Lecture 3 (All results & Proofs by Krasnov & Schlenker)

Limiting fundamental forms

$$I^* = \lim_{r\to\infty} 4e^{-2r} I_r = I((E+B)\cdot,(E+B)\cdot) = I + 2II + III$$

$$I_r = \cosh^2 r\, I + 2\sinh r \cosh r\, II + \sinh^2 r\, III$$

$$= \tfrac{1}{4}\left(e^{2r}(I+2II+III) + 2(I-III) + e^{-2r}(I-2II+III)\right)$$

$$= \tfrac{1}{4}\left(e^{2r} I^* + 2II^* + e^{-2r} III^*\right)$$

$$I^* = I + 2II + III = I((E+B)\cdot;(E+B)\cdot)$$

$$II^* = I - III = I((E+B)\cdot;(E-B)\cdot)$$

$$III^* = I - 2II + III = I((E-B)\cdot;(E-B)\cdot)$$

Def $B^* = I^{*-1} II^* = (E+B)^{-1}(E-B)$

$$I^*(B^*v,w) = I^*(v,B^*w) = II^*(v,w)$$

$$III^*(B^*v,w) = II^*(v,B^*w) = III^*(v,w)$$

Invert linear eqns

$$I = \tfrac{1}{4}(I^* + 2II^* + III^*) = \tfrac{1}{4} I^*((E+B^*)\cdot;(E+B^*)\cdot)$$

$$II = \tfrac{1}{4}(I^* - III^*) = \tfrac{1}{4} I^*((E+B^*)\cdot;(E-B^*)\cdot)$$

$$III = \tfrac{1}{4}(I^* - 2II^* + III^*) = \tfrac{1}{4} I^*((E-B^*)\cdot;(E-B^*)\cdot)$$

$$B = (E+B^*)^{-1}(E-B^*)$$

Thm Egregium (Gauss)

S smooth surface in M

$x \in S$ $\quad K =$ sect. curv of TS in TM

then $K_g = K + K_e$

Pf/ Let $\bar\nabla$ cov der. on S

$$\nabla_w v = \bar\nabla_w v + \langle \nabla_w v, n\rangle n = \bar\nabla_w v + \langle w, Bv\rangle n$$

$$\nabla_v \nabla_w z = \bar\nabla_v \bar\nabla_w z + \nabla_v(\langle z, Bw\rangle n)$$

$$= \bar\nabla_v \bar\nabla_w z + v(\langle z, Bw\rangle) n + \langle z, Bw\rangle Bv$$

$$R(v,w)z = \nabla_v \nabla_w z - \nabla_w \nabla_v z - \nabla_{[v,w]} z$$

$$= \bar R(v,w)z - \langle z, Bw\rangle Bv + \langle z, Bv\rangle Bw$$

$$+ \left(v(\langle z, Bw\rangle) - w(\langle z, Bv\rangle) - \langle z, B[v,w]\rangle\right) n$$

$$\langle R(v,w)w, v\rangle = \langle \bar R(v,w)w, v\rangle - \langle w, Bw\rangle \langle v, Bv\rangle + \langle w, v\rangle \langle v, Bw$$

$$K = K_g - \det B = K_g - K_e \quad \square$$

Lemma

$$B_r = (E + e^{-2r}B^*)^{-1}(E - e^{-2r}B^*) \simeq E - 2e^{-2r}B^*$$

Pf/

$$I_r = \frac{1}{4}\left(e^{2r} I^* + 2 II^* + e^{-2r} III^*\right) = I^*\left((e^r E + e^{-r}B^*)\cdot, (e^r E + e^{-r}B^*)\cdot\right)$$

$$II_r = \frac{1}{2}\frac{d I_r}{dr} = \frac{1}{4}\left(e^{2r} I^* - e^{-2r} III^*\right)$$

$$= I^*\left((e^r E + e^{-r}B^*)\cdot, (e^r E - e^{-r}B^*)\cdot\right)$$

$$B_r = I_r^{-1} II_r = (e^r E + e^{-r}B^*)^{-1}(e^r E - e^{-r}B^*)$$

$$= (E + e^{-2r}B^*)^{-1}(E - e^{-2r}B^*) \simeq E - 2e^{-2r}B^*$$

Cor $\quad K^* = -H^*$

Pf/

Let K_r = Gauss curve of S_r

$$K_r = \det B_r - 1 = \frac{-2e^{-2r}\,tr(B^*)}{1 + e^{-2r}tr(B^*) + e^{-4r}\det B^*}$$

$$K^* = \lim_{r\to\infty} \frac{e^{2r}}{4} K_r = -\frac{tr B^*}{2} = -H^*$$

$\square$

h

$G = g_{ij} v^i \otimes v^j$ inner prod on V

For A, B symmetric bilinear

$A = A_{ij} v^i \otimes v^j$ $\quad \langle A, B \rangle = g^{ij} g^{kl} A_{ik} B_{jl}$

$B = B_{ij} v^i \otimes v^j$ $\quad = tr(G^{-1} A G^{-1} B)$

<u>Variational formula</u>

$M = Q(X, Y)$ qf mfld, $N \subseteq M$ strictly convex ∂N smooth

Vary hyp metric on M

$$\delta W(N) = \frac{1}{2} \int \delta H + \frac{1}{2} \langle \delta I, II - HI \rangle \, da$$

$$= \frac{1}{4} \int \langle \delta II - H \, \delta I, I \rangle \, da$$

Sketch:

$$\delta W = \delta V(N) - \frac{1}{2} \int \delta H \, da - \frac{1}{2} \int H \delta (da)$$

Generalized Schlafli (Rivin-Schlenker)

$$\delta V(N) = \int_{\partial N} \partial H + \frac{1}{4} \langle \delta I, II \rangle da$$

$$\delta W = \frac{1}{2} \int \delta H da + \frac{1}{4} \int \langle \delta I, II \rangle da - \frac{1}{2} \int H \delta da$$

$da = (\det I)^{\frac{1}{2}} dv$ $\quad \delta(da) = \delta((\det I)^{1/2}) dv = \frac{1}{2} (\det I)^{-\frac{1}{2}} \delta(\det(I)) \, dv$

$$= \frac{1}{2} tr(I^{-1} \delta I) \, da = \frac{1}{2} \langle I, \delta I \rangle \, da$$

$$\delta W = \frac{1}{2} \int \delta H + \frac{1}{2} \langle \delta I, II - HI \rangle da$$

$2H = tr(B) = tr(\mathrm{I}^{-1}\mathrm{II}) =$

$2\delta H = -tr(\mathrm{I}^{-1}\delta\mathrm{I}\,\mathrm{I}^{-1}\mathrm{II}) + tr(\mathrm{I}^{-1}\delta\mathrm{II})$

$= -\langle \delta\mathrm{I}, \mathrm{II}\rangle + \langle \mathrm{I}, \delta\mathrm{II}\rangle$

$\delta W = \frac{1}{4}\int \langle \delta\mathrm{II} - H\,\delta\mathrm{I}, \mathrm{I}\rangle\, da$

$\square$

Use inversion to $\mathrm{I}^*, \mathrm{II}^*, \mathrm{III}^*$ to obtain

$\delta W = -\frac{1}{4}\int \langle \delta\mathrm{II}^* - H^*\delta\mathrm{I}^*, \mathrm{I}^*\rangle\, da^*$

$= -\frac{1}{2}\int \delta H^* + \frac{1}{2}\langle \delta\mathrm{I}^*, \mathrm{II}^* - H^*\mathrm{I}^*\rangle\, da^*$

<u>Def</u> $\mathrm{II}_0^* = \mathrm{II}^* - H^*\mathrm{I}^*$ traceless as

$tr(\mathrm{I}^{*-1}\mathrm{II}_0^*) = \langle \mathrm{I}^*, \mathrm{II}_0^*\rangle = 0$

<u>Cor</u> $W(M,h)$ max at $h =$ Poincaré

<u>Pf</u> Let I^* be metric. Vary so conformal
Fix area as vary

$\delta\mathrm{I}^* = u\mathrm{I}^*$ u function on ∂M

$F(N) = W(N) - \frac{\lambda}{2}\int_{\partial M} da^*$ Lagrange Multipliers

$$\delta F = -\frac{1}{2}\int \delta H^* + \frac{1}{2}\langle \mathrm{II}_0^*, \delta I^* \rangle \, da^* - \frac{\lambda}{2}\int \delta \, da^*$$

As $\delta I^* = u I^*$ $\quad \langle \mathrm{II}_0^*, \delta I^* \rangle = u \,\mathrm{tr}(I^* \mathrm{II}_0^*) = 0$

$\delta da^* = u \, da^* \quad H^* = -K^*$

$$\delta F = \frac{1}{2}\int (\delta K^* - \lambda u) \, da^*$$

Area const $\Rightarrow \delta \int K^* da^* = 0$

$\Rightarrow \int \delta K^* da = -\int K^* \delta da^* = -\int K^* u \, da^*$

$$\delta F = -\frac{1}{2}\int (K^* + \lambda) u \, da^*$$

Only critical at $K^* = -\lambda$ const curv.

$I^* =$ Poincare only crit value. $\square$

$$V_R(M) = W(M, h_p)$$

Follows $\delta V_R(M) = -\frac{1}{4}\int \langle \delta I^*, \mathrm{II}_0^* \rangle \, da^*$

Note

$$\delta V_R(M) = -\frac{1}{2}\int \delta H^* + \frac{1}{2}\langle \delta I^*, \mathrm{II}_0^* \rangle \, da^*$$

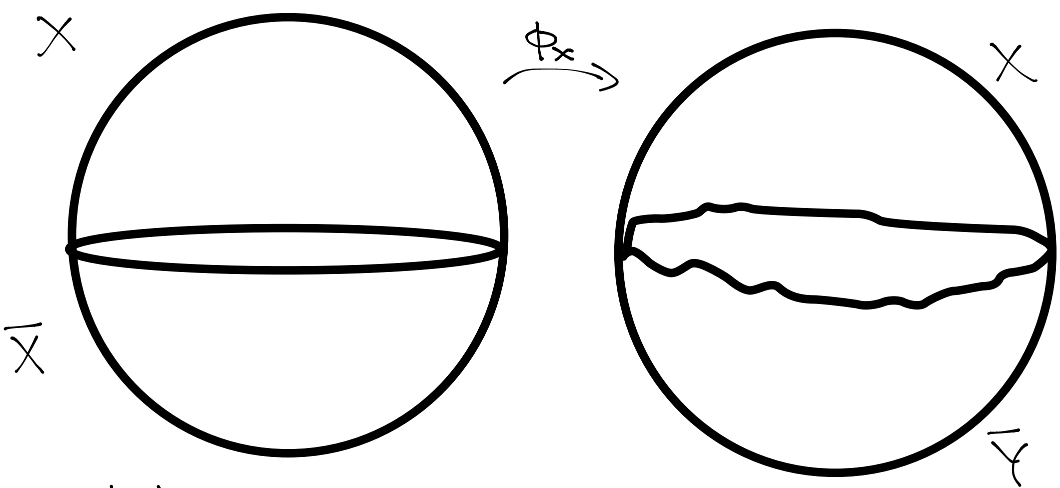
Vary along Poincare $\Rightarrow H^* = -K^* = -1 \quad \delta H^* = 0$

Schwartzian

Let $M = Q(X,Y)$ q.f. As $QF(S) \cong T(S) \times T(\bar{S})$

$T_M(QF(S)) = T_X(T(S)) \oplus T_{\bar{Y}}(T(\bar{S}))$

$T_X(T(S)) = Q(X)$ = holo. quad diffs on X



ϕ_X holo on top

$S(\phi_X)$ = Schwartzian deriv $\in Q(X) \cong T_X(T(S))$

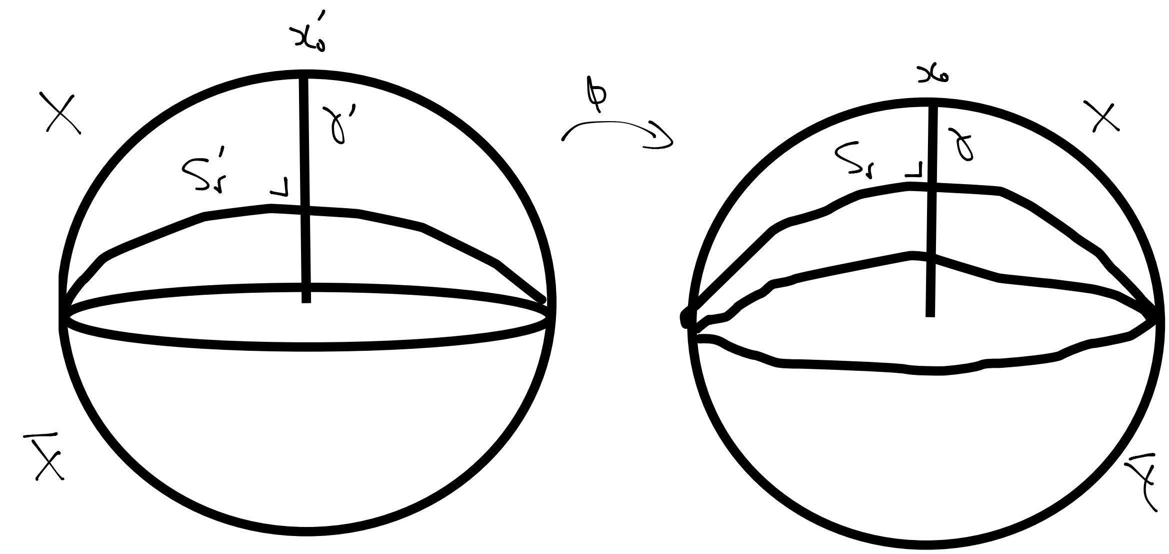
Def $S_M \in T_M^*(QF(S))$ by $S_M = (S(\phi_X), S(\phi_Y))$

$S_M(v,w) = S(\phi_X)v + S(\phi_Y)w$

Thm $dV_R(M) = S_M$

Sketch: $dV_R(M) = -\frac{1}{4}\int \langle \dot{I}^*, \mathrm{II}_0^* \rangle da^*$

as $\delta H^* = 0$



Have two Poincare metrics I'^*, I^*

ϕ isometry between $\Rightarrow \phi_* I'^* = I^*$ Also by symm $\mathbb{I}'^* = \mathbb{I}^*$

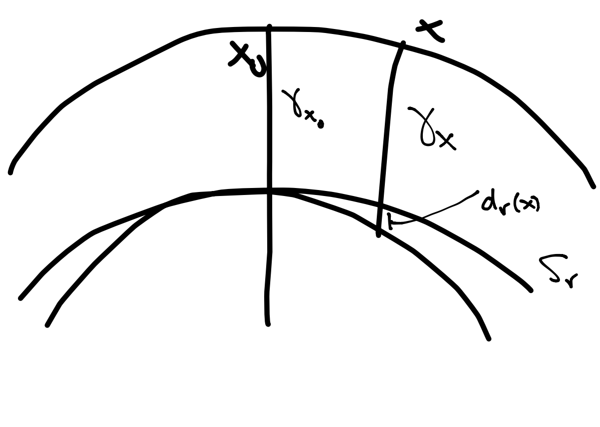
Let Φ_0 be isom of $\mathbb{H}^3$ ϕ_0 = Mobius map $\Phi_0|_{\partial \mathbb{H}^3}$

sending γ' to γ and normalize so ϕ_0 isom of I'^* & I^* at x_0'

Let $f = \phi_0^{-1} \circ \phi$ f isom I'^* to $\phi_0^* I^*$

$\Phi_0(S_r')$ & S_r tangent at $\gamma \cap S_r$

x near x_0 let $\gamma_x \perp S_r$ $d_r(x)$ = signed distance



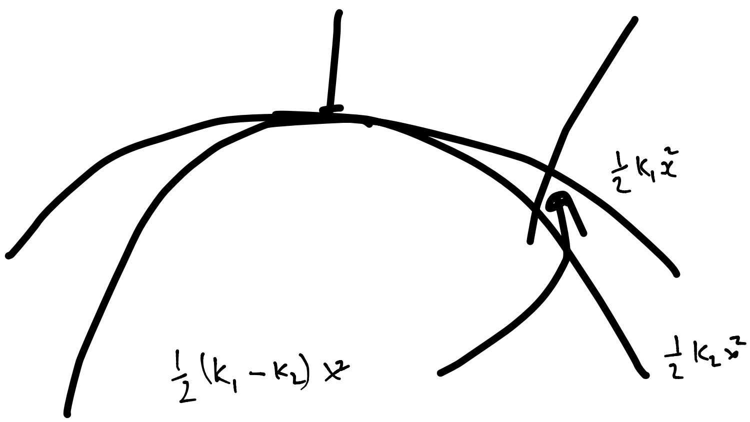
d_r smooth, deriv zero at x_0

Let $d_\infty = \lim_{r\to\infty} d_r$

Lemma

$$\mathrm{Hess}(d_\infty)(x_0) = \mathrm{II}^*_0$$

Sketch:



$d''_\rho(x_0) \cong I_\rho((B_\rho - B'_\rho)\cdot\,,\,\cdot)$

$$d''_\infty(x_0) = \lim_{\rho\to\infty} I_\rho((B_\rho - B'_\rho)v, v)$$
$$= \lim_{\rho\to\infty} 2e^{-2\rho} I_\rho((B^* - B'^*)v, v)$$
$$= \frac{1}{2}(\mathrm{II}^* - \mathrm{II}'^*) = \frac{1}{2}(\mathrm{II}^* - E) = \mathrm{II}_0$$

$\square$

I'^* conf to $\phi_0^* I^*$ $\qquad I'^* = e^{2u}\phi_0^* I^*$

Lemma $u'(x_0) = 0$ $\quad \mathrm{Hess}(u)(x_0) = \mathrm{II}^*_0$

follows from above

Lemma $\qquad \mathrm{II}^*_0 = \mathrm{Re}(S(f)) = \mathrm{Re}(S(\varphi))$

Pf/ f isom so $|f'(z)|^2 = e^{-2u}$

$\Rightarrow f(z) = z + \frac{b}{2}z^2 + \frac{c}{6}z^3 + O(z^4)$ expansion about x_0

$b = 0$ as $u'(x_0) = 0$

$$|f'(z)|^2 = \left|1 + \frac{c}{2} z^2 + O(z^3)\right|^2 = 1 + \mathrm{Re}(c z^2) + O(z^3)$$

$$\Rightarrow \mathrm{II}_0^*(x_0)(z,z) = u''(x_0)(z,z) = -\mathrm{Re}(c z^2)$$

$$\Rightarrow -\mathrm{II}_0^*(x_0) = \mathrm{Re}\left((\varphi_0^{-1} \circ \varphi)'''(x_0)\right) = \mathrm{Re}\left(S(\varphi_0^{-1} \circ \varphi)(x_0)\right) = \mathrm{Re}\left(S(\varphi)(x_0)\right)$$