Seminar: New Results on Hyperbolic Volumes

Harvard University
Lecture 6 — C. McMullen
19 November 2014

These are notes related to the paper *Normalized entropy versus volume for pseudo-Anosovs*, by Kojima and McShane.

1   Bounds for volume: summary

Let S be a fixed, oriented surface of type (g, n). Let $\psi \in \mathrm{Mod}(S)$ be a pseudo–Anosov mapping class, let M_ψ be the associated hyperbolic 3-manifold of finite volume, and let $T_\psi \in AH(S)$ be the natural $\mathbb{Z}$-covering space of M_ψ.

Given $Y, Z \in \mathrm{Teich}(S)$, we let

$$Q(Y, Z) \in \mathrm{QF}(S)$$

denote the quasifuchsian group with conformal boundary $Y \cup \overline{Z}$, compatible with markings. Thus $Q(Y, Y)$ is a Fuchsian group. Note that the boundary component $\overline{Z}$ is naturally marked by S with its orientation reversed.

Now fix $X \in \mathrm{Teich}(S)$ on the Teichmüller geodesic stabilized by ψ, and let

$$Q_{a,b} = Q(\psi^{-a}X, \psi^b X) \in \mathrm{QF}(S).$$

The most basic result is the whole discussion is:

Theorem 1.1 (Thurston) *As $a, b \to \infty$ we have*

$$Q_{a,b} \to T_\psi$$

in the algebraic topology on $AH(S)$. Moreover $Q_{a,b} \to T_\psi$ geometrically.

We then have

$$M_\psi = T_\psi / \mathbb{Z}.$$

The volume estimate for M_ψ involves 3 results. Let $N(X, Y)$ denote the convex core of $Q(X, Y)$, let vol denote hyperbolic volume, and let Rvol denote renormalized volume.

1. For all $Q(X, Y) \in \mathrm{Teich}(S)$, we have

$$\mathrm{Rvol}\, Q(X, Y) = \mathrm{vol}\, N(X, Y) + O(1).$$

2. If X_t is a smooth path in $\mathrm{Teich}(S)$, then

$$\frac{d}{dt}\,\mathrm{Rvol}\,Q(X_t, Y) \le C_0 \left| \int_{X_t} q_t \mu_t \right|,$$

where μ_t represents dX_t/dt and $q_t \in Q(X)$ is the quadratic differential coming from the complex projective structure on X_t determined by $Q(X_t, Y)$.

3. We have

$$\mathrm{vol}\,M_\psi = \lim_{a\to\infty} \frac{\mathrm{vol\,core}\,Q_{a,a}}{2a} = \lim_{a\to\infty} \frac{\mathrm{vol\,core}\,Q_{a,0}}{a}.$$

Now by a basic result in conformal mapping, due to Nehari, we have $\|Sf\|_\infty \le 3/2$ for any univalent map $f : \mathbb{H} \to \widehat{\mathbb{C}}$, where

$$\|q\|_\infty = \sup_{\mathbb{H}} \frac{|q|}{\rho^2},$$

and ρ^2 is the hyperbolic area element (constant curvature -1).

Returning to a finite volume Riemann surface X, we also have L^p normals on $Q(X)$, given by

$$\|q\|_\infty = \sup_X \rho^{-2}|q|, \quad \text{and}$$

$$\|q\|_p = \left(\int_X \rho^{2-2p}|q|^p \right)^{1/p}.$$

This gives immediately

$$\|q\|_p \le \|q\|_\infty \,\mathrm{area}(X)^{1/p}.$$

The case $p = 1$ corresponds to the Teichmüller metric, and the case $p = 2$ corresponds to the Weil–Petersson metric. Letting $q = d(\mathrm{Rvol})$, the results above show

$$\|d(\mathrm{Rvol})\|_p \le (3/2)C_0 \,\mathrm{area}(X)^{1/p}.$$

Combining these results gives:

Theorem 1.2 *The volume of the mapping–cylinder for ψ satisfies*

$$\mathrm{vol}(M_\psi) \le (3/2)C_0 \,\mathrm{area}(X)^{1/p}\,\mathrm{length}_p(\psi),$$

where $\mathrm{length}_p(\psi)$ is the minimal length, in the L^p–metric, of a loop representing $[\psi] \in \pi_1(\mathrm{Teich}(S)/\mathrm{Mod}(S))$.

Note that $\text{area}(X) = 2\pi|\chi(S)|$ by Gauss–Bonnet. The surface under consideration corresponds to *one end* of $Q_{a,0}$.

Improvement with p. The quantity

$$N(p) = \text{area}(X)^{-1/p}\|q\|_p$$

is simply the L^p norm of the *function* $|q|/\rho^2$ on the space X with respect to the *probability measure* $\rho^2/\text{area}(X)$. By Jensen's inequality, for any function f on a probability space, $\|f\|_p$ is weakly increasing as a function of p; that is, since $x^{1+\epsilon}$ is a convex function, we have

$$\left(\int |f|^p\right)^{1+\epsilon} \leq \int |f|^{p(1+\epsilon)},$$

which shows $\|f\|_p = 1 \implies \|f\|_{p(1+\epsilon)} > 1$. It follows that $N(p)$ is increasing and the function

$$N^*(p) = \text{length}_p(\psi)\,\text{area}(X)^{1/p}$$

is decreasing as a function of p. Thus the bound in Theorem **??** *improves* as $p \to \infty$, and the sharpest bound comes from $p = \infty$.

The Teichmüller metric. For $p = 1$ the length is given by the entropy $h(\psi)$, so we get

$$\text{vol}(M_\psi) \leq (3/2)C_0\,\text{area}(X)\,\text{length}_1(\psi) = (3/2)C_0\,\text{area}(X)h(\psi).$$

The Weil–Petersson metric. For $p = 2$ we have the Weil–Petersson length, and we get

$$\text{vol}(M_\psi) \leq (3/2)C_0\,\text{area}(X)^{1/2}\,\text{length}_2(\psi).$$

Note that this bound formally implies the one above, since by (**??**) we have

$$\text{length}_2(\psi) \leq \text{area}(X)^{1/2}\,\text{length}_1(\psi).$$

Cf. our first set of notes, where we wrote:

> **Exercise.** *Let $f_n \in \text{Mod}_{1,1}$ be the mapping-class for the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. Prove that $h(f_n) \to \infty$, but that the volume of the mapping cylinder, $\text{vol}(T_{f_n})$, is bounded independent of n (in fact it converges to a finite limit).*
>
> *This suggests that there might be an improvement to the bound on $\text{vol}(T_f)$ in terms of Weil–Petersson distance, still with a constant proportional to $|\chi(\Sigma)|$.*

Problem. Find practical estimates of the 'WP length' of a mapping–class, analogous to extimates of its entropy; in particular, estimates or formulas that work across a fibered face.

The L^∞–metric. The best bound is given by

$$\mathrm{vol}(M_\psi) \leq (3/2)C_0 \, \mathrm{length}_\infty(\psi).$$

The L^∞-norm on $Q(X)$ is natural from the perspective of Bers' embedding, but its metric structure on $\mathrm{Teich}(S)$ does not seem to have been studied.

2 Calculations

For the figure–eight knot we have $\psi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathrm{length}_1(\psi) = h(\psi) = \log\left(\frac{3+\sqrt{5}}{2}\right)$. A very recent calculation by Dumas gives values of $\mathrm{length}_p(\psi)$ with $p = 2$ and ∞. Altogether we have

$$
\begin{aligned}
\mathrm{length}_1(\psi) &\approx 0.9624236501, \\
\mathrm{length}_2(\psi) &\leq 1.86068 \quad \text{and} \\
\mathrm{length}_\infty(\psi) &\leq 2.80859.
\end{aligned}
$$

He also computed that the hyperbolic metric on the square torus $\mathbb{C}/\mathbb{Z}[i]$ is given by $\rho = 1.6944|dz|$ at the midpoint $z = (1+i)/2$. This implies that $\|q\|_1/\|q\|_\infty \approx 0.3483$ on the square torus. See [**?**] for the exact value of this norm.

The lengths bounds above were obtained by integrating the Weil–Petersson and L^∞ metrics over the *Teichmüller* geodesic for ψ, so the true values is definitely smaller. Experiments by Dumas show, however, they are not much smaller — curve shortening leads to length within 1% or so of those above.

Let us see what estimates these give for volume, assuming $C_0 = 1$. Since $\mathrm{area}(X) = 2\pi$, we get: They give:

$$
\begin{aligned}
\mathrm{vol}(M_\psi) &\leq (3/2)\,\mathrm{area}(X)\,\mathrm{length}_1(\psi) \approx 9.07063, \\
\mathrm{vol}(M_\psi) &\leq (3/2)\,\mathrm{area}(X)^{1/2}\,\mathrm{length}_2(\psi) \approx 6.99605, \quad \text{and} \\
\mathrm{vol}(M_\psi) &\leq (3/2)\,\mathrm{length}_\infty(\psi) \approx 4.21289.
\end{aligned}
$$

Thus passing to the Weil–Petersson metric improves (reduces) the volume upper bound by at least 22%, while passing to the L^∞ metric improves the upper bound by more than 50%.

The actual value of the volume is given by

$$\operatorname{vol}(M_\psi) = 2v_1 \approx 2.02988.$$

Working backwards, we have

$$0.481826 \le \frac{\operatorname{vol}(M_\psi)}{(3/2)\operatorname{length}_\infty(X)} \le C_0.$$

On the other hand the existing papers, such as [?, Prop 5.3], seem to require $C_0 = 1/4$. However the same paper has a factor of 4 missing in Definition 3.2, and uses the value of 6 instead of $3/2$ for Nehari's bound. The second fact should not affect the value of C_0, but the first might.

The bounds above are also compatible with $C_0 = 1/2$ and would be quite close to sharp, when one uses the L^∞ norm.

3 Volume growth of the convex core

In this section we will prove:

Theorem 3.1 *The hyperbolic volume of the mapping torus M_ψ satisfies*

$$\operatorname{vol}(M_\psi) \le \liminf \frac{\operatorname{vol}\operatorname{core} Q(\psi^{-n}X, \psi^n X)}{2n}.$$

The proof aims to use only a few well–known facts, and could easily be carried further to show equality holds above. A more detailed and technical argument, following e.g. [?], would yield an error term of size $O(1/n)$.

Geometric limits. The most important fact we will need is the following. As before let

$$Q_{a,b} = Q(\psi^{-a}X, \psi^b X)$$

as an element of $AH(S)$. I.e. $Q_{a,b}$ is a hyperbolic 3-manifold with marked fundamental group. Then we can extend the defintion to allow $a = \infty$, $b = \infty$ or both in such a way that

$$Q_{\infty,\infty} = T_\psi$$

is the fiber manifold, $Q_{a,\infty}$ and $Q_{\infty,b}$ are 1-sided limits in Bers' slices, and

$$\lim Q_{a,b} = Q_{\lim a, \lim b}$$

both algebraically *and* geometrically.

Let us recall that in $AH(S)$, geometric convergence $M_n \to M$ means the following: for any compact, connect submanifold

$$K \cong S \times I \subset M,$$

and for all n sufficiently large, there exist smooth embeddings

$$f_n : K \to M_n,$$

compatible with markings, such that f_n converges smoothly to an isometry. The precise meaning is that the pulled back hyperbolic metrics converge smoothly (C^∞) to the metric on K induced from M.

Approximate fibers. We now fix, for reference, a *fiber*

$$S_{\infty,\infty} \subset Q_{\infty,\infty} \cong T_\psi.$$

This means $S_{\infty,\infty}$ is a smooth, compact embedded surface that maps injectively into $M_\psi = T_\psi/\mathbb{Z}$ and forms a fiber of the natural fibration $M_\psi \to S^1$. Note that $S_{\infty,\infty}$ is naturally marked by S.

We say

$$S_{a,b} \subset Q_{a,b}$$

is an *approximate fiber* if there is a diffeomorphism

$$f_{a,b} : S_{\infty,\infty} \to S_{a,b}$$

that extends to a nearly isometric map between small neighborhood of the domain and range. By 'nearly isometric' we mean that f is C^∞ close to an isometry in the previous sense. We package this information in a constant $\epsilon > 0$ and say $S_{a,b}$ is a $(1+\epsilon)$–approximate fiber. In particular we require that $f_{a,b}$ is $(1+\epsilon)$ bilipschitz.

Parallel fibers. Note that the manifolds $Q_{a,b}$ and $Q_{a+1,b-1}$ are isometric. We can use this isometry to transfer an approximate fiber $S_{a+1,b-1}$ to a surface $S'_{a+1,b-1} \subset Q_{a,b}$ which we call a *parallel fiber*.

Existence of approximate fibers. Using geometric limits, it is straightforward to establish the following result.

Theorem 3.2 *Given any $\epsilon > 0$, there exists an $N, M > 0$ such that for all $a, b \geq N$:*

   1. *The convex core of $Q_{a,b}$ contains an ϵ-approximate fiber $S_{a,b}$;*

2. Any two parallel ϵ-approximate fibers bound a submanifold of $Q_{a,b}$ diffeomorphic to $S \times I$ and of volume within ϵ of $\mathrm{vol}(M_\psi)$; and

The idea of the proof is simply that (i) geometric limits provide near–isometries from $S_{\infty,\infty}$ to $S_{a,b} \subset Q_{a,b}$ for all $a, b \gg 0$; (ii) any approximate fiber is close to one constructed in this way, since the holonomy of $(S_{a,b}, p)$ is nearly determined; and (iii) any parallel to $S_{a,b}$ converges geometrically to $\Psi(S_\infty, \infty)$, where $\Psi : T_\psi \to T_\psi$ is the generator of the deck group of T_ψ/M_ψ.

The volume estimate. The main result now easily follows. Suppose $a, b \geq N$. Consider any $i, j \geq N$ with $i + j = a + b$. Then $Q_{i,j}$ contains an approximate fiber $S_{i,j}$ which we can transport by isometry to a surface $S'_{i,j} \subset \mathrm{core}\, Q_{a,b}$. The regions between adjacent surfaces give $(a + b - 2N)$ essentially disjoint copies of $S \times I$, each of volume within ϵ of $\mathrm{vol}(M_\psi)$. All these regions lie inside the convex core as well. This shows:

$$\mathrm{vol\, core}\, Q_{a,b} \geq (a + b - 2N)(\mathrm{vol}\, M_\psi - \epsilon),$$

and hence

$$\liminf \frac{\mathrm{vol\, core}\, Q_{a,b}}{a + b} \geq \mathrm{vol}\, M_\psi - \epsilon$$

for any $\epsilon > 0$. The reverse bound is similar, using the fact that the part of the convex core that is not between a pair of $S'_{i,j}$ has volume $O(1)$.