Jt. Work with K. Bromberg.

Inflexibility & volumes of $M^3 \to S^1$. (new proof of hyperbolization)

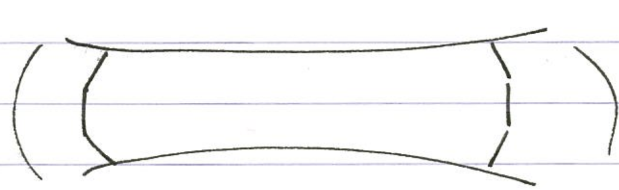
- INFLEXIBILITY idea that effect of uniform q.c. deformation decays exponentially with depth in the convex core.
- VOLUME Relation between volumes of $M^3 \to S^1$ or convex cores of quasi Fuchsian manifolds and Teichmüller or W.P. geometry.

IDEA: By keeping control over the uniformization process can refine estimates on volume & Teichmüller geometry.

Uniformization

THURSTON: $M_\psi = S \times [0,1] / (x,0) \sim (\psi(x),1)$ is hyperbolic iff ψ is pseudo-Anosov.

Idea: Analyze $Q(\psi^{-n}X, \psi^{n}X) \to$?



issues: inj radius? limit set? Geometric limit? width of core?

Modern tools make these issues less difficult.

$$\mathrm{Vol}\big(\underbrace{\mathrm{core}(\underbrace{Q(\psi^{-n}X, \psi^{n}X)}_{Q_n})}_{C_n}\big) \longrightarrow \le \underset{\downarrow \atop 3\sqrt{\pi(g-1)}}{K_S} \cdot d_{WP}(\psi^{-n}X, \psi^{n}X) + \ldots$$

$V(C_n)$

In fact more known:

$$\frac{d_{wp}(X,Y)}{K_s} - K_s' \le \mathrm{Vol}(X,Y) \le K_s\, d_{wp}(X,Y) + K_s'$$

(via geometric convergence $Q_n \to Q_\infty$?)

$\longrightarrow$ direct use

$$\text{get } \frac{\|\psi\|_{wp}}{K_s} \le \mathrm{Vol}(M_\psi) \le K_s \|\psi\|_{wp} \qquad \le K_s \sqrt{2\pi|\chi(S)|}\, \|$$

$\uparrow$ Michelle Linch?

Idea for today:

Use inflexibility to show $\exists\, K(\psi, X, S)$ s.t.

Thm (B)

$$|\mathrm{Vol}(\psi^{-n}X, \psi^{n}X) - 2n\,\mathrm{Vol}(M_\psi)| < K$$

Note that this gives a direct proof then from the quasi-Fuchsian estimate that

$$2n\,\mathrm{Vol}(M_\psi) \le \mathrm{Vol}(\psi^{-n}X, \psi^{n}X) + K \le K_s d_{wp}(\psi^{-n}X, \psi^{n}X) + K$$

$$\Rightarrow \mathrm{Vol}(M_\psi) \le K_s \cdot \|\psi\|_{wp}$$ — with Schlenker $K_s = 3\sqrt{\pi(g-1)}$

Ingredients:

- Harmonic deformation thy. "Harmonic Strain Fields" — INFLEX.
- Thickness of Convex Core and N-S dynamics of ψ.

The heart of the matter:

- To prove uniformization: find a compact subset in $Q(\psi^{-n}X, \psi^{n}X)$ and follow it along.
- Use depth in Convex core and inflexibility to show it "freezes" exponentially fast

North South dynamics of ψ on $PML(S)$.

- ψ has an attracting and repelling fixed point. $[\mu^+]$ $[\mu^-]$
- There is a K_ψ so that $\forall B, D>0$ we have $U, V \subset PML$
  i) $[\mu^-] \in U$ $[\mu^+] \in V$ so that $\alpha \in V$ $\beta \in U$ $\Rightarrow$

$$d_C(\alpha, \psi^n(\beta)) \geq K_\psi n + B.$$

  ii) $\exists W \subset PML \setminus U \cup V$ so that any subpath in $C(S)$ from α, β contains a subpath of length $> D$
  iii) $\forall \alpha \in U$ $\beta \in V$ $\gamma \in W$

$$d_C(\gamma, \psi^{-n}(\alpha)) \geq K_\psi n + B$$
$$d_C(\gamma, \psi^n(\beta)) \geq K_\psi n + B$$

Argument uses fact that can choose U, V so that any α, β in U, V necessarily satisfy $i(\alpha, \beta) > 0$

To show that we have wide cores we use the following:

<u>Thm</u> Given $L>0$ $\exists K_1, K_2 > 0$ so that (S-closed) $\alpha, \beta \in C^0(S)$ $M \in AH(S)$ we have $l_M(\alpha) < L$ $l_M(\beta) < L$ then

$$d_M(\alpha^*, \beta^*) \geq K_1 d_C(\alpha, \beta) - K_2$$

where α^* β^* are replaced by Margulis tubes if $l_M(\alpha^*) < \epsilon_3$.

- Proof - an interpolation argument. Using Simplicial hyp. Sfcs!

IDEA:

A D-coarse path in $\mathcal{C}(S)$ is a sequence $\alpha_i \in \mathcal{C}(S)$ s.t. $d_C(\alpha_i, \alpha_{i+1}) \le D$.

- Then given L, $\exists D, R$ so that if $\alpha, \beta \in \mathcal{C}(S)$ satisfy $l_M(\alpha), l_M(\beta) \le L$ then if Γ is a path s.t. Γ joins α^*, β^* then $\exists \{\alpha_i\}$ w. $l_M(\alpha_i) < L_g$ and $d_M(\alpha_i, \Gamma) < R$ and $\{\alpha_i\}$ a D-coarse path.

STANDARD INTERPOLATION ARG.

So to obtain the theorem:

- Let Γ be a geodesic joining α^*, β^*.
- Count curves in $\{\alpha_i\}$ that can lie near a unit length segment of Γ.

Given S, closed, there is a linear function f st. dist between $\delta^{\pm} \mathcal{C}(Q(X,Y))$ is bdd below by $f(d_C(X,Y))$
lim img in

$\Rightarrow$ width of core of $Q(\Psi^{-n}X, \Psi^{n}X) \ge f(K_\Psi n + B)$

INFLEXIBILITY

The basic tool is an indepth analysis of

Thm: (Reimann) Let $\Psi : M_0 \to M_1$ be a "K-quasiconformal deformation" of M_0. Then there is a 1-parameter family $(M_t) = (M, g_t)$ $t \in [0,1]$ of hyp. 3-manifolds w/ time t derivatives η_t st.

(i) The η_t are harmonic strain fields with

$\|\eta_t\|_\infty$ and $\|D_t\eta_t\|_\infty < 3k$ $\qquad k = \frac{1}{2}\log K$

(ii) $\Phi_t : M_0 \to M_t$ is $K^{3/2}$ bi-Lip and Φ_1 is homotopic to Ψ.

Key Lemma: M a complete hyp. M^3 with $\pi_1(M)$ f.g. M has no rk-1 cusps. Then if η is a harmonic strain field w/ norm of η and $D\eta$ pointwise bounded by k then

$$\int_{C(M)} \|\eta\|^2 + \|D\eta\|^2 \leq \text{area}(\partial C(M))\, k^2.$$

Use fact that $C(M)$ is exhausted by submanifolds w/ uniformly bdd area.

Using this almost there...

DEEP POINTS STAY DEEP UNDER L. BI. LIP.

THM (INFLEX): M_0, M_1 complete hyperbolic structures on 3-mfld s.t. M_1 is a K-q.c. deformation of M_0, $\pi_1(M)$ f.g. and M_0 has no rk 1 cusps. Then there is a volume pres. bi-Lip. diffeo

$$\Phi : M_0 \to M_1$$

whose pt-wise bi-Lip. constant satisfies

$$\log \text{bi-lip}(\Phi, p) \leq C_1 e^{-C_2 d(p, M_0 \setminus \text{Core}(M_0))}$$

for $p \in M_0^{\geq \epsilon}$ $C_1 C_2$ dep. only on K, ϵ, area(∂ core(M

THM (Minsky) Given S, Ψ, X there is an ϵ, so that $\forall n$ inj $Q(\Psi^{-n}X, \Psi^n X) > \epsilon$.

Indeed There is a model for Q_n from singular solv. str. over $g(\Psi^{-n}X, \Psi^n X)$.

The main idea:

The Riemann maps are volume preserving but a priori give no good metric control. inflexibility provides this:

Prop: Given $\epsilon, R, L, C > 0$ there exist $B, C_1, C_2 > 0$ such that the following holds.
Assume $K \subset Q_N$ with $\mathrm{diam}(K) < R$, $\mathrm{inj}_p(K) > \epsilon$ for each $p \in K$, and $\gamma \in \mathcal{C}(S)$ is represented by a closed curve in K of length at most L satisfying

$$\min\{ d_C(\psi^{N+n}(\gamma), \gamma), d_C(\psi^{-N-n}(\gamma), \gamma)\} \geq K_\psi n + B.$$

for all $n \geq 0$

Then $\log \mathrm{bilip}(\phi_{N+n}, p) \leq C_1 e^{-C_2 n}$

for $p \in \phi_{N+n-1} \circ \dots \circ \phi_N(K)$ and $\dfrac{C_1}{1 - e^{-C_2}} < C$.

- How to obtain volume estimate from Prop
  FOR SIMPLICITY apply Minsky's injectivity bounds: $\exists \epsilon$ $\forall n$ $\mathrm{inj} > \epsilon$.

- Fix R large enough to contain a^{10} fundamental domains $S \times [0,1]$. B_ψ
  Let $\gamma \subset B_\psi$ and take $K^+ = \psi^m(B_\psi)$ $K^- = \psi^{-m}(B_\psi)$
  so that $d_C(\psi^m(\gamma), \psi^N(\gamma)) \geq B$. γ $\ell_{Q_0}(\gamma) < L$.

- Then Φ_n converge to (vol. preserving) ~~iso~~ bilip. maps on K^+, K^-, so $\Phi_\infty(K^+)$ lies close to $(\psi^m(\gamma))^*$
  $\Phi_\infty(K^-)$ $(\psi^{-m}(\gamma))^*$

- Then geometric limit mappings exist for $[K^-, K^+]$ the interval bundle bounded by $\partial^- K^-$ $\partial^+ K^+$. after adjusting by a small isotopy (w/ vol $\to 0$) we can take K^-, K^+ by $\Phi_{N,\infty}^{-1}$ and it follows that vol K^-, K^+ in Q_∞ = that in Q_N. $= V_N \cdot (K)$.

- Then vol in Q_∞ differs from $2N \cdot \mathrm{Vol}(B_1)$ by whatever, $2m$.

- the choice of m is independent of N.