

# Lipschitz maps and nets in Euclidean space

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## 1 Introduction

In this paper we discuss the following three questions.

1. Given a real-valued function  $f \in L^\infty(\mathbb{R}^n)$  with  $\inf f(x) > 0$ , is there a bi-Lipschitz homeomorphism  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that the Jacobian determinant  $\det D\phi = f$ ?
2. Given  $f \in L^\infty(\mathbb{R}^n)$ , is there a Lipschitz or quasiconformal vector field with  $\operatorname{div} v = f$ ?
3. Given a separated net  $Y \subset \mathbb{R}^n$ , is there a bi-Lipschitz map  $\phi : Y \rightarrow \mathbb{Z}^n$ ?

When  $n = 1$  all three questions have an easy positive answer. In this paper we show that for  $n > 1$  the answer to all three questions is *no*. We also find all three questions have positive solutions if the Lipschitz condition is relaxed to a Hölder condition.

**Definitions.** A map  $\phi$  is *bi-Lipschitz* if there is a constant  $K$  such that

$$\frac{1}{K} < \frac{|\phi(x) - \phi(x')|}{|x - x'|} < K$$

for  $x \neq x'$ . A set  $Y \subset \mathbb{R}^n$  is a *net* if there is an  $R$  such that  $d(x, Y) < R$  for every  $x \in \mathbb{R}^n$ ; it is *separated* if there is an  $\epsilon > 0$  such that  $|y - y'| > \epsilon > 0$  for every pair  $y \neq y'$  in  $Y$ .

**History.** In 1965, J. Moser showed that any two positive,  $C^\infty$  volume forms on a compact manifold with the same total mass are related by a diffeomorphism [Mos]. Extensions of this result to other smoothness classes such as  $C^{k,\alpha}$  were given in [Rei1] and [DM]; see also [RY1], [RY2], and [Ye].

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Questions (1) and (2) remained open. Question (3) was posed in Gromov's 1993 book [Gr, p.23], and popularized by Toledo's review [Tol].

Recently counterexamples to (1) and (3) were discovered independently by Burago and Kleiner [BK], and the author. Here we show the linearized question (2) can be settled using a 1962 result of Ornstein (§2). The counterexample to (2) suggests the right type of  $f$  to make a counterexample to (1), as we sketch in §3. This  $f \in L^\infty$  is similar to the one constructed in [BK], to which we refer for a detailed resolution of (1). In §4 we show questions (1) and (3) are equivalent, completing the discussion of Lipschitz mappings. Finally in §5 we show questions (1-3) have positive answers in the Hölder category.

## 2 Vector fields

We begin with the infinitesimal form of the problem of constructing a map with prescribed volume distortion. That is, we study the equation

$$\operatorname{div} v = \sum \frac{\partial v_i}{\partial x_i} = f$$

on  $\mathbb{R}^n$ , where  $f$  is a real-valued function and  $\operatorname{div} v$  is the divergence of the vector field  $v$ . We will show:

**Theorem 2.1** *For any  $n > 1$  there is an  $f \in L^\infty(\mathbb{R}^n)$  which is not the divergence of any Lipschitz, or even quasiconformal, vector field.*

**Definitions.** Let  $D = (\partial/\partial x_i)$ ; then the matrix of partial derivatives of a vector field  $v$  is given by the outer product

$$(Dv)_{ij} = \frac{\partial v_i}{\partial x_j},$$

and  $\operatorname{div} v = \operatorname{tr}(Dv)$ . Similarly, letting

$$(D^2)_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},$$

we have  $\Delta f = \operatorname{tr}(D^2 f)$ .

A vector field  $v$  is *quasiconformal* if the distribution  $Sv$  lies in  $L^\infty$ , where the *conformal strain*

$$Sv = \frac{1}{2}(Dv + (Dv)^*) - \frac{1}{n}(\operatorname{tr} Dv)I$$

is the symmetric, trace-free part of  $Dv$ . Explicitly,

$$(Sv)_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{1}{n} \sum_k \frac{\partial v_k}{\partial x_k}.$$

Any Lipschitz vector field is quasiconformal.

Quasiconformal vector fields with  $\operatorname{div} v \in L^\infty$  are more general than Lipschitz vector fields, but they provide good models for infinitesimal bi-Lipschitz maps. For example,  $v(z) = iz \log |z|$  is not Lipschitz, but it generates a Lipschitz isotopy of the plane (shearing along circles). Theorem 2.1 states that even this broader class of quasiconformal vector fields is insufficient to solve  $\operatorname{div} v = f$ . (Further discussion of quasiconformal flows can be found in [Rei2] and [Mc2, Appendix A].)

**Singular integral operators.** Before proving Theorem 2.1, we mention how it fits into the general theory of singular integral operators and PDE.

Suppose  $f \in C_0^\infty(\mathbb{R}^n)$  and  $\int f = 0$ . The most straightforward solution to  $\operatorname{div} v = f$  is given by  $v = Du$ , the gradient of the solution to Laplace's equation  $\Delta u = f$ . The regularity of  $v$  is thus determined by the behavior of the operator

$$Tf = Dv = D^2 \Delta^{-1} f.$$

For example  $v$  is Lipschitz iff  $Dv = Tf \in L^\infty$ .

The operator  $T$  is a singular integral operator of Calderón-Zygmund type, whose kernel is obtained by differentiating a fundamental solution to Laplace's equation. By the general theory of such operators,  $T$  sends  $L^p$  into  $L^p$  for  $1 < p < \infty$ , but it does not preserve  $L^\infty$  or  $L^1$ .

In the case at hand, where  $f$  is in  $L^\infty$ , one can say at most that  $Dv = Tf \in BMO$  with

$$\|Dv\|_{BMO} \leq C_n \|f\|_\infty$$

(see [St, IV.4.1]). Just as vector fields with  $Dv \in L^\infty$  are Lipschitz, those with  $Dv \in BMO$  satisfy the Zygmund condition

$$\|v\|_Z = \sup_{x, y \in \mathbb{R}^n, y \neq 0} \frac{|v(x+y) + v(x-y) - 2v(x)|}{|y|} < \infty$$

(see [Mc2, Thm. A.2]). It follows that  $v$  has an  $|x \log x|$  modulus of continuity, so while  $v$  is generally not Lipschitz it is Hölder of every exponent  $\alpha < 1$ .

On the other hand, a solution to  $\operatorname{div} v = f$  is only determined up to a volume-preserving vector field  $w$ , so another solution  $v + w$  might be Lipschitz even if  $v$  is not.

To handle the kernel of the divergence operator, one is lead to argue by duality. Theorem 2.1 then reduces to a problem in  $L^1$ , which is settled by the following:

**Theorem 2.2 (Ornstein)** *For any set of linearly independent degree  $m$  differential operators on  $\mathbb{R}^n$ ,*

$$P_i = \sum_{|\alpha|=m} a_i^\alpha \frac{\partial^\alpha}{\partial x^\alpha}, \quad i = 0, \dots, k,$$

and any  $C > 0$ , there exists an  $g \in C_0^\infty(\mathbb{R}^n)$  such that

$$\|P_0 g\|_1 > C \sum_1^k \|P_i g\|_1.$$

See [Or]; we are grateful to E. Stein for this reference.

**Proof of Theorem 2.1.** The proof is by contradiction.

Suppose for every  $f \in L^\infty(\mathbb{R}^n)$  there exists a quasiconformal vector field  $v$  such that  $\operatorname{div} v = f$ . Then there is a constant  $C_n$  such that  $v$  can be chosen with

$$\|Sv\|_\infty \leq C_n \|f\|_\infty. \quad (2.1)$$

Indeed, let  $B$  be the Banach space of quasiconformal vector fields with bounded divergence, equipped with the (pseudo-)norm

$$\|v\|_B = \|Sv\|_\infty + \|\operatorname{div} v\|_\infty;$$

then the divergence map  $\operatorname{div} : B \rightarrow L^\infty(\mathbb{R}^n)$  is surjective, so (2.1) follows by the open mapping theorem.

We claim (2.1) implies, for any compactly supported smooth function  $g$ , that

$$\|\Delta g\|_1 \leq \frac{n}{n-1} C_n \|Eg\|_1.$$

Here  $E$  denotes the trace-zero part of  $D^2$ ; it satisfies

$$(D^2 g)_{ij} = (Eg)_{ij} + \frac{1}{n} (\Delta g) I_{ij}, \quad (2.2)$$

where  $I_{ij} = \delta_{ij}$  is the identity matrix.

The main point of the proof is the identity:

$$\operatorname{tr}(E(Sv)) = \sum E_{ij}(Sv)_{ji} = \frac{n-1}{n} \Delta \operatorname{div} v. \quad (2.3)$$

To check (2.3), note that

$$\operatorname{tr}((D^2)(Dv)) = \sum_{i,j} \frac{\partial^3 v_i}{\partial x_j^2 \partial x_i} = \Delta \operatorname{div} v,$$

while

$$\frac{1}{n} \operatorname{tr}((\Delta I)(Dv)) = \frac{1}{n} \Delta \operatorname{div} v;$$

so by (2.2) we have

$$\operatorname{tr}(E(Dv)) = \frac{n-1}{n} \Delta \operatorname{div} v.$$

But  $E$  is trace-zero and symmetric, so  $\operatorname{tr}(E(Dv)) = \operatorname{tr}(E(Sv))$  and we have (2.3).

Now given any  $g \in C_0^\infty(\mathbb{R}^n)$ , choose  $f \in L^\infty$  such that  $|f| = 1$  and

$$\|\Delta g\|_1 = \int f \Delta g = \int g \Delta f.$$

Choose a quasiconformal vector field with  $\operatorname{div} v = f$  and satisfying (2.1), so  $\|Sv\|_\infty \leq C_n$ . Then

$$\|\Delta g\|_1 = \int g \Delta \operatorname{div} v = \frac{n}{n-1} \int g \operatorname{tr}(E(Sv))$$

by (2.3). Integrating by parts gives

$$\int g \operatorname{tr}(E(Sv)) = \int \operatorname{tr}((Eg)(Sv)),$$

so we have

$$\|\Delta g\|_1 \leq \frac{n}{n-1} \|Eg\|_1 \|Sv\|_\infty \leq \frac{n}{n-1} C_n \|Eg\|_1.$$

But  $E$  and  $\Delta$  are linearly independent differential operators, so this inequality contradicts Ornstein's theorem.  $\blacksquare$

### 3 Maps

In this section we sketch the construction of a counterexample to (1). A similar counterexample is given in [BK, Theorem 1.2]. The  $L^1$  counterexamples given by Ornstein in [Or] are also similar in spirit.

For simplicity we will work in  $\mathbb{R}^2$ . Let  $T \subset S$  denote the square of side  $1/3$  within the unit square  $S$ . Choose  $f > 0$  to be constant on  $T$  and  $S - T$ , with  $\int_S f = 1$  and  $\int_T f = 0.99$ . Cover the edges of  $S$  and  $T$  with much smaller squares  $S_i$ , and redefine  $f|_{S_i}$  as  $f \circ h_i$ , where  $h_i : S_i \rightarrow S$  is a linear map. See Figure 1; the regions where  $f > 1$  are black.

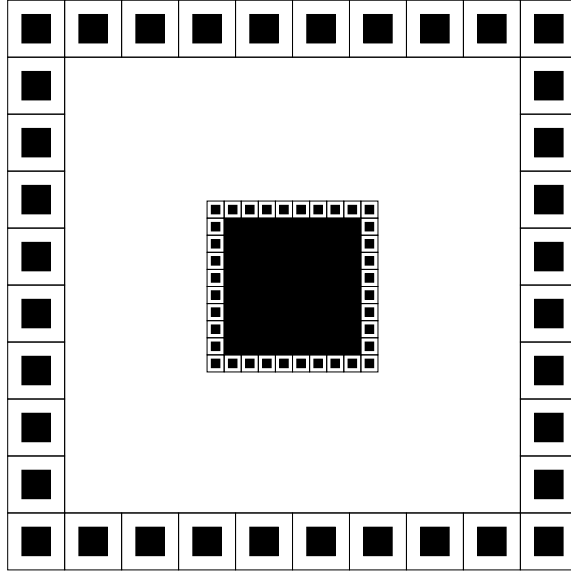


Figure 1. Non-realizable density.

Now repeat the construction along the edges of each  $S_i$ , and iterate  $j$  times to obtain  $f_j$ . As the construction is iterated, arrange that the ratio between the sizes of the squares at levels  $j$  and  $j + 1$  tends to infinity. Then  $f(x) = \lim_{j \rightarrow \infty} f_j(x)$  exists almost everywhere and is bounded above and below.

We claim  $f$  cannot be realized as the Jacobian determinant of a bi-Lipschitz homeomorphism. To see this, let  $K = \sup |\phi(a) - \phi(b)|/|a - b|$ , where the sup is over *just the edges*  $[a, b]$  of all squares at all levels  $j$ . For simplicity, suppose  $K$  is achieved on a horizontal edge  $[a, b]$  of a square  $S'$  at level  $j$ . Let  $S'_i$  denote the squares at level  $j + 1$  running along  $[a, b]$ , and let  $R = \bigcup S'_i$  be the long, thin rectangular they form.

By the triangle inequality, the horizontal edges of  $R$  are mapped to almost straight lines stretched by  $K$ . Since  $\text{area } \phi(R) = \text{area}(R)$ , the height of  $R$  is compressed by  $1/K$ . The horizontal edges of most  $S'_i$  are also stretched by  $K$ , so the perimeter of some  $S'_i$  is increased by a factor of at least  $K/2$ .

But most of the area of  $\phi(S'_i)$  is filled by  $\phi(T'_i)$ , the image of the black sub-square  $T'_i \subset S'_i$ . Since the perimeter of  $T'_i$  is  $1/3$  that of  $S'_i$ , it is stretched by a factor of about  $3K/2$  under  $\phi$ , contradicting the definition of  $K$ .

A detailed proof can be given along lines similar to those presented in [BK], to which the reader is referred for a more complete discussion.

This counterexample to (1) was motivated for us by the area-modulus inequality

$$\text{area}(T) \leq \frac{\text{area}(S)}{1 + 4\pi \text{mod}(A)} \quad (3.1)$$

where  $A$  is the annulus between two disks  $T \subset S \subset \mathbb{C}$  [Mc1, Lemma 2.17]. This inequality relates conformal distortion to distortion of relative areas. Since (3.1) comes from the isoperimetric inequality, for a rigorous proof one is lead to consider stretching along the edges and stability of geodesics as above.

## 4 Nets

In this section we show questions (1) and (3) are equivalent. In particular, a counterexample to (1) implies a counterexample to (3).

**Theorem 4.1** *The following two statements are equivalent:*

- A. *Every measurable  $f > 0$  on  $\mathbb{R}^n$  with  $f$  and  $1/f$  bounded can be realized as the Jacobian determinant of a bi-Lipschitz map.*
- B. *Every separated net  $Y \subset \mathbb{R}^n$  is bi-Lipschitz to  $\mathbb{Z}^n$ .*

**Proof of Theorem 4.1.** (B)  $\implies$  (A). Choose a net  $Y$  such that under rescaling, the measure that assigns a  $\delta$ -mass to each point of  $Y$  accumulates weakly on the measure  $\mu = f(x) dx$ . By (B) there is a bi-Lipschitz map  $\phi : Y \rightarrow \mathbb{Z}^n$ . Under suitable rescaling,  $\phi$  converges to a bi-Lipschitz homeomorphism  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with Jacobian  $f$ . Compare [BK, Lemma 2.1].

(A)  $\implies$  (B). Let  $Y \subset \mathbb{R}^n$  be a separated net. Let  $\langle C_y : y \in Y \rangle$  be the tiling of  $\mathbb{R}^n$  determined by the Voronoi cells

$$C_y = \{x : |x - y| < |x - y'| \text{ for all } y' \neq y \text{ in } Y\}.$$

Since  $Y$  is a net, we have  $\sup \text{diam } C_y < \infty$ , and  $\inf \text{vol } C_y > 0$  because  $Y$  is separated. Let

$$f(x) = \sum_{y : x \in C_y} \frac{1}{\text{vol } C_y}. \quad (4.1)$$

Then  $f$  and  $1/f$  are bounded a.e., so (A) provides a bi-Lipschitz homeomorphism  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with Jacobian determinant  $f$ . Letting  $D_y = \phi(C_y)$ , we have  $\text{vol } \phi(D_y) = 1$ .

For  $z \in \mathbb{Z}^n$  let  $E_z$  denote the unit cube centered at  $z$ . Consider the relation  $R \subset Y \times \mathbb{Z}^n$  given by the set of pairs  $(y, z)$  such that  $D_y$  meets  $E_z$ . Since  $\text{diam } D_y$  and  $\text{diam } E_z$  are bounded, the distance  $|\phi(y) - z|$  is also bounded for all  $(y, z) \in R$ .

Now think of the relation  $R$  as a multi-valued map from  $Y$  to  $\mathbb{Z}^n$ . Then for any finite set  $A \subset Y$ , we have  $|R(A)| \geq |A|$ . Indeed, the cubes labeled by  $R(A)$  cover the cells  $D_y$  labeled by  $A$ , so the inequality follows from the fact that  $\text{vol } D_y = \text{vol } E_z = 1$ . Similarly,  $|R^{-1}(B)| \geq |B|$  for any finite set  $B \subset \mathbb{Z}^n$ .

By the transfinite form of Hall's marriage theorem [Mir, Thm. 4.2.1],  $R$  contains the graph of an injective map  $\psi_1 : Y \rightarrow \mathbb{Z}^n$ . Similarly,  $R^{-1}$  contains the graph of an injective map  $\psi_2 : \mathbb{Z}^n \rightarrow Y$ . By the Schröder-Bernstein theorem [Hal, §22],  $R$  contains the graph of a bijection  $\psi : Y \rightarrow \mathbb{Z}^n$ . Since  $\sup |\psi(y) - \phi(y)| < \infty$ , the map  $\psi : Y \rightarrow \mathbb{Z}^n$  is bi-Lipschitz, proving (B). ■

The proof of (A)  $\implies$  (B) shows that for any separated net  $Y$ , the quality of a bijection  $\phi : Y \rightarrow \mathbb{Z}^n$  can be controlled by the quality of a solution to  $\det D\phi = f$ , where  $f$  is determined by the Voronoi cells as in (4.1). This fact is exploited in the next section.

## 5 Hölder maps

To conclude we show questions (1-3) have positive answers if we relax the Lipschitz condition to a Hölder condition.

**Definition.** We say  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a *homogeneous Hölder map* if there are constants  $K \geq 0$  and  $0 < \alpha \leq 1$  such that for  $|x|, |y| \leq R$  we have

$$|\phi(x) - \phi(y)| \leq KR^{1-\alpha}|x - y|^\alpha. \quad (5.1)$$

If  $\phi(x)$  satisfies (5.1), then so does  $r\phi(x/r)$  for every  $r > 0$ ; it is this sense that the Hölder condition above is homogeneous.

If  $\phi$  and  $\phi^{-1}$  both satisfy (5.1) then we say  $\phi$  is a *homogeneous bi-Hölder homeomorphism*. When  $\alpha = 1$  we obtain the class of bi-Lipschitz maps. Note that for any homogeneous bi-Hölder homeomorphism, we have

$$|\phi(y)| \asymp |y|$$

when  $|y|$  is large. To see this, set  $x = 0$  and  $R = |y|$  in (5.1).



We say a map  $\phi : Y \rightarrow Y'$  between subsets of  $\mathbb{R}^n$  is a homogeneous bi-Hölder bijection if  $\phi$  and  $\phi^{-1}$  satisfy (5.1) on their respective domains.

**Theorem 5.1** Fix  $n \geq 1$ . Then:

1. For any  $f \in L^\infty(\mathbb{R}^n)$  with  $\inf f(x) > 0$ , there is a homogeneous bi-Hölder homeomorphism  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$\text{vol}(\phi(E)) = \int_E f(x) dx \quad (5.2)$$

for all bounded open sets  $E \subset \mathbb{R}^n$ .

2. For any  $f \in L^\infty(\mathbb{R}^n)$ , there is a vector field  $v$  with Zygmund components such that  $\text{div } v = f$ .
3. For any separated net  $Y \subset \mathbb{R}^n$ , there is a homogeneous bi-Hölder bijection  $\psi : Y \rightarrow \mathbb{Z}^n$ .

**Lemma 5.2** Any radial function  $f(r) \in L^\infty(\mathbb{R}^n)$  with  $\inf f > 0$  can be realized as the Jacobian determinant of a radial bi-Lipschitz homeomorphism  $\phi(r, \theta) = (\psi(r), \theta)$ .

**Proof.** Define  $\psi : [0, \infty) \rightarrow [0, \infty)$  by

$$\frac{\psi(r)^n}{n} = \int_0^r s^{n-1} f(s) ds.$$

Then we have

$$\det D\phi = \frac{\psi'(r)\psi(r)^{n-1}}{r^{n-1}} = f(r).$$

The upper and lower bounds on  $f$  imply  $\psi(r) \asymp r$ , so by the formula above we have  $\psi'(r) \asymp 1$ . Thus  $\phi$  is bi-Lipschitz.  $\blacksquare$

**Proof of Theorem 5.1.**

(2). This statement follows from the general theory of singular integral operators, as sketched in §2. Note that a vector field  $v$  with Zygmund components has  $|x \log x|$  modulus of continuity and generates a flow whose time-one map is Hölder [Rei2, Prop. 4].

(1). This result is due to Rivière and Ye. Consider the tiling of  $\mathbb{R}^n - \{0\}$  by the dyadic annuli

$$\langle A_i = \{x : 2^i \leq |x| \leq 2^{i+1}\}, i \in \mathbb{Z} \rangle.$$

After a preliminary radial Lipschitz map, whose existence is insured by Lemma 5.2, we can assume  $\int_{A_i} f = \int_{A_i} 1$  for each  $i$ . By [RY2, Thm. 2], there exists a homeomorphism  $\phi_0 : A_0 \rightarrow A_0$  such that

- (i)  $\int_E f(x) dx = \text{vol}(\phi_0(E))$  for any open set  $E \subset A_0$ ;
- (ii)  $\phi_0(x) = x$  on  $\partial A_0$ ; and
- (iii)  $K^{-1}|x - y|^{1/\alpha} \leq |\phi(x) - \phi(y)| \leq K|x - y|^\alpha$ , where  $\alpha > 0$ ,  $K > 1$  depend only on  $\|f\|_\infty + \|1/f\|_\infty$  (compare [RY2, (2.14)]).

Since  $A_i$  is simply  $A_0$  rescaled by a factor of  $2^i$ , we can apply this result to obtain homeomorphisms  $\phi_i : A_i \rightarrow A_i$  satisfying the volume distortion equation (5.2) for  $E \subset A_i$ . The Hölder bounds in (iii) rescale to give the homogeneous bounds (5.1) for  $\phi_i$  and  $\phi_i^{-1}$ , so the  $\phi_i$  piece together to produce the desired homogeneous bi-Hölder map  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

**(3).** Let  $Y \subset \mathbb{R}^n$  be a separated net. Let  $\langle C_y \rangle$  be the Voronoi cells for  $Y$ , and let  $E_z$  denote the unit cube centered at  $z \in \mathbb{Z}^n$ . Define  $f(x) = 1/\text{vol}(C_y)$  for  $x \in C_y$  as in (4.1).

By (1) there exists a homogeneous bi-Hölder map  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  sending  $f(x) dx$  to the standard measure on  $\mathbb{R}^n$ . Letting  $D_y = \phi(C_y)$  we have  $\text{vol} D_y = 1$  and  $\text{diam} D_y = O(1 + |y|^{1-\alpha})$ , where  $\alpha$  is the exponent in (5.1). As in the proof of Theorem 4.1, Hall's marriage theorem provides a bijection  $\psi : Y \rightarrow \mathbb{Z}^n$  such that  $D_y \cap E_z \neq \emptyset$  whenever  $\psi(y) = z$ . Therefore

$$|\phi(y) - \psi(y)| \leq C(1 + |y|^{1-\alpha}) \quad (5.3)$$

for some constant  $C$ .

We claim  $\psi : Y \rightarrow \mathbb{Z}^n$  is a homogeneous bi-Hölder map. Indeed, given distinct points  $x, y \in Y$  with  $|x|, |y| \leq R$ , by (5.1) and (5.3) we have

$$\begin{aligned} |\psi(x) - \psi(y)| &\leq |\phi(x) - \phi(y)| + |\phi(x) - \psi(x)| + |\phi(y) - \psi(y)| \\ &\leq KR^{1-\alpha}|x - y|^\alpha + 2C(1 + R^{1-\alpha}) \\ &= O(R^{1-\alpha}|x - y|^\alpha) \end{aligned}$$

since  $|x - y| > \epsilon > 0$  by separation of  $Y$ . This shows  $\psi$  satisfies the homogeneous Hölder condition.

To verify the same condition for  $\psi^{-1}$ , we apply the same reasoning to the inverse image cubes  $F_z = \phi^{-1}(E_z)$ . The Hölder condition on  $\phi^{-1}$  gives  $\text{diam}(F_z) = O(1 + |z|^{1-\alpha})$ , and since  $F_z \cap C_{\psi^{-1}(z)} \neq \emptyset$  we have

$$|\psi^{-1}(z) - \phi^{-1}(z)| \leq C'(1 + |z|^{1-\alpha}).$$

Thus for distinct  $z, w \in \mathbb{Z}^n$  with  $|z|, |w| \leq R$  we have

$$\begin{aligned} |\psi^{-1}(z) - \psi^{-1}(w)| &\leq KR^{1-\alpha}|z - w|^\alpha + 2C'(1 + R^{1-\alpha}) \\ &= O(R^{1-\alpha}|z - w|^\alpha) \end{aligned}$$

since  $|z - w| \geq 1$ . Therefore  $\psi^{-1}$  also satisfies (5.1) and we are done.  $\blacksquare$

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