

1. GROUP ACTIONS ON CIRCLES

1.1. Conventions. Throughout this paper we require all homeomorphisms of the circle under consideration to be orientation-preserving.

1.2. Basic questions.

Question 1.2.1. *What kinds of groups act on circles? Given a group G , how can one understand the representations $\rho : G \rightarrow \text{Homeo}(S^1)$?*

Question 1.2.2. *Given a group action $\rho : G \rightarrow \text{Homeo}(S^1)$, what is the best possible analytic structure on S^1 compatible with the action of G ?*

1.3. Examples. Where are groups of homeomorphisms of circles found in low-dimensional topology? We list some basic examples, which will be important in the sequel.

Example 1.3.1. Groups defined in terms of the structures that they preserve. For instance:

- The group of rotations S^1
- The group of symmetric transformations $PSL(2, \mathbb{R})$
- The group stabilizing a point $\text{Homeo}(\mathbb{R})$
- The group of smooth homeomorphisms $\text{Diff}(S^1)$
- Thompson's group of homeomorphisms preserving angles which are dyadic rational multiples of 2π
- The group of quasimorphisms
- The group of homeomorphisms with locally constant derivative on an open, dense subset of S^1

and so on.

Example 1.3.2. Let Σ be a surface of negative Euler characteristic. Then $\pi_1(\Sigma)$ acts on $\tilde{\Sigma}$ and therefore on its Gromov boundary, which is topologically a circle.

Example 1.3.3. The group of self-homeomorphisms of $\tilde{\Sigma}$ which cover a self-homeomorphism of Σ also act on $S^1_\infty(\Sigma)$. The kernel of this action is the connected component of the identity in this group, $\widehat{\text{Map}}(\Sigma)$. The group $\widehat{\text{Map}}(\Sigma)$ is a central extension of the mapping class group $\text{Map}(\Sigma)$ by $\pi_1(\Sigma)$. That is, we have a short exact sequence

$$0 \longrightarrow \pi_1(\Sigma) \longrightarrow \widehat{\text{Map}}(\Sigma) \longrightarrow \text{Map}(\Sigma) \longrightarrow 0$$

One may also think of $\widehat{\text{Map}}(\Sigma)$ as $\text{Map}(\Sigma - \text{point})$.

Example 1.3.4. Let Γ be a tree with vertices Γ_0 , and let $f : \Gamma_0 \rightarrow \mathbb{R}$ be a Busemann function. Topologize the vertices Γ_0 with the discrete topology, and define $X = \Gamma_0 \times S^1$. For $v \in \Gamma_0$, denote (v, S^1) by S^1_v . Suppose for each pair x, y with $f(x) \geq f(y)$ we are given a monotone map $\phi^x_y : S^1_x \rightarrow S^1_y$ where $\phi^y_z \phi^x_y = \phi^x_z$ for any $f(x) \geq f(y) \geq f(z)$. The inverse limit of this system of monotone maps gives a circle S^1_{univ} and a monotone map $\phi^\infty_x : S^1_{\text{univ}} \rightarrow S^1_x$ for any x such that $\phi^x_y \phi^\infty_x = \phi^\infty_y$ for any $f(x) > f(y)$. Suppose G is a group of homeomorphisms of X which are compatible with the ϕ^x_y . Then G acts on S^1_{univ} .

Example 1.3.5. Let G be countable, and let $\rho : G \rightarrow \text{Homeo}(S^1)$ be some representation. Pick a point $x \in S^1$ and replace x by an interval I_x . Do the same for each translate of x , where the size of these intervals decreases geometrically. Then we can make G act on the new “blown-up” S^1 , once we choose a representation

$$\text{stab}(x) \rightarrow \text{Homeo}(I_x)$$

Example 1.3.6. Let T be an infinite planar tree with no 1-valent vertices. The planar embedding of T determines a circular ordering of the ends \mathcal{E} of T . Assume that \mathcal{E} is a perfect set (i.e. that it has no isolated points). This space of ends \mathcal{E} is a circularly ordered Cantor set, which can be canonically embedded in a circle. By blowing down the gaps in $S^1 - \mathcal{E}$ we get a new circle which we denote by $S^1_{\mathcal{E}}$.

Let G be the group of “automorphisms at infinity” of T defined as follows. An *admissible map* is a self-homeomorphism of the plane which induces an isomorphism $T - K_1 \rightarrow T - K_2$ for two compact sets $K_1, K_2 \subset T$. Two such ϕ_1, ϕ_2 are *equivalent* if there is a compact set $K \subset T$ such that $\phi_1|_{T-K} = \phi_2|_{T-K}$. The set of equivalence classes of admissible maps is the group G . Then G acts on \mathcal{E} by order-preserving homeomorphisms, and therefore also on $S^1_{\mathcal{E}}$. In general, the image of G will be much smaller than $\text{Homeo}(S^1_{\mathcal{E}})$ (for instance, it will be countable); but in many important cases it will be dense.

1.4. Euler classes and the Milnor–Wood inequality. Given a representation $\rho : G \rightarrow \text{Homeo}(S^1)$ one can construct a flat foliated circle bundle in the following way. Let BG be some $K(G, 1)$ and let EG be its universal cover. Then $EG \times S^1$ is foliated by level sets $EG \times \text{point}$, and this foliation descends to the quotient space

$$X = EG \times S^1 / (x, \theta) \sim (\alpha(x), \rho(\alpha)(\theta))$$

X is a foliated circle bundle over BG . Conversely, a foliation of a circle bundle gives a local trivialization of the bundle; equivalently, a reduction of the structure group from $\text{Homeo}(S^1)$ to $\text{Homeo}(S^1)^{\delta}$ — the group of homeomorphisms made discrete. Thus there is an equivalence

$$\text{foliated circle bundles over } BG/\text{isotopy} \longleftrightarrow \text{Hom}(G, \text{Homeo}(S^1))/\text{conjugacy}$$

Since $\text{Homeo}(S^1)$ is homotopy equivalent to S^1 , as a circle bundle X is classified by its Euler class $e(X) \in H^2(G; \mathbb{Z})$. An explicit 2-cocycle representative can be constructed as follows: pick some $p \in S^1$ and for each triple $\alpha, \beta, \gamma \in G$ define

$$e(\alpha, \beta, \gamma) = \begin{cases} 1 & \text{if } \alpha(p), \beta(p), \gamma(p) \text{ are in an anticlockwise order} \\ -1 & \text{if } \alpha(p), \beta(p), \gamma(p) \text{ are in a clockwise order} \\ 0 & \text{if at least two of } \alpha(p), \beta(p), \gamma(p) \text{ are coincident} \end{cases}$$

One can check that e is invariant under the left diagonal action of G , and that $\delta e = 0$. One can see from this that e is a *bounded* cohomology class and thus represents a nontrivial element of $H_b^2(G; \mathbb{Z})$ unless $\rho(G)$ has a global fixed point.

In particular, $\|e\|_{\infty} = 1$. If $\psi : \Sigma \rightarrow BG$ represents an element of $H_2(G; \mathbb{Z})$, then $\psi^*(X)$ is a foliated circle bundle over Σ with euler class $\psi^*(e)$. It follows that

$$|e(\psi^*[\Sigma])| \leq \|e\|_{\infty} \|\Sigma\|_1 \leq |\chi(\Sigma)|$$

In particular, we derive the well-known *Milnor–Wood inequality* (see e.g. [12]):

Theorem 1.4.1 (Milnor–Wood). *Let E be a foliated circle bundle over a surface Σ . Then*

$$|e(E)| \leq |\chi(\Sigma)|$$

This inequality is sharp, as the following well-known example shows:

Example 1.4.2. Let Σ be a closed hyperbolic surface. Then the geodesic flow on the unit tangent bundle $UT\Sigma$ is Anosov, and the stable leaves of the flow give a foliation transverse to the circles $UT_p\Sigma$. One also has

$$e(UT\Sigma) = \chi(\Sigma)$$

On the other hand, in the sharp case, a rigidity phenomenon emerges. The following theorem is from [10]:

Theorem 1.4.3 (Matsumoto). *Let Σ be a closed surface. If*

$$\phi_1, \phi_2 : \pi_1(\Sigma) \rightarrow \text{Diff}(S^1)$$

satisfy $e(\phi_1) = e(\phi_2) = \chi(\Sigma)$, then these representations are conjugate.

Remark 1.4.4. The special case in which the images of ϕ_1, ϕ_2 both lie in $PSL(2, \mathbb{R})$ was already proved by Goldman in [8].

1.5. Convergence groups.

Definition 1.5.1. Let G be a group of homeomorphisms of S^1 . G is a *convergence group* if for each infinite subset of G there is a subsequence g_i such that

- either $g_n \rightarrow g$ and $g_n^{-1} \rightarrow g^{-1}$ uniformly over S^1 for some $g \in \text{Homeo}(S^1)$;
- or
- there are points $x_0, y_0 \in S^1$ such that $g_n \rightarrow x_0$ and $g_n^{-1} \rightarrow y_0$ locally uniformly on $S^1 - y_0$ and $S^1 - x_0$ respectively

Example 1.5.2. Möbius groups are convergence groups.

Example 1.5.3. Groups of K -quasisymmetries of S^1 for some *uniform* K are convergence groups.

Observe that the property of being a convergence group is preserved under conjugation.

For such groups there is a beautiful theorem which gives a precise relationship between the topology of the dynamics of the group action and the best possible analytic quality of S^1 preserved by the action. This is the “convergence group theorem” whose proof was completed by Gabai in [6] and independently by Casson and Jungreis in [2].

Theorem 1.5.4 (Tukia, Hinkkanen, Gabai, Casson, Jungreis, etc). *A group G in $\text{Homeo}(S^1)$ is a convergence group iff it is conjugate to a subgroup of $PSL(2, \mathbb{R})$.*

The group $\text{Homeo}(S^1)$ is a topological group, with the topology of pointwise convergence. A *discrete group of homeomorphisms of S^1* is a subgroup which is discrete as a subset of this space. Such a group is necessarily countable. A discrete group G is a convergence group if and only if it acts freely and properly discontinuously on the space of ordered triples of distinct points in S^1 . In particular, there is a 3-manifold M and a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow G \longrightarrow 0$$

where the \mathbb{Z} subgroup is central. Conversely, Mess showed ([11]) that for M a 3-manifold with a central \mathbb{Z} subgroup, $\pi_1(M)/\mathbb{Z}$ is a convergence subgroup.

1.6. Visual metrics and growth metrics. For Σ a surface of negative curvature, the asymptotics of the geodesic flow identifies $UT_p\Sigma$ with $S_\infty^1(\tilde{\Sigma})$ for every p . The geodesic flow is Anosov, and the stable foliation of the flow makes $UT\Sigma$ into a foliated circle bundle. There is therefore an induced representation

$$\rho : \pi_1(\Sigma) \rightarrow \text{Homeo}(UT_p\Sigma)$$

If Σ is a surface of constant negative curvature, this representation is analytic, and lies in $PSL(2, \mathbb{R})$. However, if Σ is a surface of variable negative curvature, this representation is typically not so good. To see why this is true, consider the effect of perturbing a constant curvature metric in a small ball B . Let \tilde{B} be a lift of B to $\tilde{\Sigma}$. The geodesics emanating from \tilde{p} flowing through \tilde{B} are distorted slightly; this has the effect of perturbing the visual circle of p by some diffeomorphism ϕ which is the identity outside a small compact set. But there are also $O(e^t)$ lifts of \tilde{B} at distance t from p , with visual size approximately e^{-t} . This has the effect of perturbing the visual circle of p by diffeomorphisms ϕ^{ψ_i} where ψ_i is some (approximately linear) contraction by $O(e^{-t})$ of the support of ϕ centered at some (more or less random) direction in the visual circle of p . The limit of these iterated perturbations

$$\phi_\infty = \dots \phi^{\psi_i} \phi^{\psi_{i-1}} \dots$$

will not be C^ϵ for any $\epsilon > 0$.

The effect of the perturbations at \tilde{p} and $\alpha(\tilde{p})$ for a group element α are actually highly correlated, since geodesics from \tilde{p} and from $\alpha(\tilde{p})$ with the same limit point at infinity will converge exponentially quickly, and undergo very similar perturbations; the net result is that the action of α^{ϕ_∞} will be C^1 .

See e.g. [9] or [7] for more details.

A better metric for the action of $\pi_1(\Sigma)$ on S_∞^1 is a metric adapted to the *asymptotic geometry* of $\tilde{\Sigma}$, rather than a metric which is so sensitive to local perturbations. Such a metric is the *growth metric* on S_∞^1 .

Definition 1.6.1. Let Σ be a surface of negative curvature. Let $\Gamma \subset \Sigma$ be a graph with one vertex such that $i_* : \pi_1(\Gamma) \rightarrow \pi_1(\Sigma)$ is surjective. In particular, the lift $\tilde{\Gamma}$ of Γ to $\tilde{\Sigma}$ will be a Cayley graph for $\pi_1\Sigma$. Think of $\tilde{\Sigma} \cup S_\infty^1$ as the closed unit disk. Pick a basepoint p which is a vertex of $\tilde{\Gamma}$. For each positive integer n , let $\tilde{\Gamma}_n$ be the vertices of $\tilde{\Gamma}$ which are a combinatorial distance n from p . Define

$$\mu_n = 2\pi \frac{\sum_{v \in \tilde{\Gamma}_n} \delta_v}{|\tilde{\Gamma}_n|}$$

where δ_v is the Dirac measure with weight 1 supported at the point v . Let μ_∞ be a weak limit of μ_n . Define the *growth metric on S^1* , denoted by $g_{\Gamma,p}$ (depending on Γ, p) to be the path metric associated to the measure μ_∞ on S^1 .

Lemma 1.6.2. *The metric $g_{\Gamma,p}$ is well-defined.*

Proof: The Cayley graph of $\pi_1(\Sigma)$ is a Gromov hyperbolic metric space; in particular, the language of geodesics is regular. So there is a finite state automaton A which accepts geodesic words in $\text{Cayley}(\pi_1(\Sigma))$ — equivalently, which accepts geodesic polygonal paths in $\tilde{\Gamma}$ starting at p .

We can refine A to a new finite state automaton A' which accepts *anticlockwisest* geodesic paths in $\tilde{\Gamma}$ starting at p . Of the (finitely many) shortest paths

w_1, \dots, w_n joining p to some q , the *anticlockwisemost* is the path which, at every vertex, always takes the outgoing edge which is furthest in the anticlockwise direction from the incoming edge, amongst all geodesic paths. It is easy to verify that such a path exists and is unique — one way of defining it is as the anticlockwise hull of the union $\bigcup_i w_i$.

The set of paths accepted by A' are prefix closed, and there is a unique path from p to any q . Thus the union of these paths is a directed maximal spanning tree of $\tilde{\Gamma}$, where p is the unique vertex with only outgoing edges.

For each q in $\tilde{\Gamma}$, define $f(q) = \text{dist}(p, q)$, and define $s(q)$ to be the state of the automaton A' on reaching the point q . $s(q)$ takes on finitely many values s_1, \dots, s_m . For each n , the set $\tilde{\Gamma}_n$ defines a tiling T_n of S^1 with one “tile” of type $s(q)$ for each vertex q , in the circular ordering determined by the planar embedding of $\tilde{\Gamma}_n$ and the basepoint p . The automatic structure determines a substitution rule which describes how to obtain T_{n+1} from T_n . There is a transition matrix \mathbf{A} associated to A' whose entry a_{ij} is the number of tiles of kind s_j that are obtained from s_i . Equivalently, a_{ij} is the number of outgoing edges from a vertex q with $s(q) = s_i$ which end in vertices q' with $s(q') = s_j$. It is clear that the biggest eigenvalue λ of \mathbf{A} is real. Typically, the eigenspace V_λ is 1-dimensional, and spanned by some vector v_λ .

For a tile t , set $l(t)$ equal to $\langle s(t), v_\lambda \rangle$. Then there is a finite n such that if t_i are the subtiles obtained from $t \in T_n$,

$$\sum_i l(t_i) = \lambda t$$

For a vertex $v = \tilde{\Gamma}_n$ for such an n , $\langle s(v), v_\lambda \rangle = 0$ implies that the growth rate of the words accepted by A' which pass through v is $o(\lambda^t)$. Thus the metric on S^1 determined from T_n by setting the length of the tile t equal to $l(t)/\lambda^n$ is preserved by the substitution tiling, and is also asymptotically equal to μ_n . In particular, the measures μ_n have a unique weak limit, and their limit has full support with no atoms.

If the action of A on V_λ is diagonalizable, we can set

$$l(t) = \sum_i \langle s(t), v_\lambda^i \rangle$$

for some choice of v_λ^i an orthonormal basis for V_λ . If the action is not diagonalizable, the growth rate of the accepted words with prefix q is $O(n^{C_{s(q)}} \lambda^n)$ where $C_{s(q)}$ is a constant depending on the state of A' after processing q . If we discount q with $C_{s(q)}$ less than maximal, we can construct a piecewise-linear metric as above. \square

Remark 1.6.3. There are many free parameters in the preceding situation: we can change the combinatorics of the graph Γ . We can also set the lengths of edges of Γ to values other than 1. If the lengths are rationally related, the language of anticlockwisemost geodesics is still automatic.

Let Γ be an arbitrary planar graph, and pick two distinct vertices $v, w \in \Gamma$. Let S_n be the set of points in Γ at combinatorial distance n from v . Then S_n is naturally a circularly ordered set.

Lemma 1.6.4. *The distance function $d(p, w)$ is monotone on S_n . That is, there are $x, y \in S_n$ and a partition of S_n into $I_{n,1} = \{z : x \leq z \leq y\}$ and $I_{n,2} = \{z : y \leq z \leq x\}$ such that $d(p, q)$ is monotone (in the usual sense) on each $I_{n,i}$.*

Proof: For points $x, y, z \in S_n$, join p and q to x, y, z by anticlockwisemost geodesics $\gamma(p, x)$ and so on. If x, z, y are in anticlockwise order in S_n , and $\gamma(p, x) \cup \gamma(p, y)$ separate q from z , then $\gamma(q, z)$ intersects one of $\gamma(p, x)$ or $\gamma(p, y)$, say it intersects $\gamma(p, x)$. The triangle inequality implies that

$$\text{length}(\gamma(q, z)) \geq \text{length}(\gamma(q, x))$$

The monotonicity of $d(p, w)$ follows easily from this observation. \square

Theorem 1.6.5. *With respect to the metric $g_{\Gamma, p}$ on S_∞^1 , the action of $\pi_1(\Sigma)$ is piecewise linear. Moreover, the values of the derivatives of any element are in $\langle \lambda \rangle$, the set of powers of some algebraic λ .*

Proof: By lemma 1.6.4 the function $f(w) = d(p, w) - d(q, w)$ is monotone on each S_n . The S_n limit geometrically to S_∞^1 and the function f extends to a monotone function on S_∞^1 which takes on values from the set $\{-n, 1-n, \dots, n\}$ on intervals I_i , where there are at most two such intervals for each value.

If $q = \alpha(p)$, then the effect of α on one such interval I_i is to multiply the measure by $\lambda^{f(I_i)}$, where λ is the largest eigenvalue of the automaton A' constructed in lemma 1.6.2, and $f(I_i)$ denotes the value of f on vertices of $\tilde{\Gamma}$ limiting to I_i . \square

Question 1.6.6. *How does $\langle \lambda \rangle$ and $g_{\Gamma, p}$ depend on Γ ?*

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