Lie group quiche

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Work in progress with Greg Moore and Constantin Teleman

Recalling quiche

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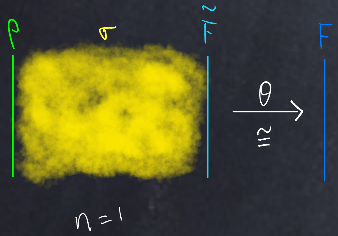

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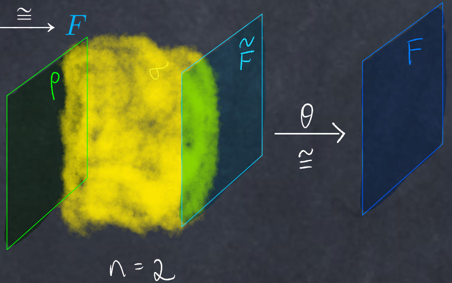

"sandwich" presentation

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- ρ is an important part of the story; σ alone does not suffice
- There is a rigorous calculus of defects in the (σ, ρ)-theory

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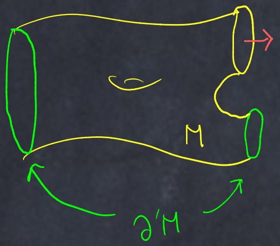
One goal: generalize these constructions to an arbitrary compact Lie group G

Finite path integral construction

Fluctuating fields: $P \to M$ principal G-bundle in σ-bulk

$s\colon \partial' M \to P\big|_{\partial' M}$ trivialization on ρ-boundary



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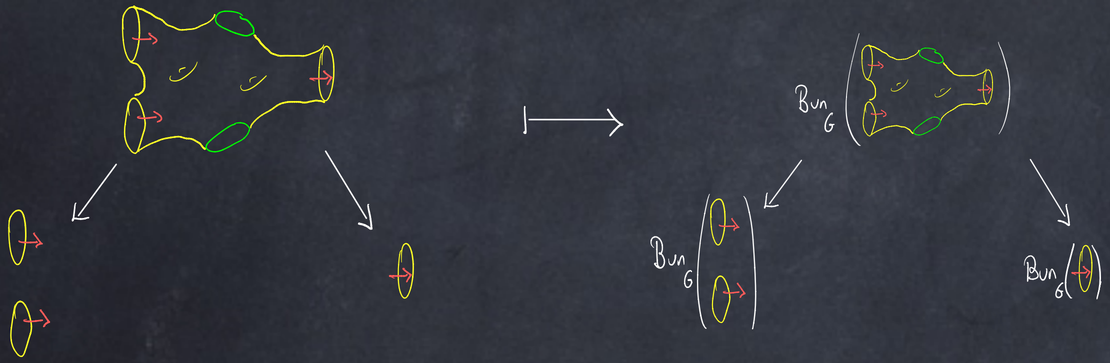

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Examples: $(n=2)$

$\mathrm{Bun}_G(S^2) \simeq * /\!\!/ G$

(D^2, S^1)

$\mathrm{Bun}_G(D^2, S^1) \simeq *$

Two cobordism hypothesis constructions

To apply the cobordism hypothesis, we need to specify:

(i) an $(n+1)$-category $\mathcal{C}$

(ii) an object $x \in \mathcal{C}$

(iii) a 1-morphism $f\colon x \longrightarrow 1$

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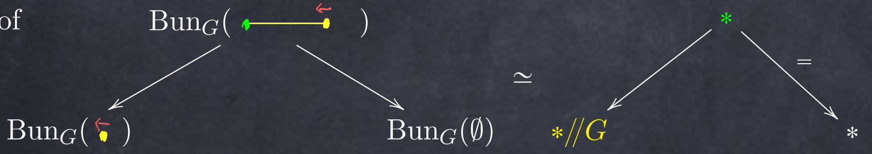

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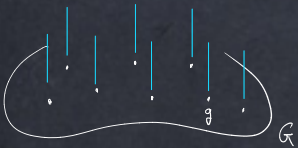

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The manipulations that follow apply to arbitrary compact Lie groups once (σ, ρ) is defined

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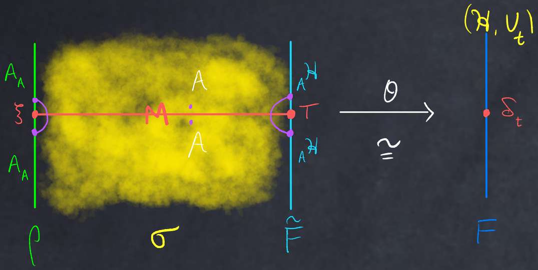

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M (A,A)-bimodule

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$T : M \longrightarrow \mathrm{End}(\mathcal{H})$

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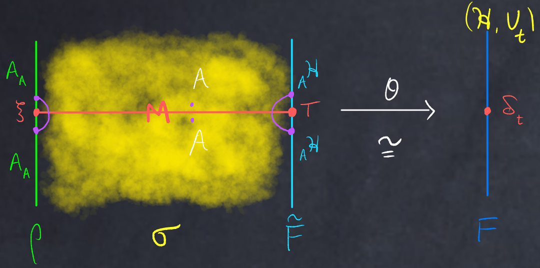

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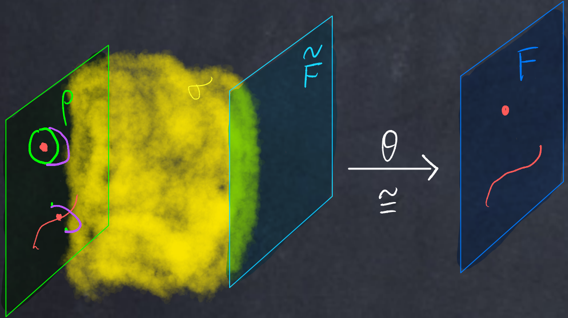

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Only transparent (identity) point defect

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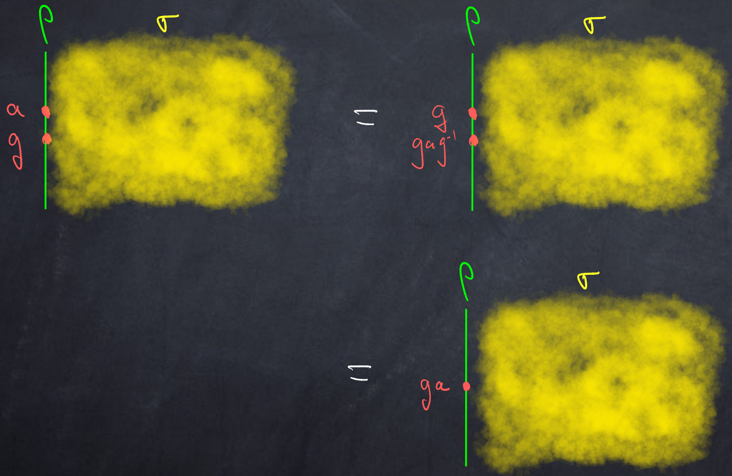
Quantizes to the category $\mathrm{Vect}(G)$



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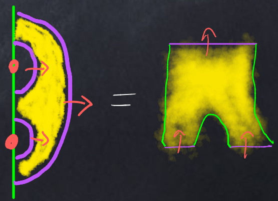

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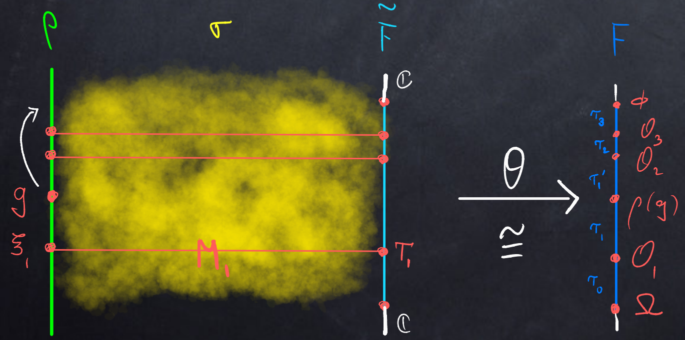


$\Omega \in \mathcal{H}, \quad \phi \in \mathcal{H}^*$

$\rho(g)$-invariant vectors

$$C_\tau = e^{-\tau H/\hbar}$$

$$\langle \phi, C_{\tau_3} O_3 \, C_{\tau_2} O_2 \, C_{\tau_1'} \rho(g) \, C_{\tau_1} O_1 \, C_{\tau_0} \Omega \rangle$$

$$= \langle \phi, C_{\tau_3} O_3^g \, C_{\tau_2} O_2^g \, C_{\tau_1'} C_{\tau_1} O_1 \, C_{\tau_0} \Omega \rangle$$

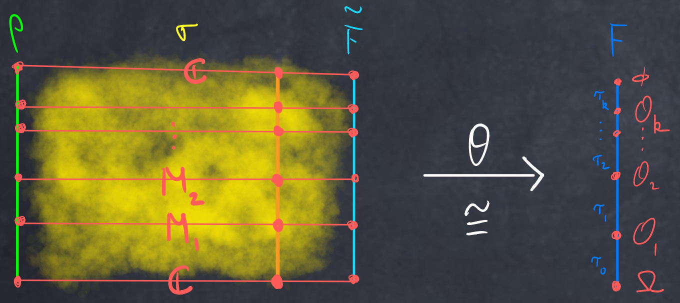
Selection rule in $n = 1$

$G \times G^{\mathrm{op}}$ acts on observables; assume $\mathcal{O}_i$ transforms in a representation M_i. Then

$$\left\langle \phi,\, e^{-\tau_k H/\hbar}\mathcal{O}_k e^{-\tau_{k-1}H/\hbar} \cdots e^{-\tau_1 H/\hbar}\mathcal{O}_1 e^{-\tau_0 H/\hbar}\,\Omega \right\rangle = 0$$

if

$$\mathbb{C} \otimes_A M_k \otimes_A \cdots \otimes_A M_1 \otimes_A \mathbb{C} = 0$$



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One construction of Lie group quiche we indicate uses this relaxation of π-finiteness

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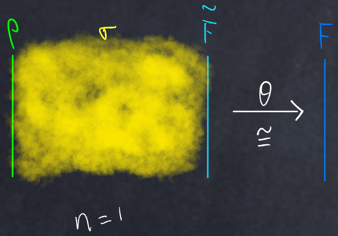

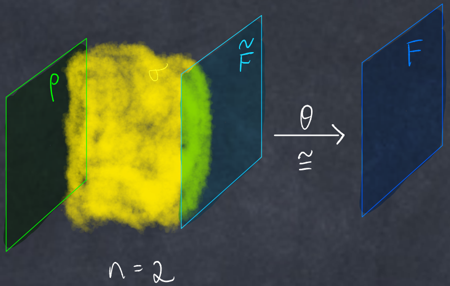

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Some related ideas appear in recent papers of Jia–Luo–Tian–Wang–Zhang and Stockall–Yu

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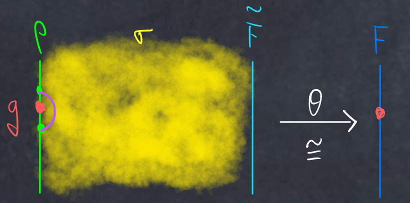

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Remark: Just as in (∞-dimensional) representation theory, dense inclusions give rise to equivalent representations. Example: the regular representation of G can be

$$\text{Fun}(G) = \bigoplus_{\substack{W \\ \text{irrep}}} W \otimes W^* \subset C^\infty(G) \subset C^0(G) \subset L^2(G) \subset C^{-\infty}(G)$$

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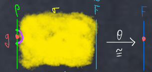

Point ρ-defects: (2) elements of $C^{-\infty}(G)$ or (3) $\mathrm{End}_{\mathrm{Rep}(G)}(\text{underlying v.s.}) \cong$ v.s. of reg rep

From BF theory to flat connections

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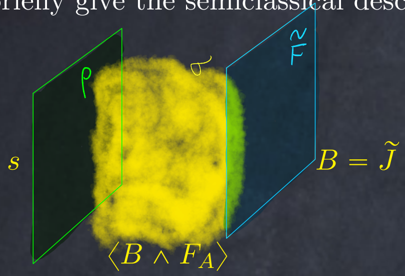

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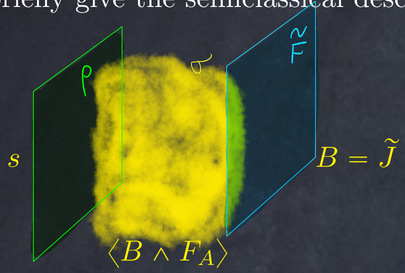

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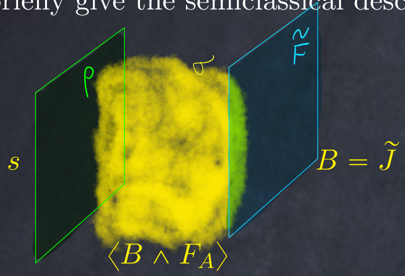

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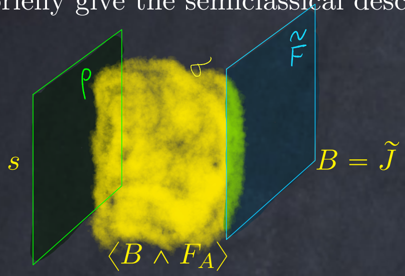

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Lesson: Build (σ, ρ) from *flat* connections as sitting inside all connections

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Deformation complex of flat connections inside all connections:

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\begin{array}{ccccccccc}
0 \longrightarrow & \Omega^0(\mathfrak{g}) & \longrightarrow & \Omega^1(\mathfrak{g}) & \longrightarrow & \Omega^2_{\text{closed}}(\mathfrak{g}) & \longrightarrow 0 \\
& \text{stacky} & \leftsquigarrow & \text{geometric} & \rightsquigarrow & \text{derived} &
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Equivalent to de Rham complex:

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The BV presentation of BF theory has fields corresponding to the de Rham complex (Cattaneo–Rossi, Cattaneo–Mnev–Reshetikhin, . . .)

(1) (Nonfinite) path integral construction of (σ, ρ)

Fluctuating fields:	$P \longrightarrow M,\ \nabla$	principal G-bundle with *flat* connection in σ-bulk
	$s\colon \partial' M \longrightarrow P\big|_{\partial' M}$	*flat* trivialization on ρ-boundary

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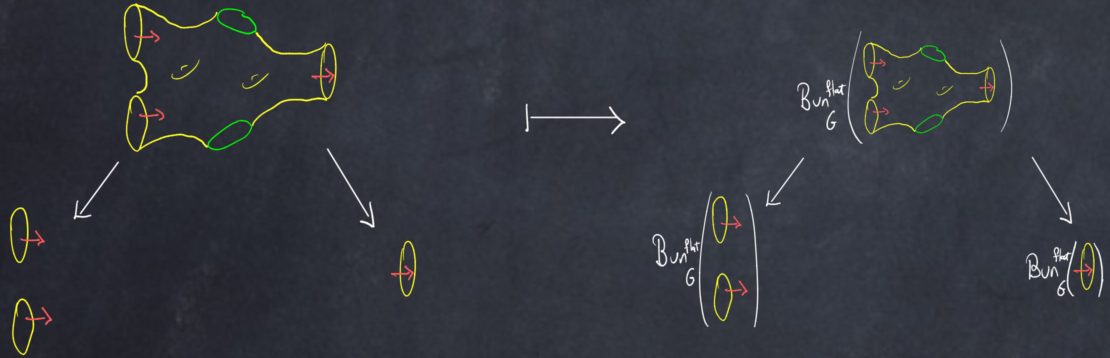

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Many choices for a *topological* vector space of functions, e.g., $n = 1$, $Y =$  :

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Equations of motion again lead to *flat* connections

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Similarly, for arbitrary n we put $\sigma(\mathrm{pt})$ a higher category of sheaves of higher categories

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$$(n = 1) \quad \mathcal{C} = \mathrm{Cat} \qquad \sigma(\mathrm{pt}) = \mathrm{Rep}(G) \qquad \rho(\mathrm{pt}) = \text{underlying vector space}$$

$\mathrm{Rep}(G)$ is an E_{n-1}-algebra in Cat for all n

This leads to a dual construction of (σ, ρ) via the cobordism hypothesis once dualizability results in progress by Scheimbauer–Steffens–Stewart–van Dyke are in place, where now we set $\sigma(\mathrm{pt}) = \mathrm{Rep}(G)$

Codimension one ρ-defects in (σ, ρ)

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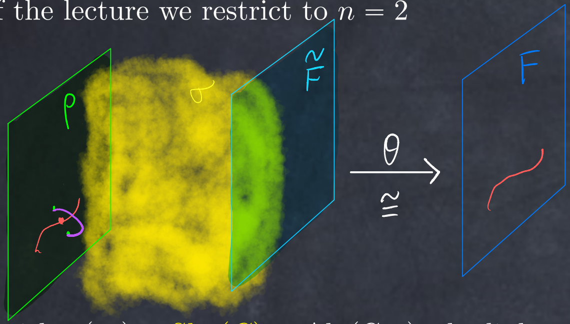

In Construction (2) with $\sigma(\mathrm{pt}) = \mathrm{Shv}(G) \in \mathrm{Alg}(\mathrm{Cat})$, the link  evaluates to the underlying category $\mathrm{Shv}(G) \in \mathrm{Cat}$

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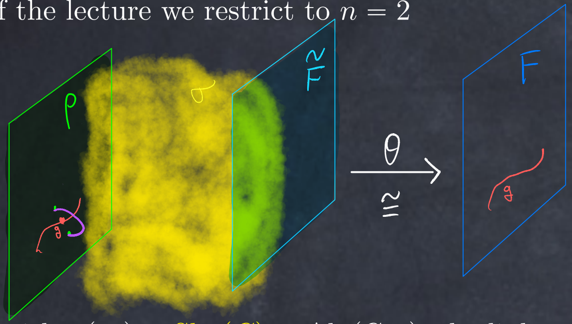

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Skyscraper sheaves $\mathbb{C}_g$, $g \in G$, are the usual codimension one defects for a group element

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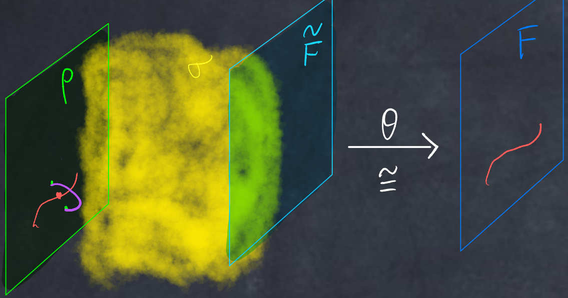

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We already saw these codimension one (point) ρ-defects in case $n = 1$

Point ρ-defects in (σ, ρ)

We compute this vector space V first using direct quantum Constructions (1) and (2), and then by geometric quantization in BF (T^*G Chern–Simons) theory

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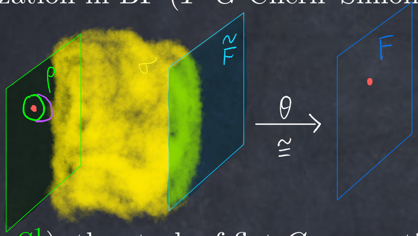
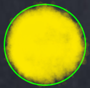


The link  is (D^2, S^1); the stack of flat G-connections (Construction (1)) is $*$

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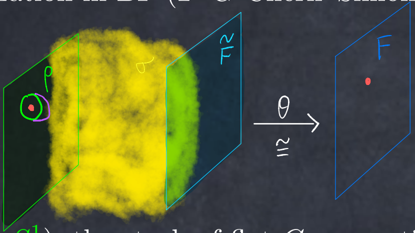

The link  is (D^2, S^1); the stack of flat G-connections (Construction (1)) is $*$

The *derived* stack of flat G-connections has tangent complex $\big(\Omega^\bullet(D^2, S^1; \mathfrak{g}), d\big)$, whose nonzero cohomology is $H^2(D^2, S^1; \mathfrak{g}) \cong \mathfrak{g}$

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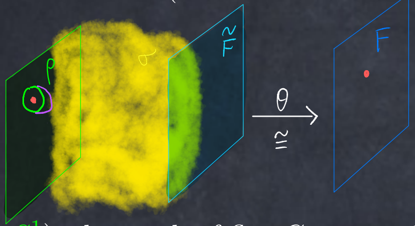

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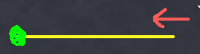
This is in degree $+1$, so the vector space of $\mathbb{C}$-valued functions on the derived stack is

$$\boxed{V = (\sigma, \rho)(D^2, S^1) = \mathrm{Sym}^\bullet\big(\mathfrak{g}^*_{\mathbb{C}}[+1]\big)},$$

which is (secretly) an exterior algebra

Construction (2) begins with $\sigma(\mathrm{pt}) = \mathrm{Shv}(G) \in \mathrm{Alg}(\mathrm{Cat})$, $(\sigma, \rho)($$) = \mathrm{Shv}(G)_{\mathrm{Shv}(G)}$

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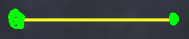
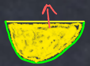
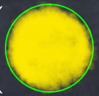
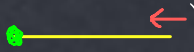
$(\sigma, \rho)($  $) = \mathrm{Shv}(G) \in \mathrm{Cat}$

$(\sigma, \rho)\Big($  $\Big) = \mathbb{C}_e \in \mathrm{Shv}(G)$

$V = (\sigma, \rho)\Big($  $\Big) = R\,\mathrm{End}(\mathbb{C}_e)$

Construction (2) begins with $\sigma(\mathrm{pt}) = \mathrm{Shv}(G) \in \mathrm{Alg}(\mathrm{Cat})$, $(\sigma, \rho)($  $) = \mathrm{Shv}(G)_{\mathrm{Shv}(G)}$

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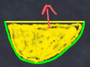
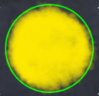
$(\sigma, \rho)\Big($  $\Big) = \mathbb{C}_e \quad \in \mathrm{Shv}(G)$

$V = (\sigma, \rho)\Big($  $\Big) = R\,\mathrm{End}(\mathbb{C}_e)$

Resolve $\mathbb{C}_e$ to compute

$$\boxed{V = \mathrm{Sym}^{\bullet}(\mathfrak{g}_{\mathbb{C}}[-1])}$$

The seemingly discrepant answers

$$V = \mathrm{Sym}^\bullet\big(\mathfrak{g}_{\mathbb{C}}^*[+1]\big)$$
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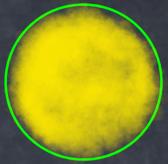

The moduli stack of solutions is $*//\mathfrak{g}^*$ (from B-field gauge transformations), while the derived moduli stack of solutions includes $H^2(D^2, S^1; \mathfrak{g}) \cong \mathfrak{g}$ in degree $+1$, which combine to the symplectic vector space

$$\mathfrak{g}^*[+1] \oplus \mathfrak{g}[-1]$$

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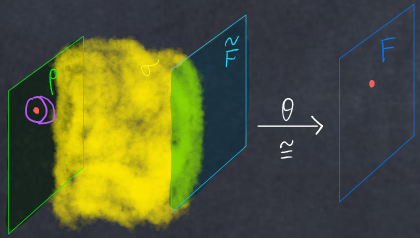

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Example: $M = E$ an elliptic curve, $G = E$ acts by translation. Let $\wp \in H^1(E; \mathcal{O}_E)$ be a principal part and fix $\xi \in \mathrm{Lie}(E)$, which induces $\widehat{\xi} \in H^0(E; TE[-1])$. Then the correlation function on S^2 with the two point defects $\widehat{\xi}, \wp$ equals the sum of the residues of $\iota_\xi\left(\wp\, dz^{\otimes 2}\right)$