Renormalized Volume for Quasifuchsian Manifolds
Outline of papers of Krasnov and Schlenker

Kleinian gp: Γ discrete subgp of $\mathrm{Iso}(\mathbb{H}^3)$
will restrict to orient preserving & torsion free
Let $x \in \mathbb{H}^3$

$L_\Gamma = \overline{\Gamma x} \cap S^2_\infty$ limit set $\qquad \Omega_\Gamma = S^2_\infty - L_\Gamma$ domain of discontinuity

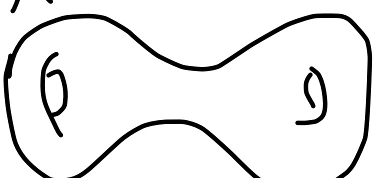
Γ acts prop disc on Ω_Γ $\qquad Z_\Gamma = \Omega_\Gamma / \Gamma$ conformal str. at ∞

$H(L_\Gamma) =$ convex hull of limit set $L_\Gamma =$ smallest convex containing all geodesics with endpts in L_Γ

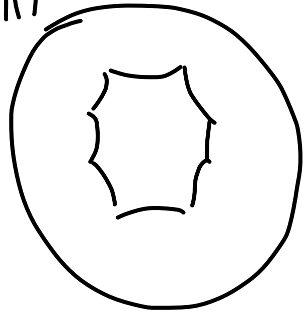
$M_\Gamma = \mathbb{H}^3/\Gamma \qquad N_\Gamma = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma \qquad C_\Gamma = H(L_\Gamma)/\Gamma$ convex core

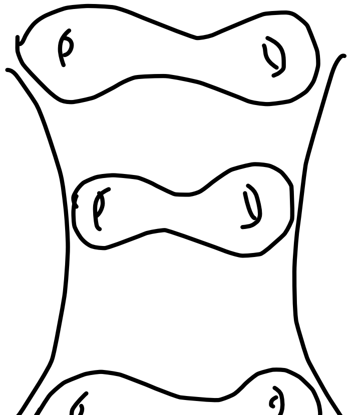
Ex/ Fuchsian: X hyp str on closed surface S. $X = \mathbb{H}^2/\Gamma \quad \Gamma \subseteq \mathrm{Iso}(\mathbb{H}^2)$

X

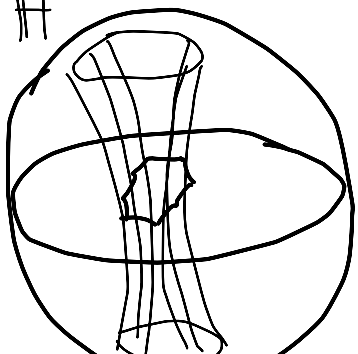


$\mathbb{H}^2$



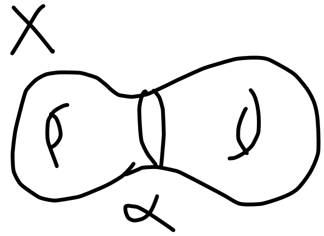


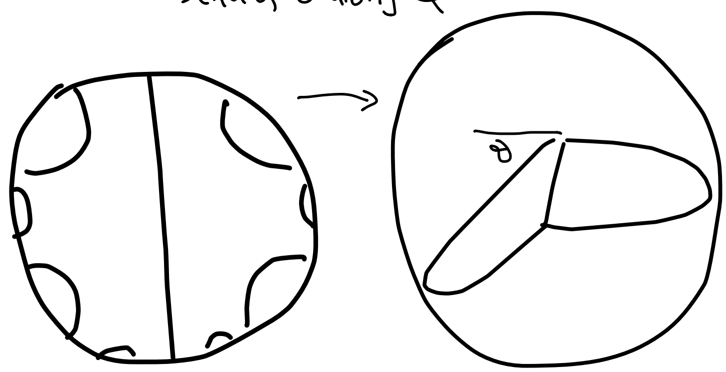
$\mathbb{H}^3$



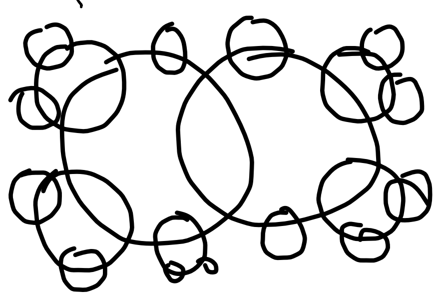
Ex/ Mickey Mouse

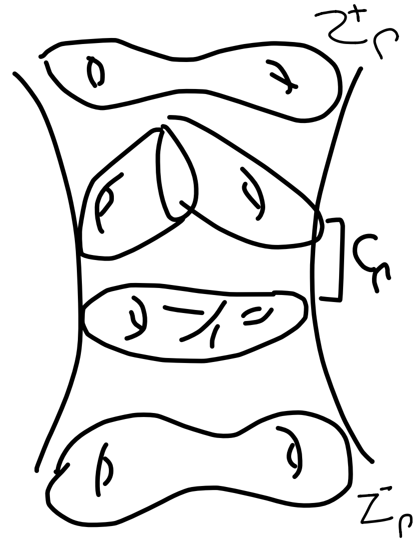
Embed with Equivariant bend of θ along α





$\Gamma \leq \mathrm{Iso}^+(\mathbb{H}^2) = PSL(2,\mathbb{R}) \leq PSL(2,\mathbb{C}) = \mathrm{Iso}^+(\mathbb{H}^3)$





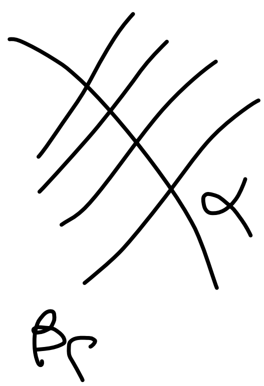
$M_\rho \cong S \times \mathbb{R}$

$L_\rho \cong S^1$ top. circle

Quasifuchsian group

∂C_ρ - has intrinsic hyp str, bent along geodesics β_ρ

β_ρ is measured lamination. Is closed subset of disjoint geodesics with transverse measure given by bending



$i(\alpha, \beta_\rho)$ total bending along α.

$L(\beta)$ = length of bending lamination $= \ell \cdot \theta$ in case of bend θ on closed geodesic of length ℓ

$QF(S)$ = space of hyp str. on $S \times \mathbb{R}$ quasiconformally conj. to fuchsian

$L_\rho \cong S^1$ top. circle $Z_\rho = Z_\rho^+ \sqcup Z_\rho^-$ both conf str on S.

Theorem (Bers Sim. Unif.)

$$QF(S) \cong \text{Teich}(S) \times \text{Teich}(\bar{S})$$

$\rho \longrightarrow (Z_\rho^+, Z_\rho^-)$ is homeo.

Families of surfaces. Let $X, Y \in \text{Teich}(S)$

$M = Q(X, Y) \in QF(S)$

If N is convex subset with $S = \partial N$ smooth

let $S_r = \partial N_r$ - bdy of r nbd of N

g_r induced metric on S_r

def $g = \lim_{r \to \infty} 4e^{-2r} g_r$ conf metric on Z_ρ

Similarly given g conf on Z_ρ can define S_r & N_r

Let $x \in \mathbb{H}^3$ & h_x visual metric on $\hat{S}^2_\infty$

For $\xi \in \Omega_\rho$ let $H_{\xi, r} = \{x \in \mathbb{H}^3 \mid h_x(\xi) \geq \frac{e^{2r}}{4} g(\xi)\}$

$S_r = \partial \left(\bigcup_{\xi \in \Omega_\rho} H_{\xi, r} \right)$ S_r bdy of nbd of ends

for r large S_r embedded.

W-Vol

$V(N) = $ vol of N

$H = $ mean curv. of $S = \partial N$

$$W(N) = V(N) - \frac{1}{2}\int_{\partial N} H\, dA$$

For h conf on Σ_ρ we define

$$W(h) = W(N_r(h)) + \pi r \chi(\partial M)$$

Thm. W is well-defined (i.e. indep of r chosen).

Cor. $W(e^{2\rho}h) = W(h) - \pi\rho\chi(\partial M)$

Pf $W(e^{2\rho}h) = W(N_r(e^{2\rho}h)) + \pi r \chi(\partial M) = W(N_{r+\rho}(h)) + \pi r\chi(\partial M)$

$= W(N_r(h)) - \pi\rho\chi(\partial M) + \pi r\chi(\partial M) = W(h) - \pi\rho\chi(\partial M)$ $\square$

Thm (Monotonicity).

If $g(x) \leq h(x)$ $\forall x \in \Sigma_\rho$ then $W(g) \leq W(h)$

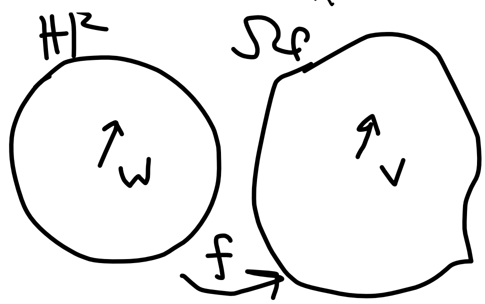
Two conformal metrics

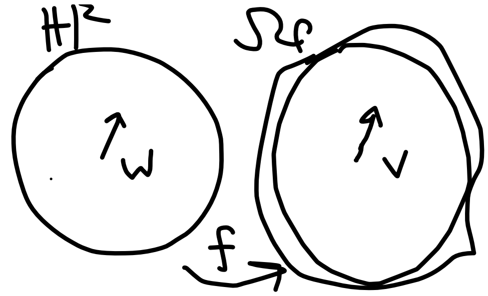
h_P - Poincare metric

$\|v\|_P = \inf\{\|w\|_h : df(w) = v,\ f: D^2 \to \Omega_\rho$ conf homeo$\}$

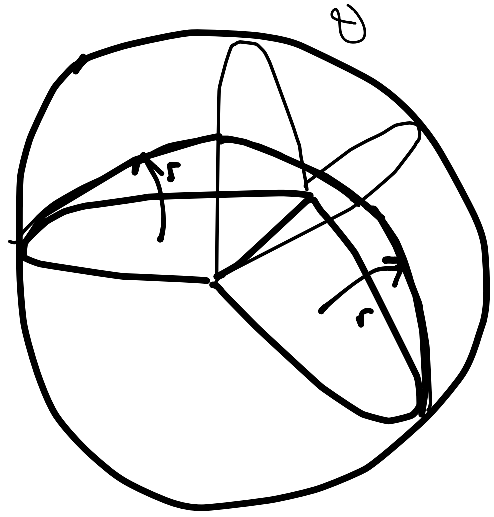
h_T = Thurston (Projective) metric

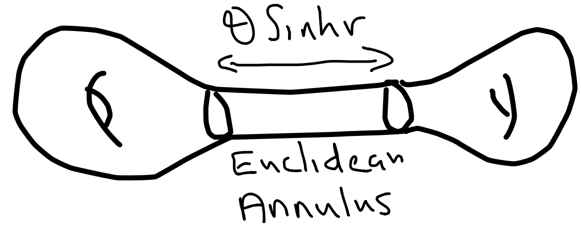
$\|v\|_T = \{\|w\|_h : df(w) = v,\ f: D^2 \to \Omega_\rho,\ f$ Möbius$\}$

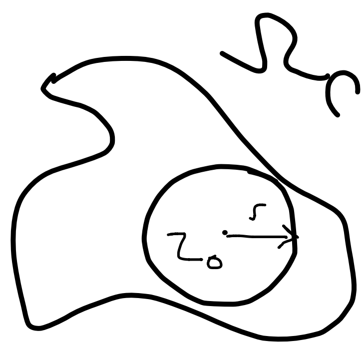




Surfaces S_r for h_T are bdy of r-nbd - C_Γ convex core - Γ







<u>Lemma</u> $\frac{1}{4} h_T \leq h_P \leq h_T$

Pf/ $h_P \leq h_T$ (by def)

Let r = max radius of Euclidean ball in Ω_Γ about z_0 Let $f(z) = z_0 + rz$ gives $h_T(z_0) \leq \frac{2}{r}$

Let $g: D^2 \to \Omega_\Gamma$ $g(0) = z_0$ uniformizing Ω_Γ h_P^D, h_P^Ω Poincare metrics on D^2, Ω_Γ

then $h_P^\Omega(z_0)|g'(z_0)| = h_P^{D^2}(0) = 2$

Koebe $\frac{1}{4}$ thm $\Rightarrow \exists$ ball radius $\frac{1}{4}|g'(0)|$ in Ω_Γ about z_0

$\Rightarrow r \geq \frac{1}{4}|g'(0)| \Rightarrow h_P^\Omega(z_0) \geq \frac{2}{|g'(0)|} \geq \frac{1}{2r} \geq \frac{1}{4} h_T(z_0)$ $\square$

Renormalized Volume: $V_R(M) \equiv W(h_p)$

<u>Cor</u>

$$W(h_T) - \pi \log 4 |\chi(\partial M)| \leq V_R(M) \leq W(h_T)$$

follows from lemma and monotonicity of W $\square$

Calc: $W(h_T) = V_C(M) - \frac{1}{4} L(\beta_\rho)$ $\qquad V_C(M) = \mathrm{vol}\ C_\rho$

Denote $W(h_T) = W^+(X,Y) + W^-(X,Y) + V_C(M)$ two ends of $M = Q(X,Y)$

$$W(h_T) = V_C(M) + \left(W^+(X,X) + W^+(Y,Y) - \pi r \chi(\partial M)\right) + \mathrm{Vol}(D_r) - \frac{1}{2}\int_{D_r} H\,dA$$

As fuchsian $W(h_T) = 0$

$$W(h_T) = V_C(M) + \frac{\ell\theta}{2}\sinh^2 r - \frac{1}{2}H(r)A(r)$$

$H(r)$ = mean curv. fn.
$A(r)$ area of wedge

$$= V_C(M) + \frac{\ell\theta}{2}\sinh^2 r - \frac{1}{2}\left(\frac{\tanh r + \coth r}{2}\right)\ell\theta \sinh r \cosh r$$

$$= V_C(M) - \frac{1}{4}\ell\theta = V_C(M) - \frac{1}{4}L(\beta_\rho) \ \square$$

Thm (B–)

Γ quasifuchsian $\Rightarrow L(\beta_\rho) \leq K|\chi(\partial M)|$ K universal

<u>Cor</u> $\exists\, k_1, k_2 > 0$ s.t. for M quasi-fuchsian

$$V_C(M) - k_1|\chi(\partial M)| \leq V_R(M) \leq V_C(M) - k_2|\chi(\partial M)|$$

S smooth in M $\quad n: S \to TM$ normal vector

$B: TS \to TS \quad Bv = -\nabla_v n$ shape operator

Let $E: TS \to TS$ be identity

Given $S = \partial N$ with I induced metric on S

In M $v, w \in T_x S$

$$0 = v(\langle n, w\rangle) = \langle \nabla_v n, w\rangle + \langle n, \nabla_v \rangle$$

$$\langle \nabla_v n, w\rangle - \langle \nabla_w n, v\rangle = \langle n, \nabla_w v - \nabla_v w\rangle = \langle n, [v,w]\rangle = 0$$

$\Rightarrow I(Bv, w) = I(v, Bw) \quad$ B self adj. wrt I $\quad$ I^{st} fund. form

$II(v,w) \equiv I(v, Bw) \quad 2^{nd}$ fund. form

$III(v,w) = II(v, Bw) \equiv I(Bv, Bw) \quad 3^{rd}$ fund. form

$H = \operatorname{tr} B \quad K_e = \det B$ extrinsic curv. $\quad K_g =$ Gauss. curv

Thm Egregium (Pf at end)

$$K = K_g - K_e$$

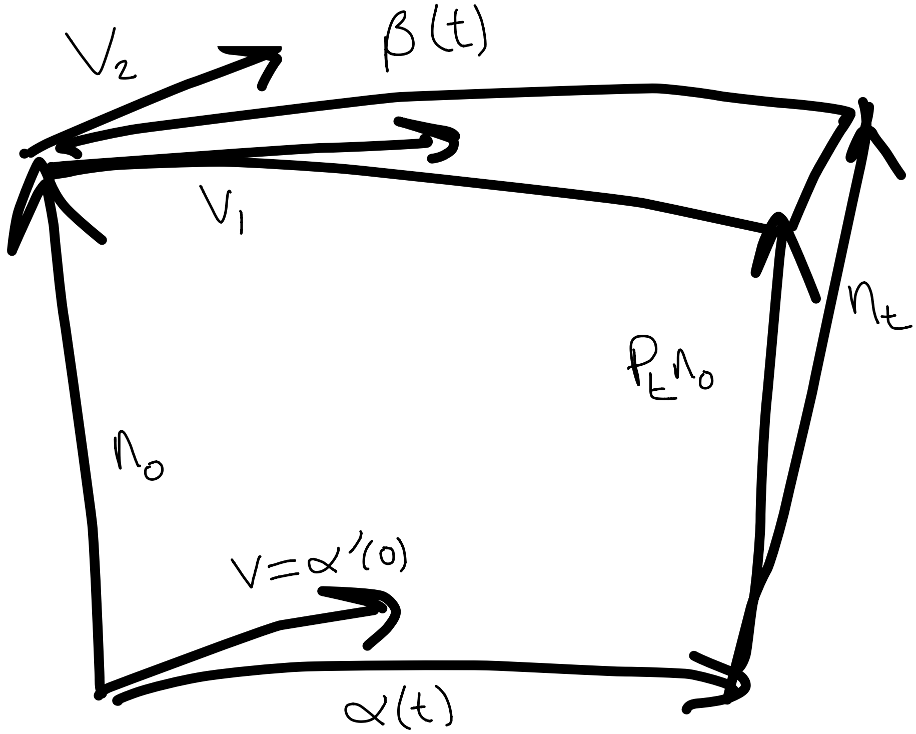
Let S_r induce metric I_r on S

<u>Lemma</u>

$$I_r(v,w) = I(C(r)v, C(r)w)$$

where $C(r) = \cosh r\, E + \sinh r\, B$

Pf



$n_r : S \to S_r$ normal map. Let $v \in TS$ α curve $\alpha'(0) = v$

$\beta(t) = n_r(\alpha(t)) = n_t$ let $\beta_1(t) = P_t n_0 =$ parallel trans. of n_0 along α

$\beta(t) \cong \beta_1(t) + \beta_2(t)$

$\beta_1(t) \cong \cosh r . v$ $\quad \beta_2(t) = n_t - P_t n_0 \cong \sinh r\, Bv$

Invariance of $W(h)$

$A(r) = $ area of S_r

$V(r) = \int_{r_0}^{r} A(x)\,dx = $ vol of region between N & N_r

$$\frac{dA}{dr} = A'(r) = \int_{S_r} \operatorname{tr} B \, dA = 2\int_{S_r} H \, dA$$

Let $W(r) = W(N_r) = V(N) + V(r) - \frac{1}{2}\int_{S_r} H\,dA$

$$W'(r) = V'(r) - \frac{1}{4}A''(r) = -\frac{1}{4}\left(A'' - 4A\right)$$

Hom. eqn $y'' - 4y = 0$ has soln e^{-2r}, e^{2r}.

Lemma $W'(r) = -\pi\chi(\partial M)$

From above can ignore terms with $\sinh 2r, \cosh 2r$

$$A(r) = \int_S \det(\cosh r\, E + \sinh r\, B)\, dA$$

$$= \int \cosh^2 r + \cosh r \sinh r (\operatorname{tr} B) + \sinh^2 r (\det B)\, dA$$

$$= \left(\tfrac{1}{2}(\cosh 2r + 1) + \tfrac{1}{2}\sinh 2r (\operatorname{tr} B) + \tfrac{1}{2}(\cosh 2r - 1)(\det B)\right) dA$$

$$W'(r) = \int \left(\tfrac{1}{2} - \tfrac{1}{2}\det B\right) dA = -\frac{1}{2}\int (\det B - 1)\, dA = -\frac{1}{2}\int K_g\, dA$$

$$= -\frac{1}{2}\left(2\pi\chi(\partial M)\right) = -\pi\chi(\partial M) \qquad \square$$

Limiting fundamental forms

$$\mathrm{I}^* = \lim_{r\to\infty} 4e^{-2r}\,\mathrm{I}_r = \mathrm{I}\big((E+B)\cdot\,,(E+B)\cdot\big) = \mathrm{I} + 2\,\mathrm{II} + \mathrm{III}$$

$$\mathrm{I}_r = \cosh^2 r\,\mathrm{I} + 2\sinh r\cosh r\,\mathrm{II} + \sinh^2 r\,\mathrm{III}$$

$$= \tfrac{1}{4}\big(e^{2r}(\mathrm{I}+2\mathrm{II}+\mathrm{III}) + 2(\mathrm{I}-\mathrm{II}) + e^{-2r}(\mathrm{I}-2\mathrm{II}+\mathrm{III})\big)$$

$$= \tfrac{1}{4}\big(e^{2r}\,\mathrm{I}^* + 2\,\mathrm{II}^* + e^{-2r}\,\mathrm{III}^*\big)$$

$$\mathrm{I}^* = \mathrm{I} + 2\mathrm{II} + \mathrm{III} = \mathrm{I}\big((E+B)\cdot\,;(E+B)\cdot\big)$$

$$\mathrm{II}^* = \mathrm{I} - \mathrm{III} = \mathrm{I}\big((E+B)\cdot\,;(E-B)\cdot\big)$$

$$\mathrm{III}^* = \mathrm{I} - 2\mathrm{II} + \mathrm{III} = \mathrm{I}\big((E-B)\cdot\,;(E-B)\cdot\big)$$

Def $B^* = \mathrm{I}^{*-1}\,\mathrm{II}^* = (E-B)(E+B)^{-1}$

$$\mathrm{I}^*(B^*v,w) = \mathrm{I}^*(v,B^*w) = \mathrm{II}(v,w)$$

$$\mathrm{III}^*(B^*v,w) = \mathrm{II}^*(v,B^*w) = \mathrm{III}^*(v,w)$$

Invert Lin. eqns

Invert linear eqns

$$\mathrm{I} = \tfrac{1}{4}(\mathrm{I}^* + 2\mathrm{II}^* + \mathrm{III}^*) = \tfrac{1}{4}\mathrm{I}^*\big((E+B^*)\cdot\,;E+B^*)\cdot\big)$$

$$\mathrm{II} = \tfrac{1}{4}(\mathrm{I}^* - \mathrm{III}^*) = \tfrac{1}{4}\mathrm{I}^*\big((E+B^*)\cdot\,;E-B^*)\cdot\big)$$

$$\mathrm{III} = \tfrac{1}{4}(\mathrm{I}^* - 2\mathrm{II}^* + \mathrm{III}^*) = \tfrac{1}{4}\mathrm{I}^*\big((E-B^*)\cdot\,;E-B^*)\cdot\big)$$