehat{\pi}$. We no longer obtain an isomorphism of twistings, but rather a twisted KO-theory class. Put differently, we can interpret α as an invertible twisted KO-class $x_\alpha \in KO^{\widehat{\pi}^*\tau_0}(\widehat{\Sigma})$; let $y_\alpha \in R^{\widehat{\pi}^*\tau_0}(\widehat{\Sigma})$ be its truncation to a twisted R-class. Then define

$$\delta = \widehat{\pi}_*(y_\alpha). \tag{3.8}$$

Remark 3.9. Recall from (**2.44**) there is an integration map

$$\int_{\widehat{\Sigma}}\colon R^{\tau^R(T\widehat{\Sigma}-2)+q}(\widehat{\Sigma}) \longrightarrow R^{q-2}(\mathrm{pt}). \tag{3.10}$$

The spin structure α, in the guise of the class y_α, is an orientation with which we define integration of untwisted classes:

$$\begin{aligned} \int_{(\widehat{\Sigma},\alpha)}\colon R^q(\widehat{\Sigma}) &\longrightarrow R^{q-2}(\mathrm{pt}) \\ z &\longmapsto \int_{\widehat{\Sigma}} y_\alpha z. \end{aligned} \tag{3.11}$$

Exercise 3.12. What is the analog of (3.11) in de Rham theory?

(3.13) *An explicit model for δ and its differential lift.* Let $\mathcal{B}_O(\Sigma) \to \Sigma$ denote the principal O_2-bundle of orthonormal frames of Σ. Recall that the orientation double cover $\widehat{\Sigma}$ carries a canonical orientation, and then observe that an orthonormal frame induces an orientation, so there is a map $\mathcal{B}_O(\Sigma) \to \widehat{\Sigma}$ which in fact is a principal SO_2-bundle, the *oriented* orthonormal frame bundle of $\widehat{\Sigma}$. The spin structure α on $\widehat{\Sigma}$ is a principal Spin_2-bundle $\widetilde{\mathcal{B}} \to \widehat{\Sigma}$ together with a map $\widetilde{\mathcal{B}} \to \mathcal{B}_O(\Sigma)$ which is a quotient map for the action of the central $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{Spin}_2$.

Let $K \to \mathcal{B}_O(\Sigma)$ be the even real flat line bundle associated to the double cover $\widetilde{\mathcal{B}} \to \mathcal{B}_O(\Sigma)$.

The spin structure α also determines an *odd* real flat line bundle $\Delta \to \Sigma$ as follows. Locally on Σ the Spin_2-action on $\widetilde{\mathcal{B}}$ extends in two ways to a Pin_2^--action on $\widetilde{\mathcal{B}}$ which covers the O_2-action on $\mathcal{B}_O(\Sigma)$. To see this observe that there are two choices for the action of each element in the non-identity component of Pin_2^-, and because we have a Spin_2-action once a choice is made for one element the choice for every other element is determined. These two local canonical actions define a global double cover of Σ; let $\Delta \to \Sigma$ be the associated real line bundle, which we take to be *odd*.

Consider the $\mathbb{Z}/2\mathbb{Z}$-graded line bundle

$$\check{\delta} : K \oplus K\Delta \longrightarrow \mathcal{B}_O(\Sigma), \tag{3.14}$$

where we find it more attractive to omit the '$\otimes$' sign: $K\Delta = K \otimes \Delta$. The definition of Δ in terms of local Pin_2^--actions gives a canonical Pin_2^--action on (3.14) which: (i) is compatible with the nontrivial grading $\mathrm{Pin}_2^- \to \mathbb{Z}/2\mathbb{Z}$ in the sense that elements in the identity component Spin_2 preserve the grading of $K \oplus K\Delta$ and elements in the non-identity component reverse it; and (ii) has the property that the center $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{Spin}_2 \subset \mathrm{Pin}_2^-$ acts by scalar multiplication on the fibers. (Recall the definition of K.) Recalling the second model of τ^{KO} in (**1.92**), we see that (3.14) represents a class $\delta \in R^{\tau_0}(\Sigma)$, and we claim it is the same class as defined in (**3.6**).

Exercise 3.15. Verify this claim. To do so, identify x_α in this model.

The following exercise shows explicitly that δ measures the failure of α to be a pin^- structure.

Exercise 3.16. Let Π be the trivial odd real line, and we use the same symbol to denote the constant line bundle with fiber Π over any space. Verify that a refinement of α to pin^- structure on Σ is an isomorphism $\Pi \to \Delta$. If it exists, show that the equivalence class of (3.14) is the zero element in $R^{\tau_0}(\Sigma)$.

Exercise 3.17. Suppose Σ has an orientation $\mathfrak{o}$. Then we can encode α as two spin structures α_ℓ, α_r on Σ which refine $\mathfrak{o}, -\mathfrak{o}$, respectively, as in (**2.57**). Prove that in this case the odd line bundle $\Delta \to \Sigma$ represents the difference $\alpha_\ell - \alpha_r$. The fact that Δ is odd reflects the different orientations underlying α_ℓ, α_r. Use the spin structure α_ℓ to trivialize $\tau^{KO}(\Sigma)$ and so identify (3.14) with the untwisted KO-class on σ represented by the super line bundle $\mathbb{R} \oplus \Delta$. This requires an explicit use of Definition 1.62 and the associated "Morita isomorphism" on twisted vector bundles.

Finally, recall that K, Δ come with a canonical flat connection, whence so to does (3.14). This flat connection lifts δ to a differential class $\check{\delta} \in \check{R}^{\tau_0}(\Sigma)$.

(3.18) *The class δ is torsion of order 8.* We use the class $\check{\delta}$ in families of surfaces, so in particular on a 3-manifold N which fibers over S^1.

Lemma 3.19. *On the 3-manifold N we have $8\check{\delta} = 0$.*

A proof in case Σ is orientable (Exercise 3.17) is buried in §4 of [25]; the general case will be a small modification based on the following universal description of a spin structure on the orientation double cover.

[25] D. S. Freed, E. Witten, *Anomalies in string theory with D-branes*, Asian J. Math, **3** (1999), 819–851, (`arXiv:hep-th/9907189`).

Exercise 3.20. Let $i\colon BSO_n \to BSO_n$ be the free involution whose quotient is BO_n. Let $\widetilde{E}$ be defined as the pullback in the diagram

$$\begin{tikzcd} \widetilde{E} \arrow[r] \arrow[d] & B\operatorname{Spin}_n \times B\operatorname{Spin}_n \arrow[d] \\ BSO_n \arrow[r, "\mathrm{id}\times i"] & BSO_n \times BSO_n \end{tikzcd} \tag{3.21}$$

The diagram is compatible with the involution i in the lower left corner and the involution which exchanges the factors in each entry of the right column. Let E denote the quotient of the induced involution on $\widetilde{E}$; the diagram produces a map $E \to BO_n$. Prove that if $V \to M$ is a rank n real vector bundle with classifying map $M \to BO_n$, then a spin structure (up to equivalence) on the double cover of M defined by orientations of V is a lift (up to homotopy) of the classifying map to E. Construct the universal version of δ.

The Pfaffian line

(3.22) *Geometric setup.* Let $f\colon M \to S$ be a fiber bundle whose fibers are closed 2-manifolds. The Riemannian metric on the family is two pieces of data: a metric on the rank 2 relative tangent bundle $T(M/S) \to M$ and a horizontal distribution on M, which is a complement to $T(M/S) \subset TM$. The fiberwise orientation double cover $\widehat{\pi}\colon \widehat{M} \to M$ is another fiber bundle $f \circ \widehat{\pi}$ of closed 2-manifolds over S, and the metric data pulls back to $f \circ \widehat{\pi}$. The spin structure α is a spin structure on the relative tangent bundle $T(\widehat{M}/S) \to \widehat{M}$. The field ϕ is a map $\phi\colon M \to X$. The fermionic functional integral over the fields ψ, χ in **(2.72)** is the pfaffian of the family of Dirac operators on the fiber of $f \circ \widehat{\pi}\colon \widehat{\Sigma} \to S$ coupled to the virtual bundle

$$\widehat{\pi}^*\bigl(\phi^*(TX) - T^*(M/S)\bigr), \tag{3.23}$$

which has a covariant derivative from the metrics on X and M/S. (This is a shorthand for the ratio of the pfaffians for Dirac coupled to each bundle separately.) Note that the bundle (3.23) is real, and because we are in dimension 2 (mod 8) the Dirac operator is complex skew-adjoint, which is why there is a pfaffian. The pfaffian is a section of the Pfaffian line bundle $L_{\mathrm{Pfaff}} \to S$, which is a hermitian line bundle with a covariant derivative. The theorems in [22] give formulas for its curvature and holonomy.

Exercise 3.24. Use the curvature formula to prove that $L_{\mathrm{Pfaff}} \to S$ is flat. (Hint: The integrand in that formula is the pullback $\widehat{\pi}^*\omega$ of a differential form $\omega \in \Omega^\bullet(M)$, since the metric data is pulled back from M, and because the deck transformation of $\widehat{\pi}$ reverses orientation, the composition $(f \circ \widehat{\pi})_* \circ \widehat{\pi}^*$ is the zero map.)

(3.25) *The holonomy formula.* Suppose $S = S^1$, and as before we use 'N' to denote the 3-manifold which is the total space of $f\colon N \to S^1$. Then the holonomy formula asserts that the holonomy is the exponential of $2\pi i$ times

$$\xi(\widehat{N})/2 \pmod 1, \tag{3.26}$$

where $\xi(\widehat{N})$ is the Atiyah-Patodi-Singer invariant (roughly half of the η-invariant) on $\widehat{N}$. The extra factor of 2 is because we take the pfaffian, not the determinant. The ξ-invariant is for the Dirac operator on N, using the bounding spin structure on the base S^1, and coupled to the virtual bundle (3.23). Because the curvature vanishes, this invariant is topological, independent of the metric data.

(3.27) *Replacing the cotangent bundle to the surface.* The relative cotangent bundle $T^*(\widehat{N}/S^1) \to S^1$ is spin, which means that its KO-class on a 3-manifold is equal to its rank, which is 2. Thus the Pfaffian line bundle, as a flat bundle, is unchanged if we replace (3.23) with

$$\hat{\pi}^*\phi^*(TX-2). \tag{3.28}$$

(3.29) *APS index theorem.* Let's consider the special case in which the relative tangent bundle of $f\colon N \to S^1$ carries an orientation $\mathfrak{o}$. That gives a section of $\hat{\pi}$, and so, combining with the spin structure on the base S^1, spin structures α_ℓ, α_r on N with opposite underlying orientation. Then (3.26) reduces to the sum

$$\frac{\xi_{\alpha_\ell}(N)}{2} + \frac{\xi_{\alpha_r}(N)}{2} \pmod 1 \tag{3.30}$$

of half ξ-invariants for the two spin structures, this time ξ-invariants on N. That sum is computed by the flat index theorem[26] as a pushforward, or integral, in KO-theory with $\mathbb{R}/\mathbb{Z}$ coefficients. We reinterpret it in differential KO-theory. As in Exercise 3.17 the difference of the spin structures is represented by the odd flat real line bundle $\Delta \to N$. According to Exercise 3.17 the class $\check{\delta}$ is represented in this situation by the bundle $\mathbb{R}\oplus\Delta \to N$ with its flat covariant derivative. The APS index theorem computes (3.30) as

$$\int_{(N,\alpha_\ell)}^{(\widecheck{KO})} \check{\delta}\cdot\phi^*[TX-2], \tag{3.31}$$

where the integral is in differential KO and uses the KO-orientation from the spin structure α_ℓ. The integral lands in $\widecheck{KO}^{-3}(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}$. Because we are in low dimension, the integral only depends on the truncation to R-cohomology, and so can be written as

$$\int_{(N,\alpha_\ell)}^{(\check{R})} \check{\delta}\cdot\phi^*[TX-10], \tag{3.32}$$

where we now interpret $\check{\delta} \in \check{R}^0(N)$, which is a flat element, and $\phi^*[TX-10] \in R^0(N)$. Also, we have used Lemma 3.19 to change '2' to '10'.

[26]adopted to KO-theory

Remark 3.33. Since $\check{\delta}$ is flat, the product in the integrand does not depend on a differential refinement of $\phi^*[TX-2]$. We use this property of the product in differential cohomology throughout. It means that all expressions we write are, in fact, products in topological (generalized) cohomology with $\mathbb{R}/\mathbb{Z}$ coefficients.

Finally, we reconfigure (3.32), which is a formula for the holonomy, to a formula for the isomorphism class of the flat line bundle $L_{\mathrm{Pfaff}} \to S$, where now we work with a family $f\colon M \to S$ over an arbitrary base S. Namely, we claim

$$[L_{\mathrm{Pfaff}}] = \int_{(M/S,\alpha_\ell)} \check{\delta}\cdot\phi^*[TX-10] \quad \in \check{R}^{-2}(S). \tag{3.34}$$

Here the integral is in differential R-theory. It remains to note that R^{-2} has nonzero homotopy groups $\pi_0 = \mathbb{Z}/2\mathbb{Z}$, $\pi_2 \cong \mathbb{Z}$ with nontrivial k-invariant (see (1.47) and (1.48)), so is represented by the Picard groupoid of $\mathbb{Z}/2\mathbb{Z}$-graded complex lines. Hence elements of $\check{R}^{-2}(S)$ are represented by $\mathbb{Z}/2\mathbb{Z}$-graded hermitian line bundles with covariant derivative over S. (There is a bit of a mismatch in the formulas and we might better exponentiate the integral on the right hand side of (3.34)—after multiplying by $2\pi i$—to interpret it as a line bundle. But we won't bother in these notes.)

(3.35) *An extension to the general case.* We can arrive at (3.34) by a more direct route, which we now employ in the general case. Thus given a family $f\colon M \to S$ with all the data in **(2.72)** and no orientation assumption, we presume a truncated differential index theorem which tells that the Pfaffian line bundle of the family of Dirac operators on $\widehat{M} \to S$ coupled to (3.28) (with '2' replaced by '10', as remarked after (3.32)) is

$$[L_{\mathrm{Pfaff}}] = \int_{(\widehat{M}/S,\alpha)} \widehat{\pi}^*\big(\phi^*[TX] - [T^*(M/S)]\big) \quad \in \check{R}^{-2}(S). \tag{3.36}$$

As in (3.11) we rewrite this as

$$[L_{\mathrm{Pfaff}}] = (f\circ\widehat{\pi})_*\Big(\check{y}_\alpha\cdot\widehat{\pi}^*\big(\phi^*[TX] - [T^*(M/S)]\big)\Big), \tag{3.37}$$

where now the integral is written as a pushforward and we use a (flat) differential refinement of y_α. The pushforward does not use any spin structure and is defined because $\check{y}_\alpha$ is twisted by the relative tangent bundle of $f\circ\widehat{\pi}$. Carry out $\widehat{\pi}_*$, use the push-pull formula, use the differential version of formula (3.8), and make the same substitutions for $T^*(M/S)$ as in (3.28) and (3.32) to derive

$$[L_{\mathrm{Pfaff}}] = f_*\big(\check{\delta}\cdot\phi^*[TX-10]\big). \tag{3.38}$$

This is our final formula for the Pfaffian line bundle.

More twisted $\check{R}$-classes on Σ

We introduce some new characters. As in Exercise 3.16 let Π be the trivial odd real line, the trivial vector space $\mathbb{R}$ regarded as odd. We use the same notation for the constant line bundle over any space.

(3.39) *The class $\breve{\epsilon}$.* First, as a variation of (3.14), consider

$$\breve{\epsilon}: K \oplus K\Delta\Pi \to \mathcal{B}_O(\Sigma). \tag{3.40}$$

This is an even flat vector bundle of rank 2, and it has an action of Pin_2^-, but now the action is purely even. So it represents a twisted class where the twisting only senses the central extension, not the orientation, and thus the twisting is $\tau_0 - w_1(\Sigma)$. We denote this class as

$$\breve{\epsilon} \in \breve{R}^{\tau_0 - w_1(\Sigma)}(\Sigma). \tag{3.41}$$

Note that the "curvature" of $\breve{\epsilon}$, in the sense of differential cohomology, is the rank of (3.40), which is the constant function 2.

Exercise 3.42. Use the explicit model of the twisting and the symmetric monoidal structure on twistings to verify that $\breve{\epsilon}$ lives in the twisted $\breve{R}$-cohomology group indicated in (3.41).

(3.43) *Twice the B-field amplitude.* Recalling (2.76) we see that we *can* integrate $\breve{\epsilon}\cdot\nu\phi^*(\breve{\beta})$ over Σ, since this product is (τ_0-1)-twisted. This integral gives a number in $\mathbb{R}/\mathbb{Z}$. That would seem to give a definition of the B-field amplitude, but the problem is that the curvature has an extra factor of 2, from the curvature of $\breve{\epsilon}$, and that means that the curvature of this integral in a family parametrized by S is the *twice* the transgression of the 3-form curvature of $\breve{\beta}$, and this is not what we want. (See the computation (3.66) below.) In other words, we have twice the B-field amplitude as a function or, exponentiating, its square as a function $S \to \mathbb{T}$. We want to take a square root of this function, and the square root is naturally a section of a flat hermitian line bundle of order 2. We give a differential cohomology version of the construction.

Exercise 3.44. Construct the square root geometrically: Given a function $h\colon M \to \mathbb{T}$ on a smooth manifold, construct a flat hermitian line bundle $L \to M$, a section s, and a trivialization of $L^{\otimes 2}$ such that $s^{\otimes 2} = h$. (Hint: Solve the universal problem $M = \mathbb{T}$ and $h = \mathrm{id}_{\mathbb{T}}$.)

(3.45) *The class $\breve{\lambda}$ and its trivialization $\breve{\zeta}$.* We easily compute that $R^1(\mathrm{pt}) = 0$, whereas

$$\breve{R}^1(\mathrm{pt}) \cong \breve{R}^0(\mathrm{pt}; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \tag{3.46}$$

and every element is flat. Define $\breve{\lambda}$ to be a representative of $1/2 \pmod 1$ in the group (3.46). We can make an explicit model as the vector space

$$\breve{\lambda}: \mathbb{R} \oplus \Pi \tag{3.47}$$

with Clifford action

$$\gamma_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.48}$$

The differential lift is defined by simply specifying the number $1/2 \in \mathbb{R}/\mathbb{Z} \cong \check{R}^1(\mathrm{pt})$. The underlying topological class in $R^1(\mathrm{pt})$ vanishes, of course, and that can be seen since there is an extra Clifford generator

$$\gamma_- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.49}$$

which graded commutes with (3.48). (The paper of Atiyah-Bott-Shapiro explains the sense in which this provides a trivialization of λ.) We claim (3.49) also provides a trivialization $\check{\zeta}$ of $\check{\lambda}$ in the differential theory without any extra data. This is because differences of differential trivializations form the group $\check{R}^0(\mathrm{pt})$, and $\check{R}^0(\mathrm{pt}) \cong R^0(\mathrm{pt})$, the latter being the group of differences of trivializations of the underlying topological element λ.

Remark 3.50. This can probably be said better in a model with superconnections, which would give some nontrivial data even over pt, but I haven't yet worked that out. Also, I don't have time to write now about what trivializations mean in the differential world. You can work that out in a general way by starting as follows. Imagine you only know about closed differential forms, not all differential forms, and you want to "invent" a theory which includes all forms. Then define a form η of degree $q-1$ on a smooth manifold M to be a closed q-form ω on the cone CM and define $d\eta$ to be the restriction of ω to $M \subset CM$. We recover an honest form by integrating ω over the generating line segments of the cone, and Stokes' theorem tells its differential. Of course, you'll want to replace CM by the cylinder $[0,1] \times M$ and use forms which vanish at one end of the cylinder. Imitate this in a model of differential "cocycles", or indeed in any geometric model of cohomology classes.

(3.51) *One more class.* Finally, we give a representative of the generator of

$$\check{R}^{-w_0+1}(B\mathbb{Z}/2\mathbb{Z}) \cong R^{-w_0+1}(B\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/8\mathbb{Z}. \tag{3.52}$$

It already appeared in **(1.142)**. It is the super vector space

$$\check{\chi} : \mathbb{R} \oplus \Pi \tag{3.53}$$

with Clifford generator (3.48) and the generator of $\mathbb{Z}/2\mathbb{Z}$ lifted to the order four transformation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.54}$$

Of course, $\check{\chi}$ is flat. We remark, as we did earlier, that χ is the Euler class of the sign representation.

Remark 3.55. Usually the Euler class of an odd rank vector bundle is torsion of order 2. That is not true for twisted Euler classes, as we see here.

(3.56) *An important isomorphism.* Recapitulating, we have the four classes

$$\begin{aligned} \check{\lambda} &\in \check{R}^{1}(\mathrm{pt}) \\ \check{\chi} &\in \check{R}^{-w_0+1}\big(B\mathbb{Z}/2\mathbb{Z}\big) \\ \check{\epsilon} &\in \check{R}^{\tau_0-w_1(\Sigma)}(\Sigma) \\ \check{\delta} &\in \check{R}^{\tau_0}(\Sigma) \end{aligned} \tag{3.57}$$

or more accurately the symbols denote geometric representatives of the underlying differential cohomology classes. Now there is a diagram of double covers

$$\begin{tikzcd} \widehat{\Sigma} \arrow[r] \arrow[d] & X_w \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ \Sigma \arrow[r, "\phi"] & X \arrow[r, "q"] & B\mathbb{Z}/2\mathbb{Z} \end{tikzcd} \tag{3.58}$$

The left diagram is the isomorphism ν, which is one of the fields on the worldsheet. More simply: the composition is the classifying map of the double cover $\widehat{\pi}$.

We construct an isomorphism

$$\check{\lambda}\cdot\check{\epsilon} \xrightarrow{\ \cong\ } (q\circ\phi)^*(\check{\chi})\cdot\check{\delta}. \tag{3.59}$$

In our model this is an isomorphism of super vector bundles

$$(\mathbb{R}\oplus\Pi)K(\mathbb{R}\oplus\Delta\Pi) \xrightarrow{\ \cong\ } (\mathbb{R}\oplus\Pi)K(\mathbb{R}\oplus\Delta) \tag{3.60}$$

over $\mathcal{B}_O(\Sigma)$. The isomorphism must be even, commute with the Clifford action of (3.48) on the first factor, commute with the Pin_2^--actions, and it must respect the flat connections. We specify the isomorphism in terms of bases, working in the fibers over a fixed point of $\mathcal{B}_O(\Sigma)$. Let e_0, e_1 be a basis of the first factor $\mathbb{R}\oplus\Pi$; a_0, b_0 a basis of $(\mathbb{R}\oplus\Delta\Pi)$; and c_0, d_1 a basis of $\mathbb{R}\oplus\Delta$. (So $a_0 = c_0$ and $b_1 = \Pi d_1$.) Define (3.60) as the map

$$\begin{aligned} e_0a_0 &\longmapsto e_0c_0 \\ e_0b_0 &\longmapsto e_1d_1 \\ e_1a_0 &\longmapsto e_1c_0 \\ e_1b_0 &\longmapsto e_0d_1 \end{aligned} \tag{3.61}$$

Exercise 3.62. Check that (3.61) satisfies the requirements listed above: commutation with this and that.

The anomalous B-field amplitude

(3.63) *The B-field line bundle and a section.* Working with a family $f\colon M \to S$ as above we define the B-field line bundle over S as

$$L_B = \int_{M/S} \check{\lambda} \cdot \check{\epsilon} \cdot \nu\phi^*(\check{\beta}) \quad \in \check{R}^{-2}(S). \tag{3.64}$$

Note that the integrand is τ_0-twisted, so the integral makes sense. The B-field amplitude is the "nonflat trivialization"

$$\int_{M/S} \check{\zeta} \cdot \check{\epsilon} \cdot \nu\phi^*(\check{\beta}) \tag{3.65}$$

of L_B. Its covariant derivative, a 1-form on S, is computed as

$$\int_{\Sigma/S} \operatorname{curv}(\check{\zeta})\, \operatorname{curv}(\check{\epsilon})\, \phi^*(\operatorname{curv}\check{\beta}) = \int_{\Sigma/S} \frac{1}{2}\cdot 2 \cdot \phi^* H = \int_{\Sigma/S} \phi^* H, \tag{3.66}$$

where $H \in \Omega^{w+3}(X)$ is the 3-form curvature of the B-field $\check{\beta}$, a twisted 3-form on X. This is the required formula for the B-field amplitude. Note that (3.66) is a *closed* 1-form, consistent with the fact that $L_B \to S$ is flat.

As an important step towards Theorem 3.1 we apply the isomorphism (3.59) to (3.64) to conclude

$$\begin{aligned} L_B &= \int_{M/S} (q\circ\phi)^*(\check{\chi}) \cdot \check{\delta} \cdot \nu\phi^*(\check{\beta}) \\ &= f_*\big(\check{\delta} \cdot \phi^*(q^*(\chi)\cdot\beta)\big), \end{aligned} \tag{3.67}$$

where we write the integral as a pushforward in the last step and also use the fact that $\check{\delta}$ is flat, so the product in differential cohomology only depends on the topological class underlying the second factor; see Remark 3.33.

Putting it all together

(3.68) *The anomaly cancellation.* To prove Theorem 3.1, the anomaly cancellation, we must show that the sum of (3.67) and (3.38) vanishes. We're off by a sign, so end up showing they're equal. (In any case the classes have order two.) This is where the twisted spin structure

$$\kappa\colon \Re(\beta) \xrightarrow{\ \cong\ } \tau^{KO}(TX - 10), \tag{3.69}$$

or rather its existence, comes into play. Using Lemma 1.157 and (1.165) we conclude from the existence of (3.69) that

$$\bar{c}\big(q^*(\chi)\cdot\beta\big) = \bar{c}[TX - 10] \quad \in r^0(X), \tag{3.70}$$

where $[TX - 10] \in R^0(X)$ is the isomorphism class of the reduced tangent bundle, defined using the quotient map $ko^0(X) \to R^0(X)$. Working on a 3-manifold $f\colon N \to S^1$, which suffices since the flat bundles are determined by their holonomies, we conclude

$$\bar{c}\phi^*\bigl(q^*(\chi)\cdot\beta\bigr) - \bar{c}\phi^*[TX - 10] = 0 \quad \in r^0(N) \tag{3.71}$$

where $\phi\colon N \to X$. The table (1.129) shows the effect of the map $\bar{c}\colon R \to r$ on homotopy groups, and the vanishing in (3.71) and the fact we are on a 3-manifold imply that the class

$$\phi^*\bigl(q^*(\chi)\cdot\beta\bigr) - \phi^*[TX - 10] \quad \in R^0(N) \tag{3.72}$$

is equal to a multiple of $1 \in R^0(N)$ which is divisible by 8. In fact, it vanishes since the "rank" of each term is zero. It follows that (3.67) and (3.38) are equal, as desired.