1. Seminar - Harvard University, Fall 2014

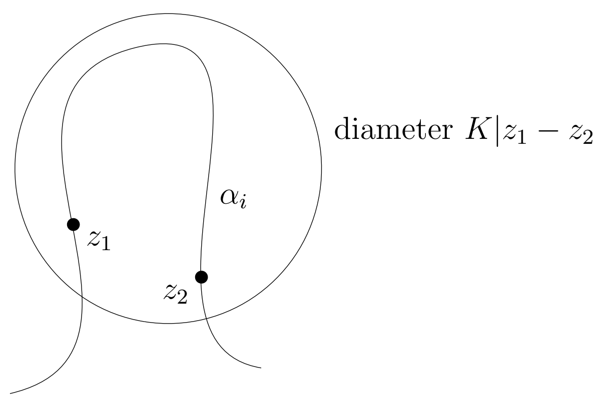
These notes on the Bers emebedding and a bound on renormalized volume for quasifuchsian manifolds are based on the paper *Normalized entropy versus volume for pseudo-Anosovs*, by Kojima and McShane.

2. Quasifuchsian manifolds and simultaneous uniformization

Let S be a smooth orientable surface of negative Euler characteristic. Set $\mathcal{QF}(S)$ to be the space of hyperbolic structures on $S \times \mathbb{R}$ which are quasiconfomally conjugate to a fuchsian structure of finite type. More precisely, if $M = \mathbb{H}^3/\Gamma \in \mathcal{QF}(S)$ then $\Gamma \leq \mathrm{PSL}(2, \mathbb{C})$ can be conjugated to $\Gamma_f \leq \mathrm{PSL}(2, \mathbb{R})$ by some quasiconfomal map f on $\mathbb{H}^3$ such that $\mathbb{H}/\Gamma_f$ is a finitely punctured compact Reimann surface.

The limit set $\Lambda_\Gamma = \bar{\Gamma x} \cap \partial\mathbb{H}^3$ (pick any $x \in \mathbb{H}^3$) is therefore a *quasicircle*, i.e. the image of S^1 under a quasiconformal map. Equivalently, a quasicircle is a Jordan curve C which admits a constant K such that for any $z_1, z_2 \in C$, the two arc $\{\alpha_1, \alpha_2\} = C \setminus \{z_1, z_2\}$ have

$$\min\{\mathrm{diam}\ \alpha_1, \mathrm{diam}\ \ \alpha_2\} \leq K|z_1 - z_2| \quad \text{(this property is called } \textit{bounded turning}\text{)}$$



We can write the domain of discontinuity $\partial\mathbb{H}^3 \setminus \Lambda_\Gamma$ of Γ as $\Omega^X \sqcup \Omega^Y$, where Ω^X has the same orientation as S and Ω^Y the opposite orientation. The spaces $Z^X = \Omega^X/\Gamma$ and $Z^Y = \Omega^Y/\Gamma$ are (marked) projective structures on S and $\bar{S}$, respectively. A *projective structure* on a surface is an atlas of charts to $\hat{\mathbb{C}}$ where the transition functions are restrictions of Möbious transformations. One of the many nice properties of projective structures is that they allow one to talk about *round circles*.

The underlying (marked) hyperbolic structures $X = \mathbb{H}/\Gamma_X, Y = \mathbb{H}^*/\Gamma_Y$ on Z^X, Z^Y can be obtained through uniformization of Ω^X and Ω^Y to $\mathbb{H}$ and $\mathbb{H}^*$ (the lower half plane), respectively. The uniformizing maps $f_X : \mathbb{H} \to \Omega^X$ and $f_Y : \mathbb{H}^* \to \Omega^Y$, called *developing maps*, are univalent and equivariant.

This construction provides us with a map $\mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S})$ by $M \mapsto (X, Y)$, where $\mathcal{T}(S)$, called Teichmüller space, is the space of all hyperbolic structures of finite type on S up to isotopy. We will call X, Y the *conformal structures at infinity* of M. Bers proved an amazing fact about this correspondence:

Theorem 2.1 (Bers Simultaneous Uniformization)**.** *There exists an analytic isomorphism*

$$\mathrm{QF}: \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \to \mathcal{QF}(S)$$

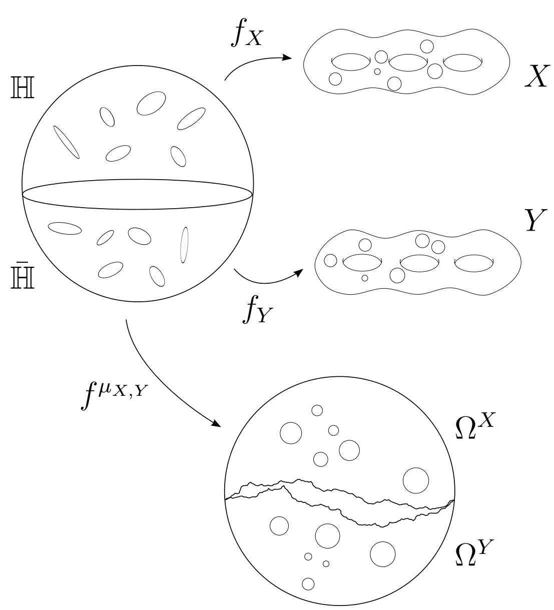
Remark 2.2. The analytic structure on $\mathcal{T}(S)$ arrises from the space of Beltrami differentials, which we will mention below. While the analytic structure on $\mathcal{QF}(S)$ is given by realizing $\mathcal{QF}(S)$ as an open subset of $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C}) /\!\!/ \mathrm{PSL}(2,\mathbb{C})$

To define $\mathrm{QF}(X,Y)$ we first fix a basepoint $X_0 = \mathbb{H}/\Gamma_0 \in \mathcal{T}(S)$. An important fact about Teichmüller space is that there are quasiconformal maps $f_X : X_0 \to X$ and $f_Y : \bar{X}_0 \to Y$ called *Teichmüller maps*, which give rise to Beltrami differentials on X_0 and $\bar{X}_0$. A *Beltrami differential* μ is a differential locally written as $\mu(z)d\bar{z}/dz$ with $\mu(z) \in L^\infty$ and $\|\mu\|_\infty < 1$. One can think of μ as an infinitesimal ellipse field of bounded eccentricity where an ellipse has minor axis tilted at angle $\arg \mu(z)/2$ and eccentricity $(1+|\mu(z)|)/(1-|\mu(z)|)$. A Beltrami differential of a quasiconformal map f is given in local coordinates by

$$\mu_f = \frac{f_{\bar{z}}}{f_z}\frac{d\bar{z}}{dz}$$

The map f will "straighten out" μ_f into an infinitesimal circle field in the image. Notice that f is conformal if and only if $\mu_f \equiv 0$.

It is an amazing fact that the inverse problem can also solved. Given a Beltrami differential μ, one can always find a quasiconformal f^μ such that $\mu = \mu_{f^\mu}$. If μ is invariant under a group on Möbious transformations, then f^μ conjugates it to an action by Möbious transformations on the image.



In our construction, we can take the Beltrami differentials $\mu_{\widetilde{f_X}}$ and $\mu_{\widetilde{f_Y}}$ on $\mathbb{H}$ and $\mathbb{H}^*$ and glue them together to get a Beltrami differential $\mu_{X,Y}$ on $\overline{\mathbb{H} \cup \mathbb{H}^*} = \hat{\mathbb{C}}$. By solving the inverse problem, we obtain a quasiconfomal map $f^{\mu_{X,Y}} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. This map $f^{\mu_{X,Y}}$ conjugates Γ_0 to a quasifuchsian group $\Gamma_{X,Y}$ such that the conformal structures at infinity of $\mathbb{H}^3/\Gamma_{X,Y} = \mathrm{QF}(X,Y)$ are indeed X and Y.

3. Bers embedding and Nehari's Theorem

We can now use the Simultaneous Uniformization Theorem to construct an embedding B_X of $\mathcal{T}(\bar{S})$ into the Banach space space of integrable holomorphic quadratic differentials $Q(X)$ over a fixed Riemann surface $X \in \mathcal{T}(S)$.

Let X be the basepoint in the construction of $\mathrm{QF}(\cdot,\cdot)$, so $f_X = \mathrm{id}$ and $\mu_{\widetilde{f_X}} \equiv 0$. It follows that for ever $Y \in \mathcal{T}(\bar{S})$, the map $f^{\mu_{X,Y}} |_{\mathbb{H}}$ is conformal and corresponds to the developing map of Z^X. To get a quadratic differential from $f^{\mu_{X,Y}} |_{\mathbb{H}}$ we use the Schwarzian derivative.

The Schwarzian derivative of a locally injective holomorphic map $f : U \to \hat{\mathbb{C}}$ is given by

$$S(f) = \left(\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right) dz^2$$

and has two key properties

- $S(A) \equiv 0$ if and only if A is a restriction of a Möbious transformation.
- $S(f \circ g) = g^* S(f) + S(g)$ where g^* pulls back the quadratic differential $S(f)$

In particular $S(A \circ f) = S(f)$ and $S(f \circ A) = A^* S(f)$ for all Möbious transformations A. Note that for a point $p \in U$, we can always find A_p such that $(A_p \circ f)'(p) = 1$ and $(A_p \circ f)''(p) = 0$, giving us that $S(f)(p) = S(A_p \circ f)(p) = (A_p \circ f)'''(p) dz^2$.

The Schwarzian derivative $S(f^{\mu_{X,Y}} |_{\mathbb{H}})$ gives a holomorphic quadratic differential on $\mathbb{H}$ which is invariant under the action of Γ_X. Taking the quotient by this action, we get an integrable homomorphic quadratic differential $B_X(Y) = S(f^{\mu_{X,Y}} |_{\mathbb{H}})/\Gamma_X$ on X for every $Y \in \mathcal{T}(\bar{S})$. The map $B_X : \mathcal{T}(\bar{S}) \to Q(X)$ is called the *Bers embedding.*

Remark 3.1. One can see that B_X is injective by showing that for any $q \in Q(X)$ we can find a locally injective $f : \mathbb{H} \to \hat{\mathbb{C}}$ with $S(f) = \widetilde{q}$ as follows. Let $u_1(z), u_2(z)$ form a basis for the space of holomorphic solutions to the differential equation

$$u''(z) + \frac{1}{2} \widetilde{q}(z) u(z) = 0$$

then $f(z) = u_1(z)/u_2(z)$ is locally injective and satisfies $S(f) = \widetilde{q}$. Checking that f is locally injective and well defined up to the action of $\mathrm{PSL}(2,\mathbb{C})$ is a good exercise.

To study the image of B_X, let us consider the L^∞ norm on $Q(X)$ given by

$$\|q\|_\infty = \sup_{x \in X} \frac{|q|}{\rho^2}(x)$$

where ρ is the hyperbolic metric on X.

Theorem 3.2 (Nehari)**.** *The image of B_X is contained in the closed ball $\overline{B}(0, 3/2)$ in $Q^\infty(X)$.*

Proof Sketch. Since B_X is constructed through the *univalent* map $f^{\mu_{X,Y}} |_{\mathbb{H}}$ it is enough to prove that $\|S(f)\|_\infty \leq 3/2$ for any univalent map f on $\mathbb{H}$. Further, by the invariance of $\|S(f)\|_\infty$ under Möbious transformations and their transitive action on $\mathbb{H}$, we only need to check the inequality at one point for a conjugate of f.

Let the conjugate of f be $F : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}}$ with $F(\infty) = \infty$, $F'(\infty) = 1$ and $F''(\infty) = 0$ and the point of interest ∞. Then

$$F(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

Claim 3.3 (Area Theorem). *For $F(z)$ as above, $|a_1|^2 + 2|a_2|^2 + \cdots \leq 1$. In particular, $|a_1| \leq 1$.*

As mentioned before, the Schwarzian for F will be

$$S(F)(z) = F'''(z)dz^2 = \left(\frac{6a_1}{z^4} + o\left(\frac{1}{z^4}\right)\right) dz^2$$

Recall that the hyperbolic metric on $\hat{\mathbb{C}} \setminus \mathbb{D}$ is given by $\rho(z) = 2|dz|/(|z|^2 - 1)$. Therefore,

$$\frac{|S(F)|}{\rho^2}(\infty) = \frac{6}{4}|a_1| \leq \frac{3}{2}$$

□

Exercise 3.4. Let $C_r = \{x \in \mathbb{C} \mid |z| = r\}$ and $F : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}}$ be a univalent function with $F(\infty) = \infty$. Call $D_{F,r}$ the region enclosed by $F(C_r)$ and prove that

$$\text{Area}(D_{F,r}) = \frac{1}{2i} \oint_{F(C_r)} \bar{z} dz$$

Use this fact to prove the area theorem above.

Remark 3.5. In more generality, a univalent functions of the form $f(z) = z + a_2 z^2 + \cdots$ of the unit disk is a called *schlicht function*. Note that $f(0) = 0$ and $f'(0) = 1$. We just saw that the space of schlicht functions maps to a compact region in $Q^\infty(X)$, so for each n there exists C_n with $|a_n| \leq C_n$ for all such f. The Bieberbach conjecture, now a theorem of de Branges, states that $|a_n| \leq n$ over the space of schlicht functions.

Remark 3.6. A theorem of Ahlfors-Weill proves that the image of B_X *contains* the ball $B(0, 1/2)$ in $Q^\infty(X)$. The "extremal" cases of these two theorem are interesting. A conformal map with $\|S(f)\|_\infty = \frac{1}{2}$ is one that maps $\mathbb{H}$ to the strip $\{z \in \mathbb{C} \mid 0 < \text{Im } x < 1\}$. On the other end, we have $f(z) = z^2$ which has $\|S(f)\|_\infty = 3/2$ and maps $\mathbb{H}$ to $\mathbb{C} \setminus \mathbb{R}^{\geq 0}$.

Using Nehari's bound, we can obtain a bound for the L^1 norm on $Q(X)$ given by

$$\|q\|_1 = \int_X |q|$$

Corollary 3.7. *The image of B_X is contained in a closed ball $\overline{B}(0, 3\pi|\chi(S)|)$ in $Q^1(X)$.*

Proof Sketch.

$$\|q\|_1 = \int_X |q| = \int_X \frac{|q|}{\rho^2}\rho^2 \leq \|q\|_\infty \int \rho^2 = \|q\|_\infty \cdot 2\pi|\chi(S)| \leq 3\pi|\chi(S)|$$

□

4. Bounds on renormalized volume

Recall that for any Riemannian metric I^* in the conformal class of $X \sqcup Y = \partial_\infty M$, there are is a foliation of the ends of M by smooth parallel surfaces S_r (bounding convex submanifolds N_r and embedded for r large enough) with induced Riemannian metrics I_r such that $\lim_{r\to\infty} e^{-2r}\mathrm{I}_r/4 = \mathrm{I}^*$. If we let I^* be the hyperbolic metric on $X \sqcup Y$, then the renomarlized volume of M is given by

$$V_R(M) = \mathrm{Vol}(N_r) - \frac{1}{4}\int_{S_r} H_r dA - \pi r|\chi \partial M|$$

where H_r is the sum of the principle curvatures of S_r. Martin Bridgeman showed in the previous seminar that $V_R(M)$ is independent of r.

One can consider the behavior of $V_R(M)$ under non-trivial deformations of I^* among hyperbolic metrics on $S \sqcup \bar{S}$ (corresponding to deformations of M under simultaneous uniformization). We can think of $\dot{\mathrm{I}}^*$ as a Beltrami differential which produces a non-trivial quasiconformal deformation of the hyperbolic metric. Krasnov and Schlenker prove the following remarkable variation formula:

Theorem 4.1 (Variation Formula)**.** *For $M \in \mathcal{QF}(S)$ with I^* the hyperbolic metric on $X \sqcup Y = \partial_\infty M$ and q the holomorphic quadratic differential associated to the projective structure $Z^X \sqcup Z^Y$ one has*

$$dV_R(M) = \frac{1}{4}\langle Re\, q, \dot{\mathrm{I}}^*\rangle$$

The inner product $\langle \cdot, \cdot \rangle$ can be interpreted in two, as it happens equivalent, ways.

- Re q and $\dot{\mathrm{I}}^*$ can be viewed as sections of a tensor bundle on $S \sqcup \bar{S}$ and one can extend the Riemannian inner product $\langle \cdot, \cdot \rangle$ to tensors via

$$\langle \alpha_1 \otimes \alpha_2, \alpha_1 \otimes \alpha_2 \rangle = \langle \alpha_1, \alpha_1 \rangle \langle \alpha_2, \alpha_2 \rangle$$

  On skew-symmetric tensors $\alpha_1 \wedge \alpha_2 = (\alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1)/\sqrt{2}$, this looks like the natural $\det[\langle \alpha_i, \alpha_j \rangle]$.
- For a quadratic differential q and Beltrami differential μ where is a natural pairing $\langle q, \mu \rangle = \int_X q\mu$, so we interpret

$$\langle \mathrm{Re}\, q, \dot{\mathrm{I}}^* \rangle = \int_X \mathrm{Re}\, q\, \dot{\mathrm{I}}^*$$

Remark 4.2. There are two remarks worth making. We will see later in this seminar that $-\mathrm{Re}\, q = \mathrm{II}^*_0$, the trace free part of the second fundamental form II^* of I^*, and that the variation formula can be stated in terms of II^*_0 in a more general setting. Secondly, when we say that $\dot{\mathrm{I}}^*$ is a non-trivial Beltrami differential, we really mean that $\dot{\mathrm{I}}^* \in Q^\perp(X) = \{\mu \mid \langle q, \mu \rangle \neq 0 \text{ for all } q \in Q(X) \setminus \{0\}\}$.

We can now use the variation formula and our bounds on the Bers embedding to obtain a bound on $V_R(M)$.

Theorem 4.3. *For* $M = \mathrm{QF}(X, Y)$

$$V_R(M) \leq \frac{3}{4}\pi|\chi(S)|d_{\mathcal{T}}(X, \bar{Y})$$

where $d_{\mathcal{T}}$ *is Teichmüller distance (given as log of the quasiconformal constant of the Teichmüller map between* X *and* $\bar{Y}$*).*

Proof Sketch. Let $d = d_{\mathcal{T}}(X, \bar{Y})$ and X_t be a unit speed Teichmüller geodesic with $X_0 = \bar{Y}$ and $X_d = X$. Set $M_t = \mathrm{QF}(X_t, Y)$, $\mathrm{I}^*_{X_t} \sqcup \mathrm{I}^*_Y$ the hyperbolic metric, and $q_{X_t} \sqcup q_Y$ the holomorphic quadric differential in $Q(X_t \sqcup Y)$. Note that $q_{X(t)} = B_{X_t}(Y)$ lies in $Q(X_t)$ and that $\dot{\mathrm{I}}^*_Y = 0$ by construction. The variation formula gives us

$$\frac{d}{dt}V_R(M_t) = \frac{1}{4}\langle \mathrm{Re}\, q_{X_t}, \dot{\mathrm{I}}^*_{X_t}\rangle + \frac{1}{4}\langle \mathrm{Re}\, q_Y, \dot{\mathrm{I}}^*_Y\rangle = \frac{1}{4}\langle \mathrm{Re}\, q_{X_t}, \dot{\mathrm{I}}^*_{X_t}\rangle =$$

$$= \int \mathrm{Re}\, q_{X_t}\, \dot{\mathrm{I}}^*_{X_t} \leq \frac{1}{4}\|q_{X_t}\|_1 \|\dot{\mathrm{I}}^*_{X_t}\|_\infty \leq \frac{3}{4}\pi|\chi(S)|$$

by Corollary 3.7 and the fact that $\|\dot{\mathrm{I}}^*_{X_t}\|_\infty \leq 1$, since $\dot{\mathrm{I}}^*_{X_t}$ is a Beltrami differential of a quasiconformal map. Integrating the inequality along X_t given the desired result.

□

Remark 4.4. Applying this bound to the mapping torus N_ϕ of a pseudo-Anosov ϕ as is done by Kojima and McShane, one gets a bound

$$\frac{8}{3}\,\mathrm{Vol}(N_\phi) \leq 2\pi|\chi(S)|\,\mathrm{ent}\,\phi$$

For the figure 8 knot complement, one gets that

$$\frac{8}{3}\,\mathrm{Vol}(N_\phi) = \frac{16}{3}v_3 \approx 5.413013$$

and

$$2\pi|\chi(S)|\,\mathrm{ent}\,\phi = 2\pi \log\frac{3+\sqrt{5}}{2} \approx 6.047086$$