Seminar
Harvard University — Fall 2014
Notes

These are notes related to the paper *Normalized entropy versus volume for pseudo-Anosovs*, by Kojima and McShane.

1 Differential geometry.

In $\mathbb{H}$, the circular arc centered at $z = 0$ has the natural, unit speed parameterization

$$\gamma(s) = \tanh(s) + i \sinh(s)$$

for $s \in \mathbb{R}$. Note that $\gamma(0) = i$. Since the perpediculars to the geodesic $x = 0$ are all circular arcs of this form, and since $\rho = 1/y$, we find:

Theorem 1.1 *The parallels at distance r from a geodesic or hyperplane expand by a factor of* $\cosh(r)$*.*

Theorem 1.2 *The geodesic curvature of a parallel to a geodesic at distance r is* $\tanh(r)$*.*

Proof. Consider an annulus $A(r)$ obtained as the tube of points at distance r from a core curve of length 1. Then by Gauss–Bonnet,

$$\text{area}(A(r)) = \int_{\partial A(r)} k\, ds = 2k(r)\cosh(r)$$

by the previous calculation. But the same calculation shows the area of $A(r)$ is $2\sinh(r) = \int_{-r}^{r} \cosh(s)\, ds$. Thus $k(r) = \tanh(r)$. ■

The Fuchsian case. In the Fuchsian case with metrics $\rho_r(z)|dz|$ on the surfaces of distance r from the totally geodesic slice, we have

$$\rho_r = \cosh(r)\rho_0.$$

The geodesic curvature of a parallel at distance r is

$$k(r) = \tanh(r),$$

and thus the mean curvature is $H_r = \tanh(r)$. The area of the parallel surface is

$$A_r = A_0 \cosh^2(r)\ A_0 e^{2r}/4.$$

We then get the volume

$$V_r = \int_0^r A_0 \cosh^2(s) ds = A_0(r + \cosh(r)\sinh(r))/2,$$

and hence

$$W_r = V_r - (1/2)A_r H_r = A_0[r/2 + \cosh(r)\sinh(r)/2 - \tanh(r)\cosh^2(r)/2] = A_0 r/2.$$

Typos. In Schlenker, the Nehari bound should be 3/2, not 6. The definition of the mean curvature H is taken to be $k_1 + k_2$, where $(k_1 + k_2)/2$ would be more usual. The renormalized metric at infinity should be $I^* = \lim 4e^{-2r} I_r = \lim(1/\cosh^2(r)) I_r$.

Warning: the surface associated to a metric by taking the envelope of horoballs is not always *embedded*.

The figure 8 knot. Let $f \in \mathrm{Mod}(\Sigma_f)$ be a pseudo-Anosov mapping–class with mapping–torus T_f. Let $\gamma_f \subset \mathrm{M}_{g,n}$ be the corresponding closed Teichmüller geodesic. The main result of Kojima–McShane is that volume is bounded by length times area:

$$\mathrm{vol}(T_f) \le C \cdot L(\gamma_f)\,\mathrm{area}(\Sigma_f),$$

with an explicit constant C (perhaps $C = 3/2$). Here $\mathrm{area}(\Sigma_f) = 2\pi|\chi(\Sigma_f)| = 2\pi(2g - 2 + n)$.

For the figure 8 knot, all the quantities are known. by $L(\gamma_f)\,\mathrm{area}(\Sigma_f)$. For the figure 8 knot, all the quantities are known. Namely, if

$$v_3 = 1.01494\ldots$$

is the volume of a regular ideal tetrahedron, then

$$\mathrm{vol}(M) = 2v_3 = 2.02988\ldots,$$

while

$$\mathrm{area}(\Sigma_f) = 2\pi = 6.2831\ldots$$

and

$$L(\gamma_f) = \log\lambda = 0.96242\ldots$$

where

$$\lambda = (3 + \sqrt{5})/2 = 2.61803\ldots,$$

which gives

$$L(\gamma_f)\,\mathrm{area}(\Sigma_f) = 6.04709\ldots$$

The bound in KM is (3/2) times this, so there is a large gap, but the figure of (3/2) is subject to revision.

Exercise. Let $f_n \in \mathrm{Mod}_{1,1}$ be the mapping-class for the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Prove that $h(f_n) \to \infty$, but that the volume of the mapping cylinder, $\mathrm{vol}(T_{f_n})$, is bounded independent of n (in fact it converges to a finite limit).

This suggests that there might be an improvement to the bound on $\mathrm{vol}(T_f)$ in terms of Weil–Petersson distance, still with a constant proportional to $|\chi(\Sigma)|$.