

Seminar
Harvard University — Fall 2018
Notes

1 Introduction

This seminar will be loosely directed at the study of dynamics and moduli spaces, both classical topics and current developments.

Its topics will range from the concrete dynamics of billiards in pentagons to the abstract theory of flows on homogeneous space to parallels with dynamics on moduli spaces of Riemann surfaces and holomorphic 1-forms.

We will start with background on extremal length, Teichmüller's theorem, and applications to complex dynamics. We will then turn to hyperbolic manifolds in dimensions 2 and 3, and Ratner's theorem, to set the stage for later developments.

Unique ergodicity. (Cf. [FLP].)

Theorem 1.1 *Let \mathcal{F} be an oriented foliation of X with dense leaves. Then the map from the space of transverse invariant measures on \mathcal{F} to $H^1(X, \mathbb{R})$ is injective.*

Proof. Consider a closed transversal $\tau = [a, b]$. Follow the leaf of \mathcal{F} through b until it comes back near to a and close it to obtain a cycle C . Then the transverse measure $\mu(\tau)$ is nearly the same as the intersection number of (\mathcal{F}, μ) with C . Thus $[\mathcal{F}, \mu] \in H^1(X, \mathbb{R})$ determines $\mu(\tau)$ for all τ and hence it determines μ . ■

Theorem 1.2 *If \mathcal{F} is invariant under a pseudo-Anosov mapping f , then it is uniquely ergodic.*

Proof. Let $M \subset H^1(X, \mathbb{R})$ denote the space of cohomology classes represented by transverse invariant measures for \mathcal{F} . We must show that M reduces to a ray through the standard class $F = [\mathcal{F}, \alpha]$. Note that F is an expanding eigenvalue for f^* .

Let $F' = [\mathcal{F}', \alpha']$ denote the orthogonal measured foliation. Then M lies in the half-space of classes with positive intersection with P . Moreover M is a convex cone with a compact base, and it is invariant under f^* . Passing to the project space $\mathbb{P}H^1(X, \mathbb{R})$, we obtain a compact convex set $\mathbb{P}M$, invariant under f^* , that contains an attracting fixed point $[F]$. This contradicts the

Schwarz lemma for the Hilbert metric on $\mathbb{P}M$, unless the latter space reduces to a point. \blacksquare

Remarks on M . Let $C(X)$ be the continuous functions on a compact metric space X , let $T : X \rightarrow X$ be a homeomorphism, and let $P \subset M(X) = C(X)^*$ be the space of T -invariant probability measures on X . Clearly P is a compact, convex set.

It is known that P is *nonempty* (since \mathbb{Z} is amenable), and that its extreme points coincide with the ergodic measures. Moreover, P is a *Choquet simplex*. That is, every point of P can be described uniquely as the barycenter of a measure on its extreme points.

In particular, when P is finite dimensional, it is a simplex.

Returning to the case of an oriented foliation \mathcal{F} as above, the cohomology classes $M \subset H^1(X, \mathbb{R})$ of transverse invariant measures, normalized to have fixed intersection number with the foliation normal to \mathcal{F} , is also a simplex.

The span of M is a Lagrangian subspace, so its dimension is at most g . In particular, there are at most g ergodic measures on \mathcal{F} . This bound is optimal — it can actually be achieved by variants of the constructions to be given below.

Failure of unique ergodicity. Most measured laminations are uniquely ergodic. Indeed, UE has full measure in $\mathbb{P}\mathcal{ML}_g$, and it is known that $\mathcal{F}(e^{2\pi i\theta}q)$ is uniquely ergodic outside an exceptional set of $\theta \in \mathbb{R}/\mathbb{Z}$ with Hausdorff dimension at most $1/2$.

Also, UE is a dense G_δ . Indeed, NUE can be written as a union of closed sets F_n where the failure of UE gives rise to a simplex in $\mathbb{P}\mathcal{ML}_g$ of diameter at least $1/n$; thus NUE is an F_σ set. So NUE is rare in both measure and category.

Nevertheless, examples of laminations in NUE can be readily constructed. This construction is in fact easier than the classical construction of NUE interval exchange transformations. Of course a finite union of two or more disjoint simple closed curves is NUE; the real goal is to find a NUE example where every leaf is dense, i.e. a *minimal*, NUE example.

Construction with laminations.

Theorem 1.3 *For every $g \geq 2$ there exists a NUE lamination $\lambda \in \mathbb{P}\mathcal{ML}_g$.*

Proof. Let $\gamma_1, \gamma_2, \dots$ enumerate all the simple closed curves on Σ_g . Since $g \geq 2$, we can choose a pair of disjoint simple closed curves α_0 and β_0 on Σ_g . Our aim is to construct a sequence of laminations

$$\lambda_n = \alpha_n \cup \beta_n$$

in $\mathbb{P}\mathcal{M}\mathcal{L}_g$, each a union of two simple curves, such that (i) α_n and β_n have distinct limits, α and β ; and (ii) $\lambda = \alpha \cup \beta$ meets γ_i for every i . The last property insures that λ is minimal, and by construction it carries two different invariant measures, α and β .

Notice that the condition $i(\lambda, \gamma_i) \neq 0$ is open. So if insure that λ_i meets γ_i , and inductively choose λ_{i+1} very close to λ_i — close enough that previous intersections with γ_j , $j \leq i$ are maintained, and even maintained in the limit λ — then λ will be minimal.

Let us detail now the inductive construction. If λ_i crosses γ_{i+1} , there is nothing to do — we set $\lambda_{i+1} = \lambda_i$.

Otherwise, γ_{i+1} is disjoint from α_i and β_i . Cut Σ_g open along β_i , choose a complicated simple curve δ_i , and reglue. Then δ_i meets α_i and γ_{i+1} but not β_i . Let $\alpha_{i+1} = \tau^n(\delta_i)$, where τ is a Dehn twist around α_i . By choosing n large, we can insure that α_{i+1} is as close to α_i as we like. It remains disjoint from β_i , and continues to meet γ_{i+1} , so we can now simply set

$$\alpha_{i+1} = \tau_{\alpha_i}^n(\delta_i), \quad n \gg 0,$$

and $\lambda_{i+1} = \alpha_{i+1} \cup \beta_i$.

A special case arises when γ_{i+1} coincides with a component of λ_i , say with α_i . But then we proceed exactly as above, with δ_i crossing α_i but not β_i .

Since we have moved α_0 and β_0 only by a tiny amount, we can insure that $\alpha \neq \beta$ in the limit. Then $\lambda = \alpha \cup \beta$ is the desired lamination. ■

Construction with connect sums. Here is a construction in the spirit of Veech's original examples [V]. (Veech actually worked with an irrational rotation on $S^1 \times \mathbb{Z}/2$ that swaps circles when x lands in a fixed interval $[0, \theta]$.)

Let $E = (\mathbb{C}/\Lambda, dz)$ be a torus equipped with a 1-form. Given an embedded arc $I = [0, z]$ in E , we can slit E open along I , and glue together two copies of the result to obtain a 1-form of genus two:

$$(X, \omega) = E \oplus_I E.$$

There is a natural map $\pi : X \rightarrow E$, branched over $\{0, z\}$, such that $\omega = \pi^*(dz)$. The preimage of I gives a loop $J \subset X$ cutting X into two pieces A and B , each a lift of $E - I$. Their areas satisfy $|A| = |B|$.

We will show:

Theorem 1.4 *If I does not lie on a closed geodesic in E , then the foliation $\mathcal{F}(e^{i\theta}\omega)$ is minimal, but not uniquely ergodic, for uncountably many values of θ .*

In fact for the θ we construct, X can be decomposed into two sets of equal measure, each a union of leaves of $\mathcal{F}(e^{i\theta}\omega)$.

To begin the construction, let $I' = [0, z']$, where $z' = z \bmod 2\Lambda$, and assume I' injects into E . Note that I and I' have the same endpoints; moreover, the cycle $I + I'$ is trivial in $H^1(E, \mathbb{Z}/2)$, so I and I' determine the same double cover of X and hence the same form (X, ω) .

Of course A' is generally different from A . But if the determinant of the parallelogram spanned by z and z' is small, then these two sets are close in the sense of measure.

Theorem 1.5 *We can order (A', B') so that that $|A \triangle A'| \leq |\operatorname{Im}(\bar{z}, z')|$.*

Proof. Consider the triangle $T \subset \mathbb{C}$ with vertices $0, z$ and $z' = z + 2\lambda$, $\lambda \in \Lambda$. Its area is given by $d/2$, where $A = |\operatorname{Im}(\bar{z}, z')|$. Since the edge from z to z' double covers its projection to E , we find that the projection of T to E gives a $\mathbb{Z}/2$ cycle T_E with $\partial T_E = I + I'$. Similarly, the preimage T_X of T_E on X satisfies $\partial T_X = J + J'$. The mass of these cycles satisfies

$$|T_X| \leq 2|T_E| \leq 2|T| = d,$$

and $\partial(A + T_X) = I + (I + I') = I'$, so $A + T_X = A'$ and the theorem follows. ■

Lemma 1.6 *If I does not lie along a closed geodesic in E , then for any $\epsilon > 0$ there exists a $z' \neq z + 2\Lambda$ such that $0 < |\operatorname{Im}(\bar{z}, z')| < \epsilon$.*

Proof. The set $\operatorname{Im}(\bar{z}(z + 2\Lambda))$ is dense in \mathbb{R} , since it is the image of Λ under a group homomorphism with trivial kernel. ■

Proof of Theorem 1.4. For concreteness let $\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$ so that E is the square torus, and choose $z_0 \neq 0$ so its slope $\operatorname{Im}(z_0)/\operatorname{Re}(z_0)$ is irrational. Then by the results above, we can choose $z_i \in z_0 + 2\Lambda$, $i = 1, 2, 3, \dots$ such that $0 < |\operatorname{Im}(\bar{z}_i z_{i+1})| < 1/2^i$, such that

$$z_i/|z_i| \rightarrow z_\infty \in S^1,$$

and such that z_∞ is not parallel to any relative period of (X, ω) . Then the foliation \mathcal{F}_∞ of $(X, |\omega|)$ at slope z_∞ is minimal, since it has no saddle connections. Moreover for each i we have a set $A_i \subset X$ of measure $|A_i| = |X|/2$, which is a union of leaves of the foliation \mathcal{F}_i of X parallel to z_i . Since

$\sum 2^{-i} < \infty$, the sets form a Cauchy sequence and converge to set $A \subset X$. This limiting set is saturated with respect to the limiting foliation \mathcal{F}_∞ , and still occupies only 1/2 the area of X , so \mathcal{F}_∞ is not ergodic (and therefore not uniquely ergodic). ■

Remark. This argument can be pushed farther to construct orientable foliations in genus g with g linearly independent ergodic measures.

2 Problems

1. Show that for every δ with $0 < \delta < 1$, there exists an $x \in S^1 = \mathbb{R}/\mathbb{Z}$ such that $E_x = \overline{2^{\mathbb{N}}x} \subset S^1$ is a Cantor set of Hausdorff dimension δ .
2. Show that every compact hyperbolic surface can be constructed by gluing together the sides of a convex hyperbolic polygon with isometries.
3. Let (X, ω) be a nonzero holomorphic 1-form with X a compact Riemann surface (of genus $g \geq 1$). Can (X, ω) always be constructed by gluing together the sides of a convex polygon (P, dz) in the plane by translation? If not, can we at least choose P to be connected?
4. Prove that the center of Mod_g is generated by the hyperelliptic involution when $g = 1, 2$ and that it is trivial for $g \geq 3$.
5. Prove that there are only finitely many types of simple closed curves on a surface of genus g — up to the action of the mapping-class group.
6. *Use the preceding fact to prove that Mod_g is generated by Dehn twists.
7. *Use the fact that Mod_g is generated by Dehn twists to prove that every compact orientable 3-manifold M is obtained by Dehn surgery on a link in S^3 . (Hint: first prove that M has a Heegaard splitting.)
8. Formulate a conjecture of the form: s is the slope of a periodic trajectory in the regular pentagon (with one edge vertical) if and only if $x \in E$, where E has something to do with $\mathbb{Q}(\sqrt{5})$. Prove one or *both directions of your conjecture.
9. **Find the periodic slopes in the regular 7-gon.

10. Let $\Gamma_n \subset \mathrm{SL}_2(\mathbb{R})$ be the (Hecke) subgroup generated by $\begin{pmatrix} 1 & 2\cos(\pi/n) \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- Describe \mathbb{H}/Γ_n as an orbifold.
 - Show that Γ_n is discrete for all $n \geq 3$.
 - Show that $\Gamma_3 = \mathrm{SL}_2(\mathbb{Z})$.
 - * Show that $\Gamma_5 \cdot \infty = \mathbb{Q}(\sqrt{5}) \cup \{\infty\}$.
 - ** Determine the cusps of Γ_7 .
11. Define a natural symplectic form on the variety of (discrete, faithful) representations of $\pi_1(\Sigma_g)$ into $\mathrm{SL}_2(\mathbb{R})$ (up to conjugacy). (Hint: what is the tangent space?)
12. Given $\lambda > 1$: use the method of extremal length to estimate above and below, with concrete constants, the value of $t > 0$ such that the upper half plane, with the points $0, 1, \lambda$ and ∞ marked, is conformally equivalent to a rectangle of the form $R = \{x+iy : x \in [0, 1], y \in [0, t]\}$, with $[0, 1]$ corresponding to $[0, 1]$.
13. Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map such that f sends $\{0, 1, \infty, a\}$ into $\{0, 1, \infty\}$. Show that a is an algebraic number.
14. Let $E = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, and let Γ denote the set of loops on E with period $a + b\tau$. Compute the extremal length $\lambda(E, \Gamma)$ and show its level sets are parallel horocycles. Explain the significance of the point on $\partial\mathbb{H}$ where these horocycles rest.
15. Prove that the map $f : \mathbb{C} \rightarrow \mathbb{C}$ given in polar coordinates by $f(r, \theta) = (r^\alpha, \theta)$ with $0 < \alpha < 1$ is K -quasiconformal, where $K = 1/\alpha$. Show that there exists a circle C defined by $|z - c| = r$ such that the maximum and minimum distances of $f(C)$ from $f(c)$ satisfy $M/m > K$.
16. Prove that a K -quasiconformal map is $1/K$ -Hölder continuous.
17. Construct explicitly, in genus g , a family of Strebel differentials of dimension $6g - 5 = 1 + \dim \mathcal{M}_g$.
18. Prove that for any finite set $A \subset \mathbb{C}$, $|A| \geq 2$, there exists a rational map f such that the critical values of f coincide with A .

19. For each $n \geq 2$, find an explicit polynomial $f_n(z)$ with postcritical set $P(f_n) = \{0, 1, n, \infty\}$. Prove that for any solution to this problem, we have $\deg(f_n) \rightarrow \infty$. *Give an explicit lower bound on $\deg(f_n)$.
20. Using the classical Jacobi elliptic functions sn , cn and dn , give an explicit formula for $f(x)$, $x \in [-1, 1]$, where f is the extremal quasi-conformal map on $\widehat{\mathbb{C}}$ that preserves the real axis, fixes ± 1 , and sends $\pm k$ to $\pm k'$, with $k, k' > 0$.
21. Let $T : X \rightarrow X$ be a homeomorphism on a compact metric space X , and let $P \subset C(X)^* = M(X)$ denote the space of T -invariant probability measures.
- Show that P is a nonempty, compact convex set, and that its extreme points E correspond to ergodic measures. Show that the elements of E are linearly independent.
- *Show that P is a Choquet simplex: every point of P is the barycenter of a unique probability measure on E .
22. Let T be a self-adjoint automorphism of a symplectic vector space S of dimension $2g$ over \mathbb{R} . This means S is equipped with a non-degenerate alternating form $[x, y]$, and $[Tx, y] = [x, Ty]$.
- (i) Prove that if $\dim(S) = 2$, then T is a multiple of the identity.
(ii) In general, prove that the (generalized) eigenspaces of T are even dimensional.
23. Let $\langle x, y \rangle$ be a Hermitian inner product on \mathbb{C}^{2g} such that \mathbb{R}^{2g} is isotropic: any two real vectors are orthogonal. Suppose $\langle Tx, y \rangle = \langle x, Ty \rangle$ and T preserves \mathbb{R}^{2g} . Prove that the eigenspaces of T are even dimensional.
24. Let f be an orientation-preserving homeomorphism of a surface S of genus g . Explain how the action of $T = f + f^{-1}$ on $H^1(S)$, with coefficients in \mathbb{R} or \mathbb{C} , gives an instance of the previous two results.
25. Show that the $\operatorname{SL}_2(\mathbb{Z})$ orbit of a point x in the torus $E = \mathbb{R}^2/\mathbb{Z}^2$ is either finite or dense, and the first case occurs iff x is torsion in the group law on E .
26. Consider the action of $N \subset \operatorname{SL}_2(\mathbb{R})$, the upper triangular group, on $E = \mathbb{R}^2/\Lambda$ for a general lattice Λ . What are the possibilities for the closure of an orbit Nx ?

27. Let $G = \mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ be the special affine group of \mathbb{R}^2 , and let Γ be its integral points, $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. (i) Show that $X = G/\Gamma$ can be identified with the universal elliptic curve over the moduli space \mathcal{M}_1 . (ii) Find all the possibilities for the orbit closure $\overline{Nx} \subset X$, where N is the upper triangular subgroup of $\mathrm{SL}_2(\mathbb{R}) \subset G$. (You may assume that fact that an N -orbit on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ is either closed or dense; or you may appeal to Ratner's theorem.)
28. Let $\Omega \subset H^1(\Sigma_g, \mathbb{C})$ be the set of complex cohomology classes $[\alpha]$ such that $(i/2) \int \alpha \wedge \bar{\alpha} = 1$.
- (i) Using the natural action of $G = \mathrm{Sp}_{2g}(\mathbb{R})$ on cohomology, show that we may identify Ω with G/H , where $H = \mathrm{Sp}_{2g-2}(\mathbb{R})$.
- (ii) Now specialize to the case $g = 2$, where $H \cong \mathrm{SL}_2(\mathbb{R})$, and let $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$. Let $\Omega_D \subset \Omega$ denote the set of cohomology classes arising as eigenforms for real multiplication by \mathcal{O}_D . Show that Ω_D is Γ -invariant and *closed* in Ω .
- (iii) Using Ratner's theorem, show that $\Gamma \cdot [\alpha]$ is dense in Ω provided $[\alpha]$ is not in $\bigcup_D \Omega_D$.
- (iv) Explain the connection to the fact that ΩE_D is closed in $\Omega \mathcal{M}_2$ and locally defined by linear equations in period coordinates.
29. Let (X, ω) be a completely periodic 1-form that has just one cylinder in each periodidirection. Show that X is a torus.
- *Construct a 1-form of higher genus that has only a single cylinder in each periodic direction.

References

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