4-manifolds with inequivalent symplectic forms and 3-manifolds with inequivalent fibrations

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Abstract

We exhibit a closed, simply connected 4-manifold X carrying two symplectic structures whose first Chern classes in $H^2(X,\mathbb{Z})$ lie in disjoint orbits of the diffeomorphism group of X. Consequently, the moduli space of symplectic forms on X is disconnected.

The example X is in turn based on a 3-manifold M. The symplectic structures on X come from a pair of fibrations $\pi_0, \pi_1 : M \to S^1$ whose Euler classes lie in disjoint orbits for the action of $\mathrm{Diff}(M)$ on $H_1(M,\mathbb{R})$.

1 Introduction

Symplectic 4-manifolds. A *symplectic form* ω on a smooth manifold X^{2n} is a closed 2-form such that $\omega^n \neq 0$ pointwise. Given a pair of symplectic forms ω_0 and ω_1 on X, we say:

(i) ω_0 and ω_1 are *homotopic* if there is a smooth family of symplectic forms ω_t, $t \in [0,1]$, interpolating between them;

(ii) ω_0 is a *pullback* of ω_1 if $\omega_0 = f^*\omega_1$ for some diffeomorphism $f : X \to X$; and

(iii) ω_0 and ω_1 are *equivalent* if they are related by a combination of (i) and (ii).

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Any symplectic form ω admits a compatible almost complex structure $J : TX \to TX$ (satisfying $\omega(v, Jv) > 0$ for $v \neq 0$). Let $c_1(\omega) \in H^2(X, \mathbb{Z})$ denote the first Chern class of the (canonical) complex line bundle $\wedge^n_{\mathbb{C}} TX$ determined by J. It is easy to see that the first Chern class is a deformation invariant of the symplectic structure; that is, $c_1(\omega_0) = c_1(\omega_1)$ if ω_0 and ω_1 are homotopic.

The purpose of this note is to show:

Theorem 1.1 *There exists a closed, simply-connected 4-manifold X which carries a pair of inequivalent symplectic forms. In fact, ω_0 and ω_1 can be chosen such that $c_1(\omega_0)$ and $c_1(\omega_1)$ lie in disjoint orbits for the action of* $\operatorname{Diff}(X)$ *on $H^2(X, \mathbb{Z})$.*

One can also formulate this result by saying that the moduli space $\mathcal{M} = (\text{symplectic forms on } X)/\operatorname{Diff}(X)$ is disconnected.

Fibered 3-manifolds. To construct the 4-dimensional example X, we first produce a compact 3-dimensional manifold M^3 that fibers over the circle in two unrelated ways.

To describe this example, we recall the correspondence between closed 1-forms and measured foliations. Let α be a closed 1-form on M, such that α and its pullback to ∂M are pointwise nonzero. Then α defines a *measured foliation* $\mathcal{F}$ of M^3, transverse to ∂M, with $T\mathcal{F} = \operatorname{Ker} \alpha$ and with transverse measure $\mu(T) = \int_T |\alpha|$. Conversely, a (transversally oriented) measured foliation $\mathcal{F}$ determines such a 1-form α. If α happens to have integral periods, then we can write $\alpha = d\pi$ for a *fibration* $\pi : M \to S^1 = \mathbb{R}/\mathbb{Z}$, and the leaves of $\mathcal{F}$ are then simply the fibers of π.

The *Euler class* of a measured foliation,

$$e(\mathcal{F}) = e(\alpha) \in H_1(M, \mathbb{Z})/(\text{torsion}),$$

is represented geometrically by the zero set of a section $s : M \to T\mathcal{F}$, such that the vector field $s|\partial M$ is inward pointing and nowhere vanishing.

Just as for symplectic forms, we say:

(i) α_0 and α_1 are *homotopic* if they are connected by a smooth family of closed 1-forms α_t, nonvanishing on M and ∂M;

(ii) α_0 is a *pullback* of α_1 if $\alpha_0 = f^* \alpha_1$ for some $f \in \operatorname{Diff}(M)$; and

(iii) α_0 and α_1 are *equivalent* if they are related by a combination of (i) and (ii).

In the 3-dimensional arena we will show:

Theorem 1.2 *There exists a compact link complement $M = S^3 - \mathcal{N}(K)$ which carries a pair of inequivalent measured foliations α_0 and α_1. In fact α_0 and α_1 can be chosen to be fibrations, with $e(\alpha_0)$ and $e(\alpha_1)$ in disjoint orbits for the action of* $\mathrm{Diff}(M)$ *on* $H_1(M,\mathbb{Z})$.

(Here and below, $\mathcal{N}(K)$ denotes an open regular neighborhood of a link K in a 3-manifold.)

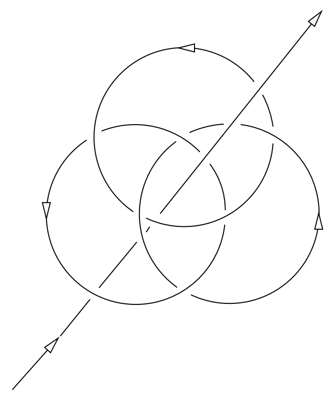

Figure 1. An axis added to the Borromean rings.

Description of the manifolds. For the specific examples we will present, the link K is obtained from the Borromean rings $K_1 \cup K_2 \cup K_3$ by adding a fourth component K_4; see Figure 1. The fourth component is the *axis* of a rotation of S^3 cyclically permuting $\{K_1, K_2, K_3\}$; it can be regarded as a vertical line in $\mathbb{R}^3$, normal to a plane nearly containing the rings.

Alternatively, we can also write $M = T^3 - \mathcal{N}(L)$, where

- $T^3 = \mathbb{R}^3/\mathbb{Z}$ is the flat Euclidean 3-torus,
- $L \subset T^3$ is a union of 4 disjoint, oriented, closed geodesics,
- (L_1, L_2, L_3) gives a basis for $H_1(T^3, \mathbb{Z})$, and
- $L_4 = L_1 + L_2 + L_3$ in $H_1(T^3, \mathbb{Z})$.

The 4-manifold X of Theorem 1.1 is the fiber-sum of $T^3 \times S^1$ with 4 copies of the elliptic surface $E(1) \to \mathbb{CP}^1$, with the elliptic fiber $F \subset E(1)$

glued along $L_i \times S^1$. The key to the example is that $\mathrm{Diff}(X)$ preserves the *Seiberg–Witten norm*

$$\|s\|_{\mathrm{SW}} = \sup\{|s \cdot t| \,:\, \mathrm{SW}(t) \neq 0\}$$

on $H^2(X, \mathbb{R})$, just as $\mathrm{Diff}(M)$ preserves the Alexander norm on $H^1(M, \mathbb{R})$. The Seiberg–Witten norm manifests the rigidity of the smooth structure on X, allowing us to check that the Chern classes $c_1(\omega_1), c_1(\omega_2)$ lie in different orbits of $\mathrm{Diff}(X)$.

On the other hand, using Freedman's work one can see that these two Chern classes *are* related by a homeomorphism of X. In fact, using the 3-torus we can write $H^2(X, \mathbb{Z})$ with its intersection form as a direct sum

$$(H^2(X, \mathbb{Z}), \wedge) \quad = \quad \left(\mathbb{Z}^6, \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)\right) \oplus (V, q),$$

where the Chern classes $c_1(\omega_1), c_1(\omega_2)$ lie in the first factor and are related by an integral automorphism preserving the hyperbolic form. By Freedman's result [FQ, §10.1], this automorphism of $H^2(X, \mathbb{Z})$ is realized by a homeomorphism of X.

Many more examples can be constructed along similar lines. For a simple variation, one can replace L_4 with a geodesic homologous to $L_1 + L_2 + (2m + 1) \cdot L_3$, $m \in \mathbb{Z}$, and replace the elliptic surface $E(1)$ with its n-fold fiber sum, $E(n)$. The manifolds M and X resulting from these variations also satisfy the Theorems above.

3-manifolds	**4-manifolds**
Measured foliations $\mathcal{F}$ of M	Symplectic forms ω on X
Fibrations $M \to S^1$	Integral symplectic forms
Fibers minimize genus	Pseudo-holomorphic curves minimize genus
Euler class $e(T\mathcal{F})$	First Chern class $c_1(\wedge^2_{\mathbb{C}} TX)$
Alexander polynomial $\Delta_M \in \mathbb{Z}[H_1]$	Seiberg–Witten polynomial $\sum \mathrm{SW}(t) \cdot t \in \mathbb{Z}[H^2]$
Alexander norm on $H^1(M, \mathbb{R})$	Seiberg–Witten norm on $H^2(X, \mathbb{R})$

Table 2.

Notes and references. Our examples exploit a dictionary between 3 and 4 dimensions, some of whose entries are summarized in Table 2.

The connection between the Thurston norm and the Seiberg–Witten invariant was developed by Kronheimer and Mrowka in [KM], [Kr2], [Kr1], while the work of Meng–Taubes and Fintushel–Stern brought the Alexander polynomial into play [MeT], [FS1], [FS2], [FS3]. Inasmuch as the Alexander polynomial is tied to the Thurston norm in [Mc2], [Mc1], (see also [Vi]), there is an intriguing circle of ideas here which might be better understood.

2 The Alexander and Thurston norms

In this section we recall the Alexander and Thurston norms for a 3-manifold, and prove that Theorem 1.2 holds for the link complement pictured in the Introduction.

The Thurston norm. Let M be a compact, connected, oriented 3-manifold, whose boundary (if any) is a union of tori. For any compact oriented n-component surface $S = S_1 \sqcup \cdots \sqcup S_n$, let

$$\chi_-(S) = \sum_{\chi(S_i)<0} |\chi(S_i)|.$$

The *Thurston norm* on $H^1(M,\mathbb{Z})$ measures the minimum complexity of a properly embedded surface $(S, \partial S) \subset (M, \partial M)$ dual to a given cohomology class; it is given by

$$\|\phi\|_T = \inf\{\chi_-(S) \,:\, [S] = \phi\}.$$

The Thurston norm extends by linearity to $H^1(M,\mathbb{R})$.

Let $B_T = \{\phi \,:\, \|\phi\|_T \leq 1\}$ denote the unit ball in the Thurston norm; it is a finite polyhedron in $H^1(M,\mathbb{R})$. A basic result is:

Theorem 2.1 *Suppose $\phi_0 \in H^1(M,\mathbb{Z})$ is represented by a fibration $M \to S^1$ with fiber S. Then:*

- *$\|\phi_0\|_T = \chi_-(S)$;*
- *ϕ_0 is contained in the open cone $\mathbb{R}_+ \cdot F$ over a top-dimensional face F of the Thurston norm ball B_T;*
- *every cohomology class in $H^1(M,\mathbb{Z}) \cap \mathbb{R}_+ \cdot F$ is represented by a fibration;*
- *the classes in $H^1(M,\mathbb{R}) \cap \mathbb{R}_+ \cdot F$ are represented by measured foliations; and*

- *the Euler class $e = e(\phi_0) \in H_1(M, \mathbb{Z})$ is dual to the supporting hyperplane to F. More precisely, $\phi(e) = -1$ for all $\phi \in F$.*

In this case we say F is a *fibered face* of the Thurston norm ball. For more details, see [Th2] and [Fr].

The Alexander norm. Next we discuss the Alexander polynomial and its associated norm. Let $G = H_1(M, \mathbb{Z})/(\text{torsion}) \cong \mathbb{Z}^{b_1(M)}$. The *Alexander polynomial* Δ_M is an element of the group ring $\mathbb{Z}[G]$, well-defined up to a unit and canonically determined by $\pi_1(M)$. It can be effectively computed from a presentation for $\pi_1(M)$ (see e.g. [CF]). Writing

$$\Delta_M = \sum_G a_g \cdot g,$$

the *Newton polygon* $N(\Delta_M) \subset H_1(M, \mathbb{R})$ is the convex hull of the set of g such that $a_g \neq 0$. The *Alexander norm* on $H^1(M, \mathbb{R})$ measures the length of the image of the Newton polygon under a cohomology class $\phi : H_1(M, \mathbb{R}) \to \mathbb{R}$; that is,

$$\|\phi\|_A = |\phi(N(\Delta_M))|.$$

From [Mc2] we have:

Theorem 2.2 *If M is a 3-manifold with $b_1(M) \geq 2$, then we have*

$$\|\phi\|_A \leq \|\phi\|_T$$

for all $\phi \in H^1(M, \mathbb{R})$; and equality holds if ϕ is represented by a fibration $M \to S^1$.

Links in the 3-torus. We now turn to the Thurston norm for link-complements in the 3-torus. Let $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ denote the flat Euclidean 3-torus. Every nonzero cohomology class $\phi \in H^1(T^3, \mathbb{Z})$ is represented by a fibration (indeed, a group homomorphism) $\Phi : T^3 \to S^1$.

Consider an n-component link $L \subset T^3$, consisting of disjoint, oriented, closed geodesics $L_1 \cup \cdots \cup L_n$. Define a norm on $H^1(T^3, \mathbb{R})$ by

$$\|\phi\|_L = \sum |\phi(L_i)|, \tag{2.1}$$

where the L_i are considered as elements of $H_1(M, \mathbb{Z})$. Let M be the link complement $T^3 - \mathcal{N}(L)$, equipped with the natural inclusion $M \subset T^3$.

Theorem 2.3 *Given* $\phi \in H^1(T^3, \mathbb{Z})$, *let* ψ *denote its pullback to* $M = T^3 - \mathcal{N}(L)$. *Then we have:*

$$\|\phi\|_L = \|\psi\|_T = \|\psi\|_A. \tag{2.2}$$

Moreover:

(a) ψ *is represented by a fibration* $\Psi : M \to S^1 \iff$

(b) $\phi(L_i) \neq 0$ *for all* $i \iff$

(c) ϕ *belongs to the open cone over a top-dimensional face of the norm ball* $B_L = \{\phi \,:\, \|\phi\|_L \le 1\} \subset H^1(T^3, \mathbb{R})$.

Proof. We begin by showing (a-c) are equivalent. If ψ is represented by a fibration $\Psi : M \to S^1$, then the fibers are transverse to ∂M and thus $\phi(L_i) \neq 0$ for all i. On the other hand, the latter condition insures that the linear fibration $\Phi : T^3 \to S^1$ associated to ϕ restricts to a fibration of M representing ψ, so we have (a) $\iff$ (b). Finally $\|\phi\|_L$ behaves linearly on $H^1(T^3, \mathbb{R})$ unless one of the terms $\phi_i(L)$ changes sign, and thus the cone on the top dimensional faces is exactly the locus where $\phi(L_i) \neq 0$ for all i, showing (b) $\iff$ (c).

To establish equation (2.2), first suppose ψ is represented by a fibration $\Psi : M \to S^1$ with fiber S. Since we may take $\Psi = \Phi|M$, we see S is a union of tori with $\sum |\phi(L_i)|$ punctures, and thus

$$\chi_-(S) = \|\psi\|_T = \sum |\phi(L_i)| = \|\phi\|_L.$$

Equality with the Alexander norm holds by Theorem 2.2.

Thus (2.2) holds on the cone over the top-dimensional faces of B_L. Since this cone is dense, (2.2) holds throughout $H^1(T^3, \mathbb{Z})$ by continuity. ∎

The Borromean rings plus axis. We now turn to the study of the 4-component link $K \subset S^3$ pictured in Figure 1. Let $M = S^3 - \mathcal{N}(K)$, and let m_i denote the meridian linking K_i positively. Then (m_1, m_2, m_3, m_4) forms a basis for $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^4$, and the Alexander polynomial Δ_M can be written as a Laurent polynomial in these variables.

Lemma 2.4 *The Alexander polynomial of* $M = S^3 - \mathcal{N}(K)$ *is given by*

$$\begin{aligned}\Delta_M(x,y,z,t) \;=\; & -4 + \left(t + \frac{1}{t}\right) - \left(xy + \frac{1}{xy} + yz + \frac{1}{yz} + xz + \frac{1}{xz}\right) \\ & + \left(xyz + \frac{1}{xyz}\right) + \left(x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z}\right),\end{aligned}$$

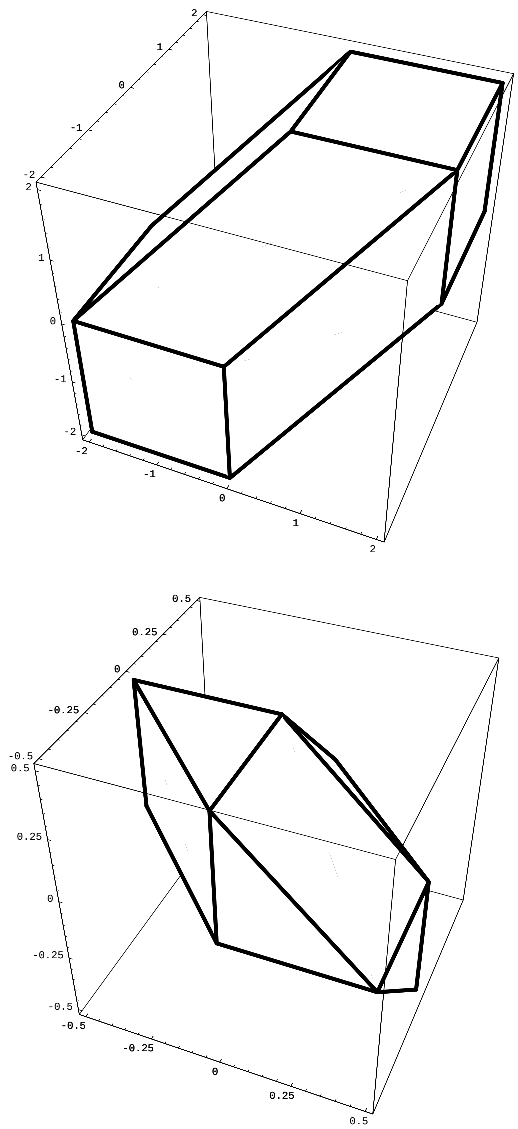

Figure 3. The Newton polygon of $\Delta_M(x, y, z, 1)$ (top), and its dual.

where $(x, y, z, t) = (m_1, m_2, m_3, m_4)$.

Proof. The projection in Figure 1 yields the Wirtinger presentation

$$\begin{aligned} \pi_1(M) \;\; = \;\; & \langle a, b, c, d, e, f, g, h, i, j, k, l \; : \\ & aj = jb, bi = ic, gc = ag, dc = ce, ae = fa, fj = jd, \\ & ge = eh, hj = ji, di = gd, jg = gk, kc = cl, le = ej \rangle. \end{aligned}$$

Here (a, b, c), (d, e, f), (g, h, i) and (j, k, l) are the edges of K_1, K_2, K_3 and K_4 respectively. Given this presentation, the calculation of Δ_M is a straightforward application of the Fox calculus [Fox]. ∎

Figure 3 shows the intersection of the Newton polygon $N(\Delta_M)$ with the (x, y, z)-hyperplane.

To bring the 3-torus into play, recall that 0-surgery along the Borromean rings determines a diffeomorphism

$$S^3 - \mathcal{N}(K_1 \cup K_2 \cup K_3) \cong T^3 - \mathcal{N}(L_1 \cup L_2 \cup L_3),$$

where (L_1, L_2, L_3) are disjoint closed geodesics forming a basis for $H_1(T^3, \mathbb{Z})$. Under this surgery, the meridians (m_1, m_2, m_3) go over to longitudes of (L_1, L_2, L_3). On the other hand, K_4 goes over to the isotopy class of a geodesic $L_4 \subset T^3$, with

$$L_4 = L_1 + L_2 + L_3 \quad \text{in } H_1(T^3, \mathbb{Z}).$$

(To check the homology class of L_4, note that in S^3 we have $\mathrm{lk}(K_i, K_4) = 1$ for $i = 1, 2, 3$.)

The meridian m_4 goes over to a meridian of L_4, so unlike (m_1, m_2, m_3) it becomes trivial in $H_1(T^3, \mathbb{Z})$. Thus we have:

$$H^1(M, \mathbb{R}) \supset H^1(T^3, \mathbb{R}) = (\mathbb{R} \cdot m_4)^{\perp}.$$

Lemma 2.5 *The action of* $\mathrm{Diff}(M)$ *on* $H^1(M, \mathbb{R})$ *preserves the subspace* $H^1(T^3, \mathbb{R})$.

Proof. Consider the Newton polygon

$$N = N(\Delta_M) \subset H_1(M, \mathbb{R}),$$

where Δ_M is given by Proposition 2.4. Since $(t + 1/t)$ is the only expression in Δ_M involving t, we have $N = N_0 + [-1, 1] \cdot t$ where

$$N_0 = N(\Delta_M(x, y, z, 1))$$

is the polyhedron in (x, y, z)-space shown in Figure 3. The vertices $\pm t$ of N are thus combinatorially distinguished: they are the endpoints of 14 edges of N (coming from the 14 vertices of N_0), whereas all other vertices of N have degree 5. Since $\mathrm{Diff}(X)$ preserves N, it also stabilizes the special vertices $\{\pm t\}$, and thus $\mathrm{Diff}(X)$ stabilizes $H^1(T^3, \mathbb{R}) = (\mathbb{R} \cdot t)^\perp = (\mathbb{R} \cdot m_4)^\perp$. ∎

Proof of Theorem 1.2. For our chosen link $L \subset T^3$, we have

$$\|\phi\|_L = |\phi(m_1)| + |\phi(m_2)| + |\phi(m_3)| + |\phi(m_1 + m_2 + m_3)|.$$

The unit ball $B_L \subset H^1(T^3, \mathbb{R})$ of this norm is shown in Figure 3 (bottom); it is dual to the convex body N_0.

Note that B_L has both triangular and quadrilateral faces. Pick integral classes $\phi_0, \phi_1 \in H^1(T^3, \mathbb{Z})$ lying inside the cones over faces F_0 and F_1 of different types, and let $\alpha_0, \alpha_1 \in H^1(M, \mathbb{Z})$ denote their pullbacks to M.

By Theorem 2.3, the classes α_0 and α_1 correspond to fibrations $M \to S^1$. On the other hand, $\mathrm{Diff}(M)$ preserves the subspace $H^1(T^3, \mathbb{R}) \subset H^1(M, \mathbb{R})$ as well as the norm $\|\phi\|_L = \|\alpha\|_T$ on this subspace. Thus $\mathrm{Diff}(M)$ preserves B_L, so it cannot send the face F_0 to F_1. The supporting hyperplanes for α_0 and α_1 in B_T thus lie in different orbits of $\mathrm{Diff}(M)$. But these supporting hyperplanes are represented by $e(\alpha_0)$ and $e(\alpha_1)$, so their Euler classes are in different orbits as well. ∎

The Thurston norm. As was shown in [Mc2], the Alexander and Thurston norms agree for many simple links. The norms agree for the Borromean rings plus axis $K \subset S^3$ as well.

To see this, note that K can be presented as the closure of a 3-strand braid wrapping once around the axis $K_4 \subset K$. A disk spanning K_4 and transverse to $K_1 \cup K_2 \cup K_3$ determines a fibered face F of the Thurston norm ball B_T. As observed by N. Dunfield, one can use the Teichmüller polynomial [Mc1] to show that for any 3-strand braid, the fibered face F coincides with a face of the Alexander norm ball B_A. In the example at hand, all the vertices of B_A are contained in $\pm F$, so we have $B_A \subset B_T$ by convexity. The reverse inclusion comes from the general inequality $\|\phi\|_A \le \|\phi\|_T$.

Further example: a closed 3-manifold. To conclude, we describe a *closed* 3-manifold N which fibers over the circle in two inequivalent ways.

Let $M = T^3 - \mathcal{N}(L) = S^3 - \mathcal{N}(K)$ be the link complement considered above. Note that the longitudes of K_1, K_2 and K_3 are all homologous to the meridian m_4 of K_4, since the components of the Borromean rings are unlinked, while each component links K_4 once. Since T^3 is obtained by 0-surgery on K, all the meridians of L are homologous to m_4.

Now let $N \to T^3$ be the 2-fold covering, branched over L, determined by the homomorphism

$$\xi : H_1(M, \mathbb{Z}) \to \{-1, 1\}$$

satisfying $\xi(m_1) = \xi(m_2) = \xi(m_3) = 1$ and $\xi(m_4) = -1$.

The pullback map $H^1(T^3, \mathbb{R}) \to H^1(N, \mathbb{R})$ is easily seen to be injective. We claim it is an isomorphism. To see surjectivity, let $N' \subset N$ be the preimage of $M \subset T^3$. Decomposing $H^1(N', \mathbb{R})$ into eigenspaces for the action of the $\mathbb{Z}/2$ deck group for $N' \to M$, we obtain an isomorphism

$$H^1(N', \mathbb{R}) \cong H^1(M, \mathbb{R}) \oplus H^1(M, \mathbb{R}_\xi),$$

where the last term represents cohomology coefficients twisted by the character ξ of $\pi_1(M)$. Since $\Delta_M(\xi) = \Delta_M(1, 1, 1, -1) = 4 \neq 0$, we have $H^1(M, \mathbb{R}_\xi) = 0$ (cf. [Mc2, §3]). Thus any cohomology class in $H^1(N, \mathbb{R})$ restricts to a $\mathbb{Z}/2$-invariant class on N', so it is the pullback of a class on T^3.

Moreover, every fibration of T^3 transverse to L lifts to a fibration of N, so we find:

Theorem 2.6 *The Thurston norm ball $B_T \subset H^1(N, \mathbb{R})$ agrees with the norm ball $B_L \subset H^1(T^3, \mathbb{R})$, and every face is fibered.*

Picking fibrations in combinatorially inequivalent faces of B_T as before, we have:

Corollary 2.7 *The closed 3-manifold N admits a pair of fibrations α_0, α_1 such that $e(\alpha_0), e(\alpha_1)$ lie in disjoint orbits for the action of* $\mathrm{Diff}(N)$ *on $H^2(N, \mathbb{Z})$.*

3 Fiber sum and symplectic 4-manifolds

In this section we recall the fiber sum construction, which can be used to canonically associate a 4-manifold $X = X(P, L)$ to a link L in a 3-manifold P. Under this construction, suitable fibrations of P give symplectic forms on $X(P, L)$, and the Alexander polynomial Δ_M of $M = P - \mathcal{N}(L)$ determines Seiberg–Witten invariants of X. It is then straightforward to prove Theorem 1.1 by taking $X = X(T^3, L)$, where $L \subset T^3$ is the 4-component link discussed in previous sections.

Fiber sum. Let $f_i : T^2 \times D^2 \to X_i$, $i = 1, 2$ be smooth embeddings of the torus cross a disk into a pair of smooth closed 4-manifolds. Let

$$X_i' = X_i - f(T^2 \times \operatorname{int} D^2);$$

it is a smooth manifold whose boundary is marked by $T^2 \times S^1$. The *fiber sum* Z of X_1 and X_2 is the closed smooth manifold obtained by gluing together X_1' and X_2' along their boundaries, such that $(x, t) \in \partial X_1'$ is identified with $(x, -t) \in \partial X_2'$. We denote the fiber sum by

$$Z = X_1 \underset{T_1 = T_2}{\#} X_2,$$

where $T_i = f(T^2 \times \{0\}) \subset X_i$; note that there is an implicit identification between the normal bundles of the tori T_i.

The fiber sum of symplectic manifolds along symplectic tori is also symplectic. More precisely, if ω_i are symplectic forms on X_i with $\omega_i > 0$ on T_i and $\int_{T_1} \omega_1 = \int_{T_2} \omega_2$, then Z carries a natural symplectic form ω with $\omega = \omega_i$ on X_i'.

For more details, see [Go], [MW], [FS1], [FS2], [FS3].

The elliptic surface $E(1)$. A convenient 4-manifold for use in the fiber-sum construction is the *rational elliptic surface* $E(1)$. The complex manifold $E(1)$ is obtained by blowing up the base-locus for a generic pencil of elliptic curves on $\mathbb{CP}^2$. Thus $E(1)$ is isomorphic to $\mathbb{CP}^2 \# 9 \overline{\mathbb{CP}^2}$; it is simply-connected and unique up to diffeomorphism. The pencil provides a holomorphic map $E(1) \to \mathbb{CP}^1$ with generic fiber F an elliptic curve, and the canonical bundle of $E(1)$ is represented by the divisor $-F$.

The projection $E(1) \to \mathbb{CP}^1$ gives a natural trivialization of the normal bundle of the fiber torus F. Since $F \subset E(1)$ is a holomorphic curve in a projective variety, there is a symplectic (Kähler) form on $E(1)$ with $\omega | F > 0$.

Each of the nine exceptional divisors gives a holomorphic section

$$s : \mathbb{P}^1 \to E(1).$$

In particular, a meridian for the fiber F is contractible in $E(1) - \mathcal{N}(F)$, since it bounds the image of a disk under s. Since $E(1)$ is simply-connected, any loop in the complement of F is homotopic to a product of conjugates of meridians, so $E(1) - \mathcal{N}(F)$ is also simply-connected.

For a detailed discussion of the topology of elliptic surfaces, see [HKK, §1] or [GS].

From links to 4-manifolds. Now let $L \subset P^3$ be a framed n-component link in a closed, oriented 3-manifold. Such a link determines:

- a 3-dimensional *link complement* $M = P - \mathcal{N}(L)$, and
- a 4-dimensional *fiber-sum* $X = X(P, L) = (P \times S^1) \underset{L \times S^1 = nF}{\#} nE(1)$.

To describe the fiber-sum in more detail, note that each component L_i of L determines a torus

$$T_i = L_i \times S^1 \subset P \times S^1,$$

and the framing of L_i provides a trivialization of the normal bundle of T_i. Take n copies of the elliptic surface $E(1)$ with fiber F; as remarked above, the projection $E(1) \to \mathbb{CP}^1$ provides a natural trivialization of the normal bundle of F. Finally, choose an orientation-preserving identification between $L \times S^1$ and nF. The fiber-sum $X(P, L)$ is then defined using these identifications.

It turns out that every orientation-preserving diffeomorphism of F extends to a diffeomorphism of $E(1)$, preserving the normal data; indeed, the monodromy of the fibration $E(1) \to \mathbb{CP}^1$ is the full group $SL_2(\mathbb{Z})$. Thus the diffeomorphism type of $X(P, L)$ is the same for any choice of identification between $L \times S^1$ and nF.

Proposition 3.1 *The fiber-sum X is simply-connected if $\pi_1(M)$ is normally generated by $\pi_1(\partial M)$ (e.g. if M is homeomorphic to a link complement in S^3).*

Proof. When the simply-connected manifolds $n(E(1) - \mathcal{N}(F))$ are attached to $M \times S^1$ along $\partial M \times S^1$, they kill $\pi_1(\partial M \times S^1)$ by van Kampen's theorem. Since the latter groups normally generate $\pi_1(M \times S^1)$, the resulting manifold X is simply-connected. ∎

Promotion of cycles. The fiber-sum construction furnishes us with an inclusion $M \times S^1 = (P \times S^1)' \subset X$.

Proposition 3.2 *The map*

$$i : H_1(M, \mathbb{R}) \to H^2(X, \mathbb{R}),$$

sending a 1-cycle $\gamma \subset M$ to the Poincaré dual of $\gamma \times S^1 \subset X$, is injective.

Proof. The map i is a composition of three maps:

$$H_1(M) \to H_2(M \times S^1) \to H_2(X) \to H^2(X).$$

The first arrow is part of the Künneth isomorphism, and the last comes from Poincaré duality, so they are both injective. As for the middle arrow

$$H_2(M \times S^1) \to H_2(X),$$

we can use the exact sequence of the pair $(X, M \times S^1)$ to identify its kernel with

$$H_3(X, M \times S^1) \cong H_3(nE(1), nF) \cong H^1(nE(1) - nF) = 0.$$

Here we have used excision, Poincaré duality and the simple-connectivity of $E(1) - F$. Thus all three arrows are injective, and so i is injective. ■

Corollary 3.3 *For an n-component link, we have*

$$b_2^+(X(P, L)) \geq b_1(M) \geq n.$$

Here $b_2^+(X)$ denotes the rank of the maximal subspace of $H_2(X, \mathbb{R})$ on which the intersection form is positive-definite.

Proof. Since 1-cycles in general position on M are disjoint, the intersection form on $H^2(X, \mathbb{R})$ restricts to zero on $i(H_1(M, \mathbb{R}))$. But the intersection form is non-degenerate, so it must admit a positive (and negative) subspace of dimension at least $b_1(M) = \dim i(H_1(M, \mathbb{R}))$.

For the second inequality, just note that we have $b_1(M) \geq b_1(\partial M)/2 = n$. Indeed, by Lefschetz duality, the kernel of $H_1(\partial M) \to H_1(M)$ is Lagrangian, so the image has dimension n. ■

From fibrations to symplectic forms. A central point for us is that suitable fibrations α of P give rise to symplectic structures ω on $X(P, L)$.

Theorem 3.4 *For any fibration $\alpha \in H^1(P, \mathbb{Z})$ transverse to L, there is a symplectic form ω on $X(P, L)$ with*

$$c_1(\omega) \quad = \quad i(e(\alpha|M)).$$

Proof. Let $\alpha = d\pi$ be the closed 1-form representing a fibration $\pi : P \to S^1$ transverse to L.

Pick a closed 2-form β on M such that β restricts to an area form on each leaf of $\mathcal{F}$. (One can construct such a form by representing the monodromy of the fibration by an area-preserving map.) As observed by Thurston, for $\epsilon > 0$ sufficiently small, the closed 2-form

$$\omega_0 = \alpha \wedge dt + \epsilon\beta$$

is a symplectic form on $P \times S^1$, nowhere vanishing on $L \times S^1$ [Th1]. (Here $[dt]$ is the standard 1-form on $S^1 = \mathbb{R}/\mathbb{Z}$, and α and β have been pulled back to the product).

By scaling the Kähler form, we can provide the ith copy of $E(1)$ with a symplectic form ω_i such that $\int_F \omega_i = \int_{L_i \times S^1} \omega$. Then as mentioned above, ω_0 and (ω_i) joined together under fiber-sum to yield a symplectic form ω on X.

Let $K \to X$ denote the canonical bundle of (X, ω). We will compute $c_1(K)$ by constructing a section $\sigma : X \to K$.

Let $M = P - \mathcal{N}(L)$. As an oriented $\mathbb{R}^2$-bundle, $K|(M \times S^1)$ is isomorphic to the pullback of $T\mathcal{F}$ from M. Let $s : M \to T\mathcal{F}$ be a section such that $s|\partial M$ is inward pointing and nowhere vanishing. Then the zero set of s is a 1-cycle γ representing the Euler class $e(\alpha|M) \in H_1(M, \mathbb{R})$. Pulling back s, we obtain a section $\sigma_0 : M \times S^1 \to K$ with zero set $\gamma \times S^1$.

Now consider the 4-manifold $E(1)' = E(1) - \mathcal{N}(F)$ attached to $M \times S^1$ along $T_i \times S^1$. If we have $\omega_i(F) > 0$, then $K|E(1)'$ is just the pullback of the canonical bundle of $E(1)$. Since $-F$ is a canonical divisor on $E(1)$, there is a nowhere vanishing section $\sigma_i : E(1)' \to K$, namely the restriction of a meromorphic 2-form on $E(1)$ with divisor $-F$.

We claim σ_0 and σ_i fit together under the gluing identification between $T_i \times S^1$ and $F \times S^1$. To check this, we use the framings to identify $K|T_i \times S^1$ and $K|F \times S^1$ with the trivial bundle over $T^2 \times S^1$. Under this identification,

$$\sigma_0 : T^2 \times S^1 \to \mathbb{C}^*$$

is homotopic to the projection $T^2 \times S^1 \to S^1 \subset \mathbb{C}^*$, since the vector field $s|T_i$ runs along the meridians of ∂M. Similarly,

$$\sigma_i : T^2 \times S^1 \to \mathbb{C}^*$$

is homotopic to $1/\sigma_0$, because of the simple pole along F. Since $T_i \times S^1$ is identified with $F \times S^1$ using the involution $(x,t) \sim (x,-t)$ on $T^2 \times S^1$, the two sections correspond under gluing.

In the case where we have $\omega_i(F) < 0$, both homotopy classes are reversed, so σ_0 and σ_i still agree under gluing. Thus σ_0 and (σ_i) join together to form a global section $\sigma : X \to K$ with no zeros outside $M \times S^1$. It follows that $c_1(X, \omega)$ is Poincaré dual to $\gamma \times S^1$; equivalently, that $c_1(\omega) = i(\alpha|M)$. ■

The Seiberg–Witten polynomial. A central feature of the fiber-sum $X = X(P, L)$ is that its Seiberg–Witten polynomial is directly computable.

Assume that X is simply-connected and $b_2^+(X) > 1$. Then the Seiberg–Witten invariant of X can be regarded as a map

$$\mathrm{SW} : H^2(X, \mathbb{Z}) \to \mathbb{Z},$$

well-defined up to a sign and vanishing outside a finite set. This information is conveniently packaged as a Laurent polynomial

$$\mathcal{SW}_X \;=\; \sum_t \mathrm{SW}(t) \cdot t \in \mathbb{Z}[H^2(X, \mathbb{Z})].$$

Theorem 3.5 *Suppose M is the complement of an n-component link $L \subset P$, and $\pi_1(\partial M)$ normally generates $\pi_1(M)$. Then $X = X(P, L)$ is simply-connected, we have $b_2^+(X) \geq n$, and*

$$\mathcal{SW}_X = \pm \sum a_t \cdot i(2t),$$

where $\Delta_M = \sum a_t \cdot t$ is the symmetrized Alexander polynomial of M.

Remarks. This Theorem was established by Fintushel and Stern in the special case where (P, L) is obtained by a certain surgery on a link in S^3 [FS2, Thm. 1.9].[1] To obtain the symmetrized Alexander polynomial, one multiplies $\Delta_K(t)$ by a monomial to arrange that its Newton polygon is centered at the origin. The exponents in the symmetrized polynomial may be half-integral.

Proof. To compute $\mathcal{SW}_X$, we regard X as the union of manifolds $X_0 = M \times S^1$ and $X_i = E(1) - \mathcal{N}(F)$, $i = 1, \ldots, n$, glued together along their boundary. For such manifolds one can define a *relative* Seiberg–Witten polynomial $\mathcal{SW}_{X_i} \in \mathbb{Z}[H^2(X_i, \partial X_i; \mathbb{Z})$, such that

$$\mathcal{SW}_X = \mathcal{SW}_{X_0} \cdot \mathcal{SW}_{X_1} \cdots \mathcal{SW}_{X_n},$$

using the natural map $H^2(X_i, \partial X_i) \to H^2(X)$ to compute the product. For this gluing formula, developed by Morgan, Mrowka, Szabo and Taubes, see [FS2, Thm. 2.2] and [Ta].

Now for each $X_i = E(1) - \mathcal{N}(F)$, the relative polynomial is simply 1. To see this, just apply the product formula above to the K3 surface

[1] Note: contrary to [FS2, p. 371]: the cohomology classes $[T_j]$ in their formula for $\mathcal{SW}_X$ are always linearly independent in $H^2(X, \mathbb{R})$, by Proposition 3.2 above.

$Z = E(1)\#_F E(1)$, which satisfies $\mathrm{SW}_Z = 1$. (This well-known property of K3 surfaces follows, for example, from equations (4.17) and (4.20) in Witten's original paper [Wit].)

Thus we have $\mathcal{SW}_X = \mathcal{SW}_{X_0} = \mathcal{SW}_{M \times S^1}$. Finally the Seiberg-Witten polynomial for $M \times S^1$ is given in terms of Δ_M by the main result of [MeT], yielding the formula for $\mathcal{SW}_X$ above.

To see $\pi_1(X) = \{1\}$ and $b_2^+(X) \geq n$, apply Proposition 3.1 and Corollary 3.3 above. ∎

Proof of Theorem 1.1. Using the Seiberg–Witten invariants to control the action of $\mathrm{Diff}(X)$, it is now easy to give an example of a simply-connected 4-manifold X with inequivalent symplectic forms.

For a concrete example, let $X = X(T^3, L)$ for the 4-component link $L \subset T^3$ studied in the preceding section, and choose any framing of L. As we have seen, the link-complement $M = T^3 - \mathcal{N}(L)$ is homeomorphic to the exterior $S^3 - \mathcal{N}(K)$ of the Borromean rings plus axis. In particular, $\pi_1(M)$ is the normal closure of $\pi_1(\partial M)$, so X is simply-connected and we have $b_2^+(X) \geq 4$.

Let m_i, $i = 1, \ldots, 4$ be the basis for $H_1(M, \mathbb{Z})$ coming from the meridians of $K \subset S^3$. Then the classes $t_i = i(m_i)$ form a basis for $i(H_1(M, \mathbb{Z})) \subset H^2(X, \mathbb{Z})$. By Theorem 3.5, we have:

> *The Seiberg–Witten polynomial of X is given by*
>
> $$\mathcal{SW}_X = \Delta_M(t_1^2, t_2^2, t_3^3, t_4^2),$$
>
> *where $\Delta_M(x, y, z, t)$ is given by Lemma 2.4.*

In particular, the Newton polygons satisfy $N(\mathcal{SW}_X) = 2i(N(\Delta_M))$.

Now identify $H_1(T^3, \mathbb{R})$ with the subspace of $H_1(M, \mathbb{R})$ spanned by (m_1, m_2, m_3), and let

$$N_0 = N(\Delta_M) \cap H_1(T^3, \mathbb{R}).$$

As we have seen before, any vertex v of N_0 is dual to a fibered face F of the Thurston norm on $H^1(M, \mathbb{R})$; indeed, v is dual to a fibration pulled by from T^3. All fibrations ϕ in the cone over F have the same Euler class e, which satisfies

$$\|\phi\|_T = 2\phi(v) = -\phi(e);$$

thus $e = -2v$.

By Theorem 3.4, the vertex

$$i(e) = i(-2v) \in 2i(N_0)$$

is the first Chern class of a symplectic structure on X. Since $v \in N_0$ was an arbitrary vertex, we have:

> *Every vertex of $2i(N_0) \subset N(\mathcal{SW}_X)$ is the first Chern class of a symplectic structure on X.*

Now pick a pair combinatorially distinct vertices

$$v_0, v_1 \in 2i(N_0) \subset N(\mathcal{SW}_X).$$

More precisely, referring to Figure 3 (top), we see $2i(N_0)$ has vertices of degrees 3 and 4; choose one of each type. Then v_0 and v_1 have degrees 5 and 6 as vertices of $N(\mathcal{SW}_X)$, since

$$N(\mathcal{SW}_X) \;\; = \;\; 2i(N_0) + [-2, 2] \cdot t_4$$

is simply the suspension of $2i(N_0)$. As a consequence, no automorphism of $H^2(X, \mathbb{R})$ stabilizing $N(\mathcal{SW}_X)$ can transport v_0 to v_1.

To complete the proof, choose symplectic forms on X with $c_1(\omega_0) = v_0$ and $c_1(\omega_1) = v_1$. Then the Chern classes of ω_0 and ω_1 lie in distinct orbits for the action of $\mathrm{Diff}(X)$ on $H^2(X, \mathbb{R})$, since diffeomorphisms preserve the Newton polygon of the Seiberg-Witten polynomial. In particular, ω_0 and ω_1 are inequivalent symplectic forms on X. ■

Question. Could it be that $\mathrm{Diff}(X)$ actually preserves the submanifold $M \times S^1 \subset X$ up to isotopy?

Further example: skirting gauge theory. To conclude, we sketch an *elementary* example of a 4-manifold X carrying a pair of inequivalent symplectic forms — but with $\pi_1(X) \neq 1$. By elementary, we mean the proof does not use the Seiberg–Witten invariants; instead, it uses the fundamental group.

To construct the example, simply let $X = N \times S^1$, where N is the closed 3-manifold discussed at the end of §2.

By considering N as a covering of T^3 with a $\mathbb{Z}/2$-orbifold locus along L, one can show that $\pi_1(N)$ has trivial center. It follows that $\pi_1(S^1)$ is the center of $\pi_1(X)$, and thus the projection

$$\pi_1(X) \to \pi_1(N)$$

is canonical. In particular, every diffeomorphism of X induces an automorphism of $\pi_1(N)$.

Now let α_0, α_1 be fibrations of N whose Euler classes are in different orbits for the action of $\mathrm{Aut}(\pi_1(N))$ on $H_1(N, \mathbb{Z})$. (These classes exist as before, because the Alexander polynomial is functorially determined by $\pi_1(N)$, and hence preserved by automorphisms.) Then the Euler classes $e(\alpha_0), e(\alpha_1)$ lie in disjoint orbits for the action of $\mathrm{Diff}(X)$ on $H_1(N) = H_1(X)/H_1(S^1)$.

Now as we have seen above, each α_i gives a symplectic form ω_i on X with $c_1(\omega_i)$ dual to $e(\alpha_i) \times S^1$. Since the Euler classes lie in disjoint orbits for the action of $\mathrm{Diff}(X)$, so do these Chern classes. In particular, ω_0 and ω_1 are inequivalent symplectic forms on X. ■

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