Fully local Chern–Simons theories

Dan Freed
Harvard University

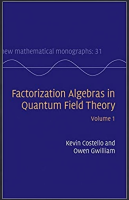
June 22, 2026

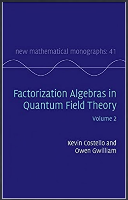
Joint work with Claudia Scheimbauer and Constantin Teleman

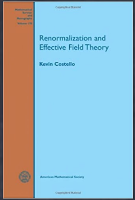
arXiv:2601.05518 and arXiv:2603.11291

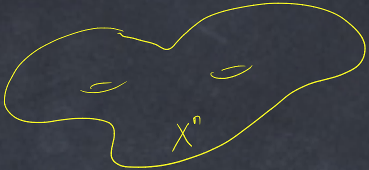
Geometric axiom systems for Wick-rotated QFT

The more recent one, in which observables are primary, is due to Costello-Gwilliam, with many examples constructed using Costello's treatment of perturbation theory









I will focus on the other, in which states are primary, due to Graeme Segal for 2d CFT and shortly afterwards in a variation by Atiyah for topological theories (TFT)

The definition of conformal field theory

Graeme Segal

I shall propose a definition of 2-dimensional conformal field theory which I believe is equivalent to that used by Friedan et al. The idea arises from joint work with Quillen.

§1 The definition

The category $\mathcal{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n\geq 0}$, where C_n is the disjoint union of a set of n

MICHAEL F. ATIYAH

Topological quantum field theory

Publications mathématiques de l'I.H.É.S., tome 68 (1988), p. 175-186.

It will be clear how much this whole subject rests on the ideas of Witten. In formulating the axiomatic framework in § 2, I have also been following Graeme Segal who produced a very similar approach to conformal field theories [10]. Finally it seems appropriate to point out the major role that *cobordism* plays in these theories. Thus René Thom's most celebrated contribution to geometry has now a new and deeper relevance.

Wick-rotated QFT as a linear representation

Graeme Segal (mid 1980's): Wick-rotated QFT is a representation of a bordism category

There are two "discrete parameters" that specify the species of bordism category: $n, \mathcal{F}$

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Man_n	category of smooth n-manifolds and local diffeomorphisms
sSet	category of simplicial sets

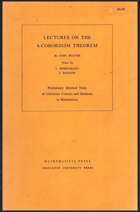
Definition: A *Wick-rotated field* is a sheaf

$$\mathcal{F}\colon \mathrm{Man}_n^{\mathrm{op}} \longrightarrow \mathrm{sSet}$$

Examples: Riemannian metrics, G-connections, $\mathbb{R}$-valued functions, M-valued functions, orientations, spin structures, gerbes, ...

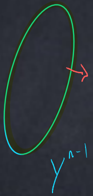
$\mathcal{F}$ can be a *collection* of fields; $\mathcal{F}(M)$ is the simplicial set of fields on an n-manifold M

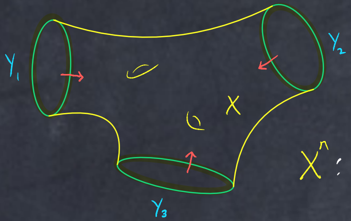
Axiom System: $\mathrm{Bord}_n(\mathcal{F})$ bordism category



n dimension of spacetime

$\mathcal{F}$ background fields (orientation, Riemannian metric, . . .)





$X^n : Y_1 \sqcup Y_2 \sqcup Y_3 \to \emptyset^{n-1}$

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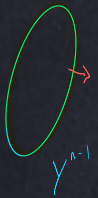
n dimension of spacetime

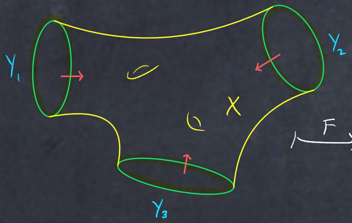
$\mathcal{F}$ background fields (orientation, Riemannian metric, ...)

$t\mathrm{Vect}$ linear category of topological vector spaces and linear maps

$F\colon \mathrm{Bord}_n(\mathcal{F}) \longrightarrow t\mathrm{Vect}$ linear representation of bordism category



$\xmapsto{\ F\ } F(Y)$



$X^n : Y_1 \sqcup Y_2 \sqcup Y_3 \to \emptyset^{n-1}$

$\xmapsto{\ F\ } \Big(F(X) : F(Y_1) \otimes F(Y_2) \otimes F(Y_3) \longrightarrow \mathbb{C} \Big)$

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Unitarity is encoded via an additional reflection positivity structure (not used here)

Outline

(1) Yang–Mills + Chern–Simons

(2) Reshetikhin–Turaev

(3) Fully local topological topological field theories

(4) Construction via cobordism hypothesis and other results

(5) Concluding remarks

Yang–Mills + Chern–Simons

G	compact Lie group
λ	(cocycle for a) class in $H^4(BG; 2\pi\mathbb{Z})$, the *level*
e	positive real number

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λ determines a G-invariant symmetric bilinear pairing $\langle -, - \rangle$ on Lie G and, for each G-connection A on a principal G-bundle $P \to X$ a 3-form

$$\gamma(A) = \langle A \wedge F(A) \rangle - \frac{1}{6} \langle A \wedge [A \wedge A] \rangle \qquad \in \Omega^3(P)$$

The *Chern-Simons invariant* $CS_\lambda(A) \in \mathbb{R}/2\pi\mathbb{Z}$ is (roughly) the integral of $\gamma(A)$

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The YM+CS action on a closed 3-manifold X is

$$S_{e,\lambda}(A) = \frac{1}{4e^2} \int_X \langle F_A \wedge *F_A \rangle \; - \; \sqrt{-1}\, CS_\lambda(A)$$

Some sheaves on Man_3

We use the following (simplicial) sheaves on Man_3:

$$\begin{aligned} \mathcal{F}_{\mathrm{Riem}}: M &\longmapsto \mathrm{Riem}(M), && \text{the set of Riemannian metrics} \\ \mathcal{F}_{w_1}: M &\longmapsto \mathrm{Orient}(M), && \text{the set of orientations} \\ \mathcal{F}_{w_1,w_2}: M &\longmapsto \mathrm{Spin}(M), && \text{the groupoid of Spin structures} \end{aligned}$$

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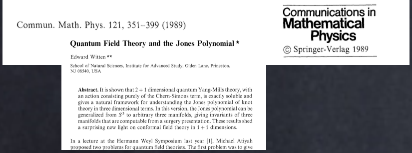
The following conjectures are distilled from papers and lectures of Witten


Commun. Math. Phys. 121, 351–399 (1989)

Communications in Mathematical Physics

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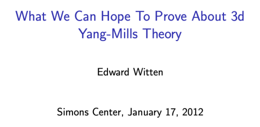
Quantum Field Theory and the Jones Polynomial *

Edward Witten **

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

In a lecture at the Hermann Weyl Symposium last year [1], Michael Atiyah proposed two problems for quantum field theorists. The first problem was to give



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(ii) If λ is nondegenerate, the singular limit $e \to \infty$ exists and defines a field theory

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(iv) The projectivity (anomaly) α_λ of $\overline{F}_\lambda$ extends to $\tilde{\alpha}_\lambda$, a 4-dimensional invertible field theory of oriented manifolds whose partition function on a closed oriented 4-manifold W is

$$\tilde{\alpha}_\lambda(W) = \exp\left(\frac{2\pi i c(\lambda)}{24}\langle p_1(W), [W]\rangle\right)$$

where $c = c(\lambda) \in \mathbb{Q}$ is the *central charge* associated to the level λ

Witten maneuver

So far a linear nontopological theory F_λ and a projective topological descent $\overline{F}_\lambda$

$$
\begin{tikzcd}
F_\lambda \quad \mathfrak{F}_{\mathrm{Riem},w_1} \arrow[dr] & \\
& \overline{F}_\lambda \quad \mathfrak{F}_{w_1}
\end{tikzcd}
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The *Witten maneuver* constructs a linear *topological* lift Z_λ of $\overline{F}_\lambda$ over a different sheaf

$$\begin{tikzcd}
 & \mathcal{F}_{\mathrm{Riem},w_1,p_1} \quad F_\lambda \otimes \gamma_c \arrow[dl] \arrow[dr] & \\
F_\lambda \quad \mathcal{F}_{\mathrm{Riem},w_1} \arrow[dr] & & \mathcal{F}_{w_1,p_1} \quad Z_\lambda \arrow[dl] \\
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\end{tikzcd}$$

We will briefly explain p_1*-structures* and the nontopological invertible theory γ_c

$$
\begin{tikzcd}
& \color{white}\mathcal{F}_{\mathrm{Riem},w_1,p_1} \quad F_\lambda \otimes \gamma_c \arrow[dl] \arrow[dr] & \\
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Remark: If β is an *invertible topological* theory over $\mathcal{F}_{w_1,p_1}$, then $Z_\lambda \otimes \beta$ is also a linear topological lift of $\overline{F}_\lambda$; $\{\beta\}$ form a cyclic group $\mu\!\!\!/_6$ (no reflection positivity)

$$\begin{tikzcd}[column sep=small] & \mathcal{F}_{\mathrm{Riem},w_1,p_1} \quad {\color{yellow}F_\lambda \otimes \gamma_c} \arrow[dl] \arrow[dr] & \\ {\color{yellow}F_\lambda} \quad \mathcal{F}_{\mathrm{Riem},w_1} \arrow[dr] & & \mathcal{F}_{w_1,p_1} \quad {\color{yellow}Z_\lambda} \arrow[dl] \\ & {\color{yellow}\overline{F}_\lambda} \quad \mathcal{F}_{w_1} & \end{tikzcd}$$

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A p_1-*structure* on a smooth oriented 3-manifold X is a lift of the classifying map of its oriented (orthonormal) frame bundle:

$$\begin{tikzcd} & \mathcal{X}_3 \ (\text{homotopy fiber}) \arrow[d] \\ X \arrow[ur, "s"] \arrow[r, "\tau_M"'] & BSO_3 \arrow[d, "p_1"] \\ & K(\mathbb{Z},4) \end{tikzcd}$$

Tangential Chern–Simons theory (fractional)

The Chern–Simons form of the Levi–Civita connection Θ^{LC} of a Riemannian 3-manifold X lives on the total space of the principal SO_3 frame bundle $\mathcal{B}_{\mathrm{SO}}(X) \to X$:

$$\gamma(X) = \langle \Theta^{\mathrm{LC}} \wedge F(\Theta^{\mathrm{LC}}) \rangle - \frac{1}{6} \langle \Theta^{\mathrm{LC}} \wedge [\Theta^{\mathrm{LC}} \wedge \Theta^{\mathrm{LC}}] \rangle \quad \in \Omega^3\big(\mathcal{B}_{\mathrm{SO}}(X)\big)$$

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A p_1-structure descends $\gamma(X)$ to $\phi(X) \in \Omega^3(X)$, said here in terms of sheaves on Man_3:

$$\begin{tikzcd}
\mathcal{F}_{\mathrm{Riem},p_1} \arrow[rr, dashed, "\phi"] \arrow[d] & & \Omega^3 \arrow[r, "d"] \arrow[d] & \Omega^4_{\mathrm{cl}} \arrow[d, equal] \\
\mathcal{F}_{\mathrm{Riem}} \arrow[r, "\Theta^{\mathrm{LC}}"] \arrow[d, "p_1"'] & \mathcal{F}_{B_\nabla \mathrm{SO}_3} \arrow[r, "\Gamma"] & \widehat{Z}^4 \arrow[r, "\omega"] \arrow[d, "\mu"] & \Omega^4_{\mathrm{cl}} \\
Z^4 \arrow[rr, equal] & & Z^4 &
\end{tikzcd}$$

$\widehat{Z}^4$ is a sheaf of *differential cocycles*

Tangential Chern–Simons theory (fractional)

The theory γ_c, $c \in \mathbb{R}$, is constructed by integrating $c\phi/24$ and exponentiating; the partition function on a closed oriented Riemannian 3-manifold X with p_1-structure is

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Remark: The Witten maneuver can also be executed on the (anomalous) free chiral spinor field. The resulting 3d topological anomaly theory is the invertible field theory extension of the Atiyah–Patodi–Singer expression for the *Adams e-invariant*

Outline

(1) Yang–Mills + Chern–Simons

(2) Reshetikhin–Turaev

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Reshetikhin-Turaev construction

First rigorous $F_{(\mathrm{SU}_2,\lambda)}$, done via surgery presentations and Kirby calculus

Invent. math. 103, 547–597 (1991)

Inventiones mathematicae

Invariants of 3-manifolds via link polynomials and quantum groups

N. Reshetikhin[1] and V.G. Turaev[2]

[1] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
[2] LOMI, Fontanka 27, Leningrad 191011, USSR

Oblatum 20-XII-1989 & 7-VII-1990

1. Introduction

The aim of this paper is to construct new topological invariants of compact oriented 3-manifolds and of framed links in such manifolds. Our invariant of (a link in) a closed oriented 3-manifold is a sequence of complex numbers parametrized by complex roots of 1. For a framed link in S^3 the terms of the sequence are equale to the values of the (suitably parametrized) Jones polynomial of the link in the corresponding roots of 1. Thus, for links in S^3 our invariants are essentially equivalent to the Jones polynomial [Jo].

Note that in general we do not know if our invariant of (a framed link in) a closed oriented 3-manifold may be described as the sequence of values of a certain polynomial in the roots of unity.

In the case of manifolds with boundary our invariant is a (sequence of) finite dimensional complex linear operators. This produces from each root of unity q a 3-dimensional topological quantum field theory (see [A1]). In particular, for each q we associate with every closed oriented surface a finite dimensional complex linear space. We construct a projective action of the modular group of the surface in this space.

Reshetikhin-Turaev construction

First rigorous $F_{(\mathrm{SU}_2,\lambda)}$, done via surgery presentations and Kirby calculus

They use oriented manifolds without p_1-structure or framing, so obtain a *projective* rather than *linear* theory

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Shortly after, Walker gave a construction of a linear theory

On Witten's 3-manifold Invariants

Kevin Walker *†

February 28, 1991

[PRELIMINARY VERSION #2]

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Definition: A *braided fusion category* T is

- a finite semisimple category T, linear over $\mathbb{C}$
- a braided tensor structure on T
- rigidity (duals)

There is a *nondegeneracy* condition as well.

Definition: A braided fusion category T is *nondegenerate* if

$$\{x \in T : \sigma(y, x) \otimes \sigma(x, y) = \mathrm{id}_{x \otimes y} \text{ for all } y \in T\} = \mathrm{Vect} \cdot 1$$

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Example: T has 3 simple objects $1, \sigma, \psi$ up to iso, with *fusion rules*

$$\sigma^2 = 1 + \psi$$
$$\sigma\psi = \sigma$$
$$\psi^2 = 1$$

and a sample (self-) braiding

$$\begin{tikzcd} \sigma \otimes \sigma \arrow[r] \arrow[d, equal] & \sigma \otimes \sigma \arrow[d, equal] \\ 1 \oplus \psi \arrow[r] & 1 \oplus \psi \end{tikzcd} \qquad \begin{array}{ll} 1 \longrightarrow 1 & -\exp(-2\pi i(3/16)) \\ \psi \longrightarrow \psi & \exp(2\pi i(1/16)) \end{array}$$

Other data has a similar combinatorial flavor

Questions

Two very different starting points: (1) Chern–Simons invariants based on (G, λ)
(2) a nondegenerate braided fusion category T

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(Known in principle and in practice for G finite)

Higher Algebraic Structures and Quantization

Daniel S. Freed

Department of Mathematics
University of Texas at Austin

April 22, 1993

Very Abstract. We derive (quasi-)quantum groups in 2+1 dimensional topological field theory directly from the classical action and the path integral. Detailed computations are carried out for the Chern-Simons theory with finite gauge group. The principles behind our computations are presumably more general. We extend the classical action in a $d+1$ dimensional topological theory to manifolds of dimension less than $d+1$. We then "construct" a generalized path integral which in $d+1$ dimensions reduces to the standard one and in d dimensions reproduces the quantum Hilbert space. In a $2+1$ dimensional topological theory the path integral over the circle is the category of representations of a quasi-quantum group. In this paper we only consider finite theories, in which the generalized path integral reduces to a finite sum. New ideas are needed to extend beyond the finite theories treated here.

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(This is our next topic)

Question 3: How to construct T from (G, λ)?

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Question 1: How are these two starting points related?
(Known in principle and in practice for G finite)

Question 2: Where does T show up in the theory?
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Our recent work completes the line of development that starts with (2)

Outline

(1) Yang–Mills + Chern–Simons

(2) Reshetikhin–Turaev

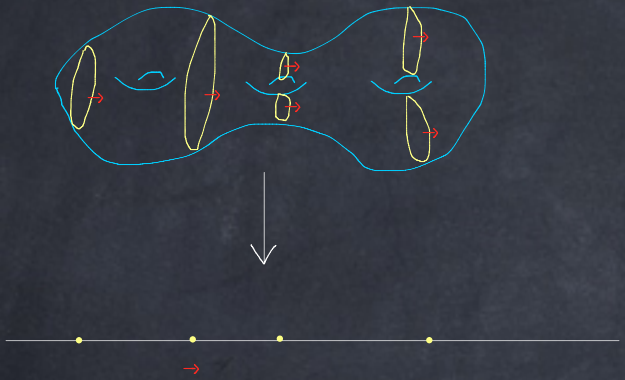
(3) Fully local topological topological field theories

(4) Construction via cobordism hypothesis and other results

(5) Concluding remarks

Down in dimension; up in category number

The ordinary bordism 1-category can be viewed as using a single (Morse) function to slice



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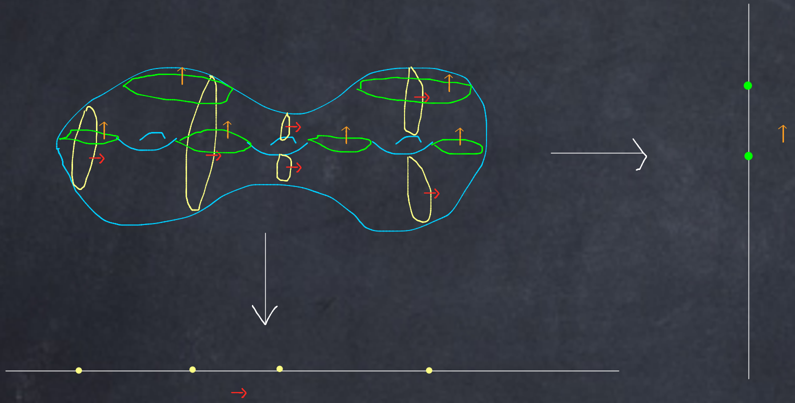

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So in dimension 3 we seek to extend $F\colon \mathrm{Bord}_{\langle 2,3\rangle}(\mathcal{F}) \to \mathrm{Vect}$ to a homomorphism

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Walker and Turaev (in his book, based on joint work with Reshetikhin) construct this extension to a $(1,2,3)$-theory starting from algebraic data

Question 2: Where does T show up in the theory?

Recall that T is a nondegenerate braided fusion category, and we have a theory

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Hence we obtain the linear category

$$T := F(S^1)$$

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The tensor structure and braiding are derived from the following in the bordism category:

Fully local topological field theory

Slicing 3-manifolds with 3 transverse functions will complete the progression

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Remark: A construction as a fully local *anomalous* TFT was given by Haïoun

Disambiguation

The following two concepts in topological field theory are sometimes confused:

- the object attached to a point
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They are different! An object in a 3-category $\mathcal{C}$ has no intrinsic structure or meaning. On the other hand, if $x', x \in \mathcal{C}$ then $\mathrm{Hom}_{\mathcal{C}}(x', x)$ is a 2-category. If $\mathcal{C}$ is symmetric monoidal, and a topological field theory attaches $x \in \mathcal{C}$ to a point, then $\mathrm{Hom}_{\mathcal{C}}(1, x)$ is the 2-category of boundary theories

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Bottom line: Full locality and existence of nonzero boundary theories are different

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Higher-Dimensional Algebra and
Topological Quantum Field Theory

John C. Baez and James Dolan

Department of Mathematics
University of California
Riverside CA 92521

February 5, 1995

Abstract

The study of topological quantum field theories increasingly relies upon concepts from higher-dimensional algebra such as n-categories and n-vector spaces. We review progress towards a definition of n-category suited for this purpose, and outline a program in which n-dimensional TQFTs are to be described as n-category representations. First we describe a 'suspension' operation on n-categories, and hypothesize that the k-fold suspension of a weak n-category stabilizes for $k \geq n+2$. We give evidence for this hypothesis and describe its relation to stable homotopy theory. We then propose a description of n-dimensional unitary extended TQFTs as weak n-functors from the 'free stable weak n-category with duals on one object' to the n-category of 'n-Hilbert spaces'. We conclude by describing n-categorical generalizations of deformation quantization and the quantum double construction.

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Cobordism Hypothesis: Evaluation on a point is an equivalence

$$\begin{aligned} \mathrm{TFT}(\mathcal{C}) &\longrightarrow (\mathcal{C}^{\mathrm{fd}})^{\sim} \\ F &\longmapsto F(\mathrm{pt}) \end{aligned}$$

Baez–Dolan conjecture; Hopkins–Lurie, Lurie, Schommer-Pries, Ayala-Francis, . . .

On the Classification of Topological Field Theories (Draft)

Jacob Lurie

May 10, 2009

Our goal in this article is to give an expository account of some recent work on the classification of topological field theories. More specifically, we will outline the proof of a version of the *cobordism hypothesis* conjectured by Baez and Dolan in [2].

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The cobordism hypothesis is a deep result in differential topology; the guts is contratibility of a space of generalized Morse functions

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Drinfeld centers

Goal: Construct a 3-category $\mathcal{C}$ such that: for all nondegenerate braided fusion categories T there exists $X \in \mathcal{C}$ fully dualizable and the resulting TFT

$$F_X : \mathrm{Bord}_{\langle 0,1,2,3 \rangle} \to \mathcal{C}$$

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In this case F_Φ is called a *Turaev–Viro theory* (Douglas–Schommer-Pries–Snyder)

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Definition: Let $\mathrm{Pic}(\mathbb{F})$ be the 4-category of invertible $\mathbb{F}$-modules

Definition: Let $\mathrm{GL}_1\mathbb{B} \subset \mathbb{B}$ be the 4-groupoid of invertible braided fusion categories

Then there is a map

$$\begin{aligned} \mathrm{GL}_1\mathbb{B} &\longrightarrow \mathrm{Pic}(\mathbb{F}) \\ T &\longmapsto \mathbb{F}_T \end{aligned}$$

where $\mathbb{F}_T$ is the 3-cat of fusion categories equipped with an invertible T-module structure

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$$\bigoplus_T \mathbb{F}_T$$

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But this kind of construction—choosing representatives—is not rigid enough to determine a symmetric monoidal structure on the direct sum

Instead, we proceed by a higher algebra version of the *crossed product algebra* construction to obtain the following theorem

Theorem (F.–Scheimbauer–Teleman): There exists a (universal) symmetric monoidal 3-category $E\mathbb{F}$ such that

(i) $E\mathbb{F}$ has all duals and adjoints

(ii) There exists a central extension

$$\boldsymbol{\mu}_6 \longrightarrow \widetilde{W} \longrightarrow W$$

and a direct sum decomposition $E\mathbb{F} = \bigoplus_{\widetilde{w} \in \widetilde{W}} \mathbb{F}_{\widetilde{w}}$

(iii) $\mathbb{F}_0 = \mathbb{F}$ and each $\mathbb{F}_{\widetilde{w}}$ is an invertible $\mathbb{F}$-module. Furthermore, for $T \in \mathrm{GL}_1\mathbb{B}$, $\widetilde{w} \in \widetilde{W}$ such that the image of w in W is the Witt class of T, we have $\mathbb{F}_{\widetilde{w}} = \mathbb{F}_T$

(iv) For $\zeta \in \boldsymbol{\mu}_6$, the invertible $\mathbb{F}$-module $\mathbb{F}_\zeta$ is generated by an invertible object $U_\zeta \in \mathbb{F}_\zeta$. Furthermore, U_ζ generates an *invertible* 3-framed TFT that factors uniquely to an invertible (w_1, p_1)-theory

Crossed product algebra

A	(discrete) algebra
G	(discrete) groups
A_g	invertible (A, A)-bimodule for each $g \in G$

Associative algebra structure on $G \ltimes A = \bigoplus_{g \in G} A_g$, requires *coherent* maps $A_{g_1} \otimes A_{g_2} \to A_{g_1 g_2}$

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$$\alpha \colon G \longrightarrow \mathrm{Aut}(A)$$

and defines the bimodule A_g via $a_L \cdot b \cdot a_R = [\alpha(g^{-1})(a_L)]\, b\, a_R \quad (a_L, b, a_R \in A)$

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$G \ltimes A$ is *commutative* if (i) A is commutative, (ii) G is abelian, (iii) either α is trivial or we relax to a "homomorphism" of Picard groupoids

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For any connective spectrum $\mathcal{X}$ there is a map $H \colon \mathbb{S} \otimes \pi_0\mathcal{X} \longrightarrow \mathcal{X}$, and for $\mathrm{GL}_1\mathbb{B}$ we have

$$\pi_0 \,\mathrm{GL}_1\mathbb{B} = W$$
$$\pi_{\{1,2,3\}} \,\mathrm{GL}_1\mathbb{B} = 0$$
$$\pi_4 \,\mathrm{GL}_1\mathbb{B} = \mathbb{C}^\times$$

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We try to fit m into

$$\begin{tikzcd} & HW \arrow[dr, dashed, "m"] & \\ \mathbb{S}\otimes W \arrow[ur, "H"] \arrow[rr, "\varphi"] & & \mathrm{GL}_1\mathbb{B} \end{tikzcd}$$

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- We construct a "super/fermionic" version $ES\mathbb{F}$, in which case the extension of the super Witt group is by μ_{24}

Further results

Any $X \in ES\mathbb{F}$ determines a fully local 3-framed theory

$$\mathscr{T}_X : \mathrm{Bord}_{\langle 0,1,2,3\rangle} \longrightarrow \mathfrak{F}_{\text{3-framing}}$$

for which

- we define a 3d central charge in $\mathbb{C}/6\mathbb{Z}$
- we investigate factoring through tangential structures $\mathfrak{F}_{\text{3-framing}} \longrightarrow \mathfrak{F}$

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Theorem: For $X \in E\mathbb{F}$, a ribbon/modular structure on $T = \mathscr{T}_X(S^1_{\text{bounding}})$ descends $\mathscr{T}_X$ to a theory over $\mathfrak{F}_{w_1,p_1}$

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$$\mathscr{T}_X : \ \mathrm{Bord}_{\langle 0,1,2,3\rangle} \longrightarrow \mathfrak{F}_{\text{3-framing}}$$

for which

- we define a 3d central charge in $\mathbb{C}/6\mathbb{Z}$
- we investigate factoring through tangential structures $\mathfrak{F}_{\text{3-framing}} \longrightarrow \mathfrak{F}$

Theorem: Let $F \in \mathcal{S}\mathbb{F}$ be a superfusion category; then $\mathscr{T}_F$ factors through $\mathfrak{F}_{w_1,w_2}$

Theorem: For $X \in E\mathbb{F}$, a ribbon/modular structure on $T = \mathscr{T}_X(S^1_{\text{bounding}})$ descends $\mathscr{T}_X$ to a theory over $\mathfrak{F}_{w_1,p_1}$

Theorem: Let F be a simple fusion category. The following are equivalent:

- a spherical structure on F
- an SO_3-invariance structure on F together with compatible SO_2-invariance structure on the regular module ${}_F F$

Outline

(1) Yang–Mills + Chern–Simons

(2) Reshetikhin–Turaev

(3) Fully local topological topological field theories

(4) Construction via cobordism hypothesis and other results

(5) Concluding remarks

Scorecard

Two very different starting points: (1) Chern–Simons invariants based on (G, λ)
(2) a nondegenerate braided fusion category T

✓ **Question 1:** How are these two starting points related?
(Known in principle and in practice for G finite)

✓ **Question 2:** Where does T show up in the theory?

Question 3: How to construct T from (G, λ)?
(Known for many special G but not in general)

Question 4: Use geometry of Chern–Simons for (G, λ) to construct a fully local TFT
(Only very partial progress)

✓ **Question 5:** Construct a fully local TFT from a nondegenerate braided fusion category T

Stepping back, we observe:

> Aspects of geometric QFT are well-developed, but it is time to infuse analytic techniques into the geometric/topological/algebraic gains of the past few decades

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> Aspects of geometric QFT are well-developed, but it is time to infuse analytic techniques into the geometric/topological/algebraic gains of the past few decades

To that end we are planning a special year in ’27–’28 at the Harvard CMSA:

Quantum Fields, Analysis, and Geometry

Sourav Chatterjee (Stanford)
Clay Cordova (U Chicago)
Dan Freed (Harvard)
Anton Kapustin (Caltech)
Constantin Teleman (UC Berkeley)

`https://cmsa.fas.harvard.edu/event/cmsaqft_2027/`