

# Kleinian groups and John domains

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## Abstract

We characterize when John domains arise in the setting of Kleinian groups.

## 1 Introduction

A region  $U$  in the Riemann sphere is a *John domain* if every point in  $U$  can be reached from a fixed basepoint by a flexible cone with a definite angle at its vertex.

John domains were introduced by Fritz John in his study of strain and the stability of quasi-isometries [John]. A Jordan curve cuts the sphere into a pair of John domains if and only if it is a quasicircle [Pom, Thm 5.9]. Thus a simply-connected John domain is like a one-sided quasidisk.

In this paper we give a new characterization of John domains in terms of 3-dimensional hyperbolic geometry (§2). From this perspective the John condition becomes an asymptotic quasi-isometry invariant in the sense of Gromov [Gr].

Recently Carleson, Jones and Yoccoz found that the John condition is directly related to expansion in conformal dynamics [CJK]. These authors show the basin of infinity for a polynomial  $f(z)$  is a John domain if and only if  $f(z)$  has no parabolic orbits and no critical point in the Julia set accumulates on itself under forward iteration.

Here we provide a complement to this dynamical theorem in the setting of Kleinian groups. We characterize exactly when a component of the domain of discontinuity is a John domain (§3), and also when it is uniformly connected (§4). Our results are motivated by the analogies between iterated

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rational maps and Kleinian groups that have emerged in the past decade; see [Sul2] and [Mc1] for part of the dictionary.

In §5 we provide examples and computer images illustrating the results below. We also amplify on the distinction between limit sets and Julia sets, by giving examples where both are dendrites, but of radically different geometry.

**Statement of results.** Let  $\Gamma$  be a nonelementary, finitely generated *Kleinian group*, that is a discrete subgroup of conformal automorphisms of the Riemann sphere  $S_\infty^2 = \partial\mathbb{H}^3$ . The sphere is naturally partitioned into a *limit set*  $\Lambda$ , where the dynamics of  $\Gamma$  is chaotic, and a *domain of discontinuity*  $\Omega$ , where the orbits of  $\Gamma$  are discrete. These sets can be complex in shape and topology, but they are also homogeneous and self-similar, by  $\Gamma$ -invariance.

Let  $U \subset S_\infty^2$  a component of  $\Omega$  with stabilizer  $\Gamma_U \subset \Gamma$ . Then we have:

**Theorem 1.1** *The component  $U$  is a John domain iff*

- (a)  $\Gamma_U$  is geometrically finite, and
- (b) every parabolic element of  $\Gamma_U$  stabilizes a round disk in  $U$ .

Condition (b) means every cusp of the 3-manifold  $\mathbb{H}^3/\Gamma_U$  is represented by a cusp of the Riemann surface  $U/\Gamma_U$ .

**Corollary 1.2** *The component  $U$  is a simply-connected John domain iff it is a quasidisk.*

A region  $V$  is *uniformly connected* if for any sequence of Möbius transformations, any Hausdorff limit of  $g_n(V)$  is connected.

**Theorem 1.3** *The component  $U$  is uniformly connected iff there is no parabolic element in  $\Gamma_U$  stabilizing a pair of tangent round disks in  $U$ .*

Note that  $\Gamma_U$  is allowed to be geometrically infinite. The parabolic condition rules out a cylinder in  $\mathbb{H}^3/\Gamma_U$  joining a pair of cusps of  $U/\Gamma_U$ .

**Corollary 1.4** *A simply-connected component of the domain of discontinuity of a finitely generated Kleinian group is always uniformly connected.*

In contrast, uniform connectivity often fails to hold for the Fatou set of a rational map. Thus Theorem 1.3 and its Corollary highlight a difference between these two types of conformal dynamical systems.

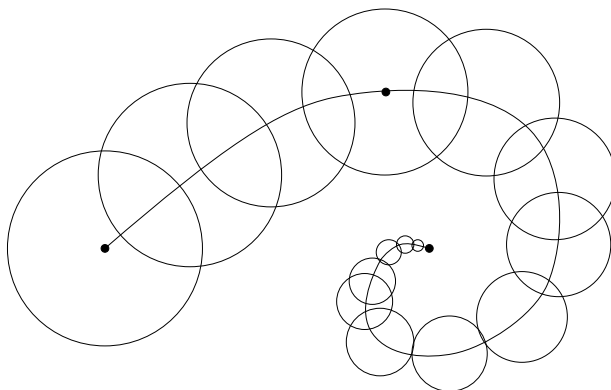
The questions addressed here emerged from joint work with Mike Freedman [FM]. See [BV] for more on John domains and Julia sets. Basic facts about hyperbolic manifolds used in the sequel can be found in [Th], [BP] and [Rat].

## 2 John domains

Let  $\mathbb{H}^n$  denote hyperbolic  $n$ -space and  $S_\infty^{n-1}$  its sphere at infinity. A region  $U \subset S_\infty^{n-1}$  is a *John domain* if there is an  $a \in U$  and an  $\epsilon > 0$  such that for any  $b \in U$ , there is a path  $p : [0, 1] \rightarrow U$  with  $p(0) = a$ ,  $p(1) = b$  and

$$d(p(t), \partial U) > \epsilon \cdot d(p(t), b) \quad (2.1)$$

for all  $t \in [0, 1]$ . Distances above are measured in the spherical metric.



The John condition.

The John condition means  $b$  can be reached from  $a$  by a flexible cone with definite angle at  $b$ . In a John domain, any point can play the role of the basepoint  $a$  (possibly after changing  $\epsilon$ ).

The notion of a John domain was introduced in [John, p.402]. Various equivalent definitions are compared in [NV]. Here we use the version adapted to domains in the sphere.

To prove Theorem 1.1, it is convenient to have a definition of John domains that involves hyperbolic geometry. In this section we will show:

**Theorem 2.1** *Let  $U \subset S_\infty^{n-1}$  be an open connected set whose complement contains at least 2 points. Let  $\widehat{U}$  be the associated boundary component of a unit neighborhood of the convex hull of  $\partial U$  in  $\mathbb{H}^n$ .*

*Then  $U$  is a John domain iff  $\widehat{U}$  is quasi-starlike.*

**Convex hulls and starlike sets.** Let  $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup S_\infty^{n-1}$  denote the compactification of hyperbolic space by the sphere at infinity. We will use the interval notation

$$[x, y] \subset \overline{\mathbb{H}^n}$$

to denote the geodesic joining a pair of points in  $\overline{\mathbb{H}^n}$ ; the endpoints are included, even if they lie in  $S_\infty^{n-1}$ .

A set  $K \subset \overline{\mathbb{H}^n}$  is *convex* if  $a, b \in K \implies [a, b] \subset K$ . The smallest convex set containing a given set  $E$  is its *convex hull*, denoted  $\text{hull}(E)$ . Given a closed convex set  $K$ , the *nearest point projection*

$$\pi_K : \overline{\mathbb{H}^n} \rightarrow K$$

sends  $x$  to the point  $\pi_K(x)$  closest to  $K$ . For  $x \in \mathbb{H}^n$  closeness is measured with the hyperbolic metric; for  $x \in S_\infty^{n-1} - K$ ,  $\pi_K(x)$  is the point where a horoball inflated at  $x$  first touches  $K$ .

For a region  $U \subset S_\infty^{n-1}$ , let  $K = \mathcal{N}_1(\text{hull}(\partial U))$  be the closed unit neighborhood of the convex hull of  $\partial U$ , and let

$$\widehat{U} = \pi_K(U).$$

Then  $\widehat{U}$  is the part of  $\partial K$  in  $\mathbb{H}^n$  that faces  $U$ , and  $\pi_K : U \rightarrow \widehat{U}$  is a homeomorphism.<sup>1</sup>

A set  $X \subset \mathbb{H}^n$  is *starlike* if there exists an  $a \in X$  such that  $[a, b] \subset X$  for all  $b \in X$ . This condition is like convexity from a single point. Similarly,  $X$  is *quasi-starlike* if there is an  $a \in X$  and  $R > 0$  such that for any  $b \in X$ , there is a path  $p : [0, 1] \rightarrow X$  with  $p(0) = a$ ,  $p(1) = b$  and

$$d(p(t), [a, b]) < R \tag{2.2}$$

for all  $t \in [0, 1]$ .

**Proof of Theorem 2.1.** Suppose  $U$  is a John domain with basepoint  $a$ . Let  $K = \mathcal{N}_1(\text{hull}(\partial U))$  and normalize by a Möbius transformation so that  $\widehat{a} = \pi_K(a) = 0$  in the Poincaré ball model for  $\mathbb{H}^n$  as the unit ball in  $\mathbb{R}^n$ . With this normalization, it is easy to approximate  $\widehat{x} = \pi_K(x)$  to within a bounded hyperbolic distance; namely

$$\widehat{x} \approx (1 - r(x))x, \tag{2.3}$$

where  $r(x) = d(x, \partial U)$ . To see this, just note that supporting hyperplanes for  $K$  correspond to round disks in  $S_\infty^{n-1} - \partial U$ .

Given  $\widehat{b} \in \widehat{U}$ , choose  $b \in U$  with  $\pi_K(b) = \widehat{b}$  and let  $p : [0, 1] \rightarrow U$  be a path from  $a$  to  $b$  satisfying (2.1). We claim  $\widehat{p}(t) = \pi_K(p(t))$  satisfies (2.2).

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<sup>1</sup>We take  $K = \mathcal{N}_1(\text{hull}(\partial U))$  because the projection  $U \rightarrow \text{hull}(\partial U)$  can be far from injective; consider the case where  $\partial U$  is a circular arc in  $S_\infty^2$ .

Indeed, the hyperbolic metric blows up like  $1/(1 - \rho)$  in polar coordinates on the ball, so

$$d(\widehat{p}(t), [\widehat{a}, b]) \approx \frac{d(p(t), b)}{r(p(t))} < \frac{1}{\epsilon}$$

by the John condition and (2.3). The broken geodesic  $[\widehat{a}, \widehat{b}] \cup [\widehat{b}, b]$  makes an angle of at least  $90^\circ$  at  $\widehat{b}$ , so  $[\widehat{a}, \widehat{b}] \subset \mathcal{N}_1([\widehat{a}, b])$ . Finally  $r(p(t)) > \epsilon r(b)/2$  which implies

$$d(\widehat{a}, \widehat{p}(t)) < d(\widehat{a}, \widehat{b}) + O(\log(1/\epsilon)).$$

Thus the projection of  $\widehat{p}(t)$  to  $[\widehat{a}, b]$  lies close to  $[\widehat{a}, \widehat{b}]$ , and we find

$$d(\widehat{p}(t), [\widehat{a}, \widehat{b}]) < R$$

with  $R \approx 1/\epsilon$ . Thus  $\widehat{U}$  is quasi-starlike.

Conversely, suppose  $\widehat{U}$  is quasi-starlike from  $\widehat{a} \in \widehat{U}$ , normalized as before so  $\widehat{a} = 0$ . Then for  $a, b \in U$  corresponding under  $\pi_K$  to  $\widehat{a}, \widehat{b} \in \widehat{U}$ , let  $p = \pi_K^{-1} \circ \widehat{p}$ , where  $d(\widehat{p}(t), [\widehat{a}, \widehat{b}]) < R$ . Then

$$\frac{d(p(t), b)}{r(p(t))} \approx d(\widehat{p}(t), [\widehat{a}, b]) < R + 1,$$

so the John condition for  $U$  is verified with  $\epsilon \approx 1/(R + 1)$ . ■

**Quasi-convexity.** Let us say  $X \subset \mathbb{H}^n$  is *quasi-convex* if there exists an  $R$  such that any  $a, b \in X$  are joined by a path  $p : [0, 1] \rightarrow X$  with  $d(p(t), [a, b]) < R$ . The following result is fairly well-known.

**Theorem 2.2** *A simply-connected region  $U \subset S_\infty^2$  is a quasidisk iff  $\widehat{U}$  is quasiconvex.*

**Sketch of the proof.** If  $\widehat{U}$  is quasiconvex, then it is quasi-isometric to a hyperbolic plane and so  $\partial\widehat{U} \subset S_\infty^2$  is a quasicircle by [GH, Prop. 7.14]. Conversely, if  $U$  is a quasidisk, then Poincaré geodesics in  $U$  project to quasi-geodesics in  $\widehat{U}$ , so  $\widehat{U}$  is quasi-convex. ■

In particular for simply-connected regions  $U \subset S_\infty^2$  we have:

$U$ is a quasidisk	$\iff$	$\widehat{U}$ is quasi-convex,
$U$ is a John disk	$\iff$	$\widehat{U}$ is quasi-starlike.

### 3 Kleinian groups

In this section we prove the following more precise version of Theorem 1.1.

**Theorem 3.1** *Let  $U$  be a component of the domain of discontinuity of a nonelementary, finitely-generated Kleinian group  $\Gamma$ . Then the following are equivalent:*

1.  $U$  is a John domain.
2.  $\widehat{U}$  is quasi-starlike.
3.  $\widehat{U}$  is quasi-convex.
4.  $\Gamma_U$  is geometrically finite, and every parabolic in  $\Gamma_U$  stabilizes a round disk in  $U$ .

**Remark.** The John condition fails dramatically when  $\Gamma_U$  is geometrically infinite, since then  $\text{H. dim}(\partial U) = 2$  by a result of Bishop and Jones [BJ].

The proof of Theorem 3.1 is elementary apart from the use of:

**Theorem 3.2 (Ahlfors Finiteness Theorem)** *If  $\Gamma$  is a finitely generated Kleinian group with domain of discontinuity  $\Omega$ , then  $\Omega/\Gamma$  is a finite union of hyperbolic Riemann surfaces of finite area.*

See [Ah], [Gre], [Bers1], [McS].

It is worth noting that  $\Gamma_U$  is almost determined by  $U$ . Indeed, let  $\text{Aut}(U)$  be the group of all Möbius transformations stabilizing  $U$ . Suppose a component  $U$  of  $\Omega$  is not a round disk; then  $\text{Aut}(U)$  is discrete, and it contains  $\Gamma_U$  with finite index because  $U/\Gamma_U$  covers  $U/\text{Aut}(U)$ . So at least in principle, most properties of  $\Gamma_U$  are reflected in the geometry of  $U$ .

**Proof of Theorem 3.1.** First some preliminary reductions. By passing to a subgroup of finite index, we may assume  $\Gamma$  is orientation-preserving and torsion-free. By the Ahlfors Finiteness Theorem,  $U/\Gamma_U$  has finite area, and thus the limit set of  $\Gamma_U$  is  $\partial U$ . Therefore we can also assume  $\Gamma = \Gamma_U$  and  $\Lambda = \partial U$ .

Following §2, let

$$\begin{aligned} K &= \mathcal{N}_1(\text{hull}(\partial U)), \\ \widehat{U} &= \pi_K(U) \subset \partial K, \\ K(M) &= K/\Gamma \text{ and} \\ U(M) &= \widehat{U}/\Gamma. \end{aligned}$$

Then  $U(M) \subset \partial K(M)$  is the component of the boundary of a unit neighborhood of the convex core of  $M$  that faces  $U$ .

(1)  $\iff$  (2). This is Theorem 2.1.

(2)  $\implies$  (4). Suppose  $\widehat{U}$  is quasi-starlike from some basepoint  $a$ . Then there exists an  $R$  such that for any  $\gamma \in \Gamma$ , the geodesic segment  $[a, \gamma a]$  is contained within an  $R$ -neighborhood of  $\widehat{U}$ . Since  $\Gamma$  acts by isometries, we have  $[\gamma a, \delta a] \subset \mathcal{N}_R(\widehat{U})$  for all  $\gamma, \delta \in \Gamma$ . But  $\Gamma a$  accumulates densely on  $\partial U$ , so any geodesic with endpoints in  $\partial U$  is also contained in  $\mathcal{N}_R(\widehat{U})$ . Any point in  $\text{hull}(\partial U)$  is within a universally bounded distance of a geodesic with endpoints in  $\partial U$ , so  $K = \mathcal{N}_1(\text{hull}(\partial U))$  is contained in an  $S$ -neighborhood of  $\widehat{U}$ ,  $S = R + O(1)$ . Passing to the quotient by  $\Gamma$  we find

$$K(M) \subset \mathcal{N}_S(U(M)).$$

Since  $U(M)$  has finite area, the thick part of  $K(M)$  is compact and thus  $M$  is geometrically finite. Also the cuspidal parts of  $K(M)$  lie within a bounded distance of  $U(M)$ , so every cusp in  $M$  has rank one and is represented by a cusp of  $U/\Gamma$ . Therefore any parabolic  $\gamma \in \Gamma$  stabilizes a round disk in  $U$ . (This last condition can also be seen directly by considering a John cone in  $U$  touching the fixed-point of  $\gamma$ ; the  $\gamma$ -orbit of this cone contains a round disk and is contained in  $U$ .)

(4)  $\implies$  (2). This is the main implication in the proof. For simplicity we first suppose  $\Gamma = \Gamma_U$  is geometrically finite without cusps. Then  $K(M)$  and  $U(M)$  are closed manifolds.

Choose a finite 0-complex  $U_0 \subset U(M)$  such that any point in  $U(M)$  can be moved slightly to belong to  $U_0$ . Extend  $U_0$  to a finite 1-complex  $M_1 \subset K(M)$  such that any path in  $K(M)$  can be moved slightly to run along the edges of  $M_1$ . (For example one can take  $M_1$  to be the 1-skeleton of a very fine triangulation.)

Since  $\Gamma = \Gamma_U$ , the morphism  $\pi_1(U(M)) \rightarrow \pi_1(K(M))$  is surjective. By elementary homotopy theory, the inclusion

$$i : (M_1, U_0) \hookrightarrow (K(M), U_0)$$

can be deformed, as a map of pairs, to a map

$$h : (M_1, U_0) \rightarrow (U(M), U_0).$$

Since  $M_1$  is compact, the homotopy  $H : [0, 1] \times M_1 \rightarrow K(M)$  between  $i$  and  $h$  need only move points some bounded distance  $R$ ; that is, we can choose  $H$  such that the length of  $H([0, 1], x)$  is less than  $R$  for all  $x$ .

Now given  $a, b \in \widehat{U}$  lying over vertices in  $U_0$ , project the geodesic  $[a, b]$  to a parameterized path  $q : [0, 1] \rightarrow K(M)$ . Move the path slightly, keeping its endpoints fixed in  $U_0$ , so it runs along  $M_1$ . Then  $h \circ q : [0, 1] \rightarrow U(M)$  admits a bounded homotopy, *rel* endpoints, to  $q$ . Thus its lift

$$p = \widetilde{h \circ q} : [0, 1] \rightarrow \widehat{U}$$

joins  $a$  to  $b$  and satisfies

$$d(p(t), [a, b]) < R$$

for all  $t$ . Since any  $a, b \in \widehat{U}$  can be moved slightly to lie over  $U_0$ , we have shown that  $\widehat{U}$  is quasi-starlike.

**The case of cusps.** We now treat the case where  $\Gamma$  is geometrically finite, possibly with cusps. Assuming all parabolics of  $\Gamma$  are represented by cusps on  $U/\Gamma$ , we will again show  $\widehat{U}$  is quasi-starlike.

Since  $\Gamma$  is geometrically finite, standard horoball neighborhoods of the cusps of  $M$  meet  $K(M)$  in a finite number of rank one cuspidal pieces  $\langle K_i(M) : i = 1, \dots, n \rangle$ , each quasi-isometric to  $C \times [0, 1]$  where

$$C = \{z \in \mathbb{H} : \text{Im}(z) \geq 1\} / \langle z \mapsto z + 1 \rangle$$

is a standard cusp on a hyperbolic surface. The cusp  $K_i(M)$  meets  $\partial K(M)$  in two components, corresponding to  $C \times \{0, 1\}$ . At least one of these components,  $U_i(M)$ , belongs to  $U(M)$ , since the corresponding parabolic subgroup stabilizes a round disk in  $U$ .

Removing the cusps, we obtain a pair of compact manifolds

$$\begin{aligned} K^*(M) &= \overline{K(M) - \bigcup K_i(M)}, \\ U^*(M) &= U(M) \cap K^*(M) \end{aligned}$$

homotopy equivalent to  $(K(M), U(M))$ . Since  $\pi_1(U^*(M)) \rightarrow \pi_1(K(M))$  is surjective, we can construct a pair of complexes  $(M_1, U_0) \hookrightarrow (K^*(M), U^*(M))$  as before, such that the inclusion is homotopic to  $h : (M_1, U_0) \rightarrow (U^*(M), U_0)$ .

Now pick a basepoint  $a \in \widehat{U}$  lying over  $U_0$ , and consider any  $b \in \widehat{U}$ . Let  $q : [0, 1] \rightarrow K(M)$  be the projection to  $M$  of the geodesic  $[a, b]$ . To verify that  $\widehat{U}$  is quasi-starlike (and hence that  $U$  is a John domain), it suffices to show  $q$  admits a uniformly bounded isotopy, *rel* endpoints, to a path in  $U(M)$ .

First suppose  $b$  lies over a point in  $U_0$ . Each cusp  $K_i(M)$  admits a bounded retraction to  $U_i(M)$ ; use these to adjust  $q$  by a bounded homotopy so it stays within  $U(M) \cup K^*(M)$ . Next move  $q$  slightly within  $K^*(M)$  so it



runs along  $M_1$ . Then  $h \circ q$  is contained in  $U(M)$ , and boundedly homotopic to  $q$ , so we have verified the quasi-starlike condition for  $b$ .

Now suppose  $b$  lies over a point in  $U^*(M)$ . Then  $b$  can be moved slightly to lie over a point in  $U_0$ , and the preceding argument applies.

Finally suppose  $b$  lies over a cusp  $K_i(M)$ . Then we must take care to choose  $U_i(M)$  to be the component of  $K_i(M) \cap U(M)$  into which  $b$  projects. (Potentially  $K_i(M) \cap U(M)$  has two components.) With this choice, the retraction of  $K_i(M)$  to  $U_i(M)$  fixes  $b$ , and the bounded homotopy from  $q$  to a path in  $U(M)$  is constructed as before.

**(2)  $\iff$  (3).** Once  $\widehat{U}$  is quasi-starlike from a basepoint  $a$ , it is also quasi-starlike (with the same constant) from any other basepoint in  $\Gamma a$ . When  $U/\Gamma$  is compact this immediately implies  $\widehat{U}$  is quasi-convex. But the result also holds when  $U/\Gamma$  has cusps, by an analysis of the thin part similar to that above.  $\blacksquare$

**Proof of Corollary 1.2.** If  $U$  is a simply-connected John domain, then  $\widehat{U}$  is quasi-convex by the preceding result, and therefore  $U$  is a quasidisk by Theorem 2.2.

Alternatively, one may use the fact that a geometrically finite surface group without accidental parabolics is quasifuchsian (cf. [Bers2], [Msk]).  $\blacksquare$

## 4 Uniform connectivity

In this section we prove Theorem 1.3, showing  $U$  is uniformly connected unless it has a double cusp.

**Definition.** Let  $U, U_n \subset S_\infty^2$  be open sets. We say  $U_n \rightarrow U$  in the Hausdorff topology if

- (a) any compact set  $K \subset U$  is contained in  $U_n$  for all  $n \gg 0$ , and
- (b) if a fixed neighborhood  $V$  of  $x$  is contained in  $U_n$  for infinitely many  $n$ , then  $x \in U$ .

Equivalently,  $U_n \rightarrow U$  if  $(S_\infty^2 - U_n) \rightarrow (S_\infty^2 - U)$  in the usual Hausdorff topology on closed subsets of the sphere [Haus].

A set  $U$  is *uniformly connected* if  $\lim g_n(U)$  is connected (or empty) for any sequence of Möbius transformations  $g_n$  such that  $g_n(U)$  converges.

An alternative definition, displaying the uniformity more directly, is as follows:  $U$  is uniformly connected if there is a function  $\delta(\epsilon) > 0$  such that for  $x_1, x_2 \in U$  and  $\epsilon > 0$ , if  $d(x_1, x_2) = s$  and  $B(x_i, \epsilon s) \subset U$ ,  $i = 1, 2$ , then

there is a path  $p : [0, 1] \rightarrow U$ , joining  $x_1$  to  $x_2$ , with  $d(x_1, p(t)) < s/\delta(\epsilon)$  and  $d(p(t), \partial U) > \delta(\epsilon)s$  for all  $t$ .

**Proof of Theorem 1.3.** As before, we can assume  $\Gamma = \Gamma_U$  and  $\Gamma$  is a torsion-free.

Suppose there is a parabolic element  $\gamma \in \Gamma$  stabilizing a pair of round disks in  $U \subset \widehat{\mathbb{C}} \cong S_\infty^2$ . After a Möbius change of coordinates we can assume  $\gamma(z) = z/(1+z)$  and

$$\{z : |z \pm ir| < r\} \subset U$$

for some  $r > 0$ . Since  $\Gamma$  is nonelementary, by iterating  $\gamma$  we find the limit set contains the sequence  $\langle 1/(k+w), k \in \mathbb{Z} \rangle$  for some  $w \in \mathbb{C}$ . Thus if we blowup around the origin with the Möbius transformations  $g_n(z) = nz$ , we find that  $g_n(U) \rightarrow \mathbb{C} - \mathbb{R}$  and thus  $U$  is not uniformly connected.

For the converse, suppose any parabolic stabilizes at most one round disk in  $U$ , and  $g_n(U) \rightarrow V$  in the Hausdorff topology. We will show that  $V$  is connected.

It is not hard to check that  $g_n(\widehat{U}) \rightarrow \widehat{V}$  in the Hausdorff topology on closed subsets of  $\mathbb{H}^3$ . Let 0 denote the origin in the ball model for  $\mathbb{H}^3 \cong \mathbb{B}^3 \subset \mathbb{R}^3$ , and let  $g_n(x_n) = 0$ . If  $d(x_n, \widehat{U}) \rightarrow \infty$ , then  $d(0, g_n(\widehat{U})) \rightarrow \infty$  and thus  $\widehat{V} = \emptyset$ . In this case,  $V = \emptyset$  or  $|S_\infty^2 - V| = 1$  (according to whether  $x_n$  stays on the convex or concave side of  $\widehat{U}$ ). So  $V$  is connected.

If  $d(x_n, \widehat{U})$  does not tend to infinity, we can pass to a subsequence such that  $d(x_n, \widehat{U})$  is bounded, and indeed we can assume  $x_n \in \widehat{U}$  by a minor modification of  $g_n$ . Consider the image  $[x_n]$  of  $x_n$  in

$$U(M) = \widehat{U}/\Gamma \subset M = \mathbb{H}^3/\Gamma.$$

By the Ahlfors Finiteness Theorem,  $U/\Gamma$  is a hyperbolic surface of finite area, so the part of  $U(M)$  outside the cusps of  $M$  is compact. If  $[x_n]$  has a convergence subsequence in  $U(M)$ , then there are  $\gamma_n \in \Gamma$  such that a subsequence of  $g_n\gamma_n$  converges to  $g \in \text{Isom}(\mathbb{H}^3)$ ; since  $\gamma_n(U) = U$ , we have  $g_n(U) \rightarrow g(U) = V$  and thus  $V$  is connected.

Finally suppose  $[x_n] \in U(M)$  tends to infinity in  $U(M)$ . Then after passing to a subsequence,  $x_n$  tends to a definite cusp of  $M$ . By assumption, the corresponding parabolic subgroup of  $\Gamma$  stabilizes only one round disk in  $U$ , and thus  $U(M)$  meets a horoball neighborhood of the cusp in only one component. It follows that  $\widehat{V} = \lim g_n(\widehat{U})$  is connected, and therefore  $V$  is connected. ■

**Proof of Corollary 1.4.** We have an exact sequence

$$1 \rightarrow \pi_1(U) \rightarrow \pi_1(U/\Gamma_U) \rightarrow \Gamma_U \rightarrow 1.$$

If a single parabolic in  $\Gamma_U$  stabilizes a pair of round disks in  $U$ , then there are two peripheral loops on  $U/\Gamma_U$  mapping to the same element of  $\Gamma_U$ , and thus  $\pi_1(U) \neq 1$ . ■

**Local connectivity.** It is at present unknown if  $\partial U$  is always locally connected when  $U$  is a component of the domain of discontinuity of a finitely-generated Kleinian group.

Some care is required to construct a uniformly connected domain  $U$  such that  $\partial U$  is not locally connected. For a typical example, take  $U = \mathbb{C} - S$  where  $S$  is a square with strips removed,

$$S = [0, 2] \times [0, 2] - \bigcup_1^\infty (a_n, a_n + a_n^2) \times [0, 1), \quad a_n = 1/(2^n)^n.$$

Since  $a_n/a_{n+1} \rightarrow \infty$ , at most one strip in  $U$  is visible in any Hausdorff limit, and thus  $U$  is uniformly connected. On the other hand  $\partial U$  is not locally connected where the strips accumulate.

By Theorem 1.3, any failure of local connectivity in Kleinian groups must similarly involve narrow fjords at very different scales.

## 5 Examples

1. Figure 1 depicts the limit set of a geometrically finite group lying in Bers' boundary for the Teichmüller space of a surface of genus two. The unbounded component  $U$  of  $\Omega$  is  $\Gamma$ -invariant, and  $U/\Gamma$  is a surface of genus 2; the remainder of  $\Omega/\Gamma$  is comprised of a pair of punctured tori. Thus  $\Gamma$  has accidental parabolics that are not represented in  $U$ , so by Theorem 2.1,  $U$  is not a John domain.

The failure of the John condition is evident at each parabolic fixed-point; for example, the point of tangency between two circles in the center of the picture cannot be reached by a John cone contained in  $U$ .

The parameters for this example were provided by Jeff Brock.

2. Figure 2 depicts the limit set of a geometrically finite Kleinian  $\Gamma$  isomorphic to the HNN-extension  $\Gamma(2)*_{\mathbb{Z}}$ . Here  $\Gamma(2)$  is a Fuchsian group

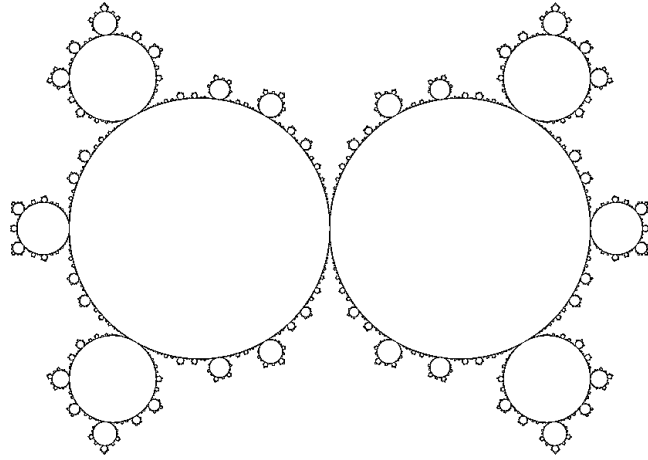


Figure 1. Failure of the John condition.

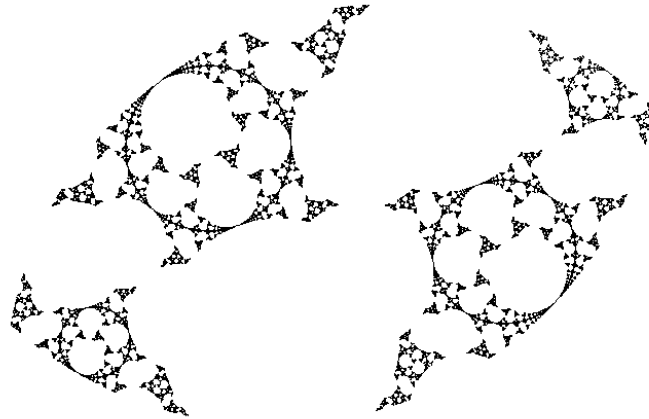


Figure 2. An infinitely connected John domain with parabolics.

uniformizing the triply-punctured sphere, and  $\mathbb{Z} = \langle h \rangle$  is generated by a hyperbolic element with one fixed point in each component of  $\Omega(\Gamma(2))$ .

The quotient Riemann surface  $\Omega/\Gamma$  is a torus with 3 punctures. Since all 3 cusps of  $\mathbb{H}^3/\Gamma$  are represented on  $\Omega/\Gamma$ , all components of  $\Omega$  are John domains.

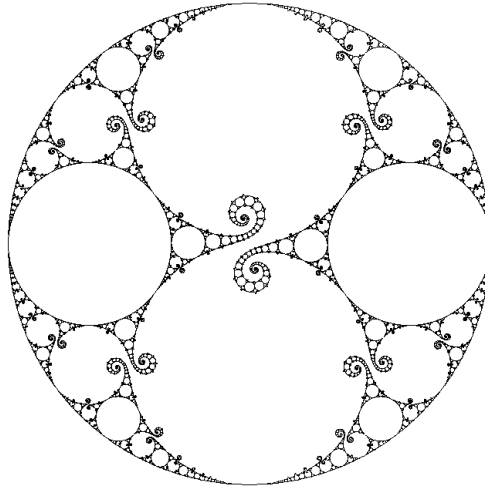


Figure 3. Bottlenecks.

3. Figure 3 shows the limit set of a group  $\Gamma \cong \mathbb{Z} * \mathbb{Z}$  in Maskit's embedding of the Teichmüller space of a punctured torus. The domain of discontinuity has a single invariant component  $U$ ; the remaining components of  $\Omega$  are round disks. The domain  $U$  has a 'bottleneck' in the center of the picture, due to a nearly parabolic element in  $\Gamma$ . The pair of spiraling arms in the center of the picture converge to the fixed-points of this almost-parabolic element.

Nevertheless,  $U$  is uniformly connected by Corollary 1.4. Although one can make examples with arbitrarily narrow bottlenecks, in any *fixed* example there is a uniform modulus of connectivity. Because  $U/\Gamma$  is a finite surface, only a finite number of types of bottlenecks are present in any given picture.

In this example  $U/\Gamma$  is a punctured torus, and the rest of  $\Omega/\Gamma$  is a triply-punctured sphere. The puncture of the torus accounts for only one of the three cusps of the triply-punctured sphere, so  $U$  is not a John domain. The failure of the John condition can be seen in the picture at 3 o'clock and 9 o'clock, where  $U$  is pinched between a pair of tangent circles.

The parameters for this group were obtained with the aid of a computer program written by David Wright [Wr].

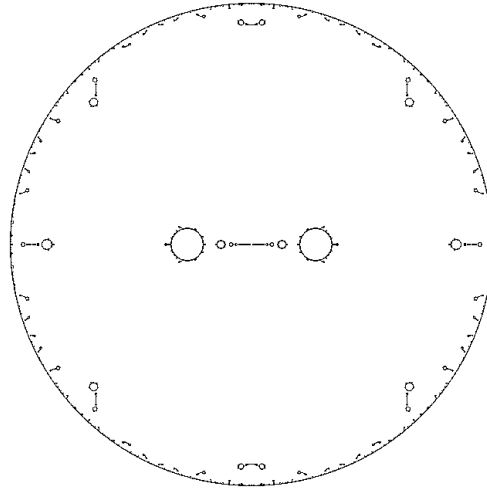


Figure 4. Failure of uniform connectivity.

4. Figure 4 shows the limit set of a typical group  $\Gamma$  with a component  $U \subset \Omega$  that is *not* uniformly connected. In this example  $\Gamma \cong \Gamma' * \mathbb{Z}$  where  $\Gamma'$  is a Fuchsian group of genus 2 and  $\mathbb{Z} = \langle p \rangle$  is generated by a parabolic element. The fixed point of  $p$  is in the center of the picture and also in the center of one component of  $\Omega(\Gamma')$ . The quotient  $U/\Gamma$  is a surface of genus 2 with two punctures, both corresponding to the same cusp of  $M = \mathbb{H}^3/\Gamma$ . Under expansion of the picture about the fixed-point of  $p$ ,  $U$  converges to the disconnected domain  $\mathbb{C} - \mathbb{R}$ , and thus  $U$  is not uniformly connected.
5. Figure 5 depicts the Julia set  $J(f)$  for  $f(z) = z^2 + c$  where  $c \approx$

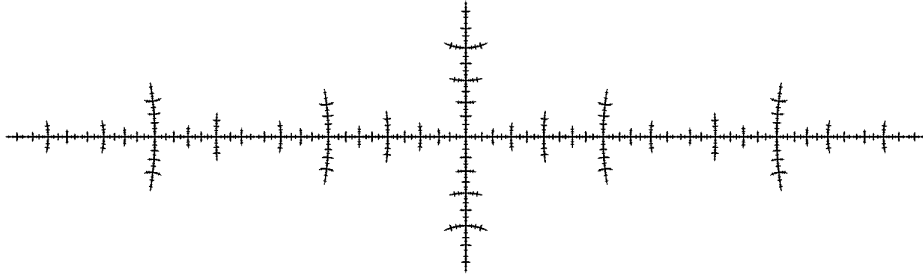


Figure 5. A Julia dendrite.

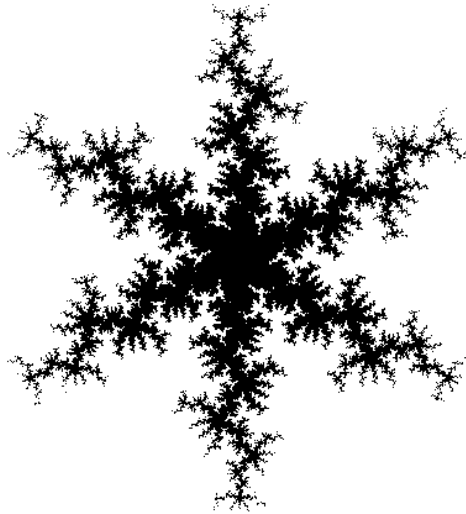


Figure 6. A Kleinian dendrite.

$-1.54369\dots$  is chosen so  $f^3(0) = f^4(0)$ . Here  $J(f)$  is a locally connected dendrite.

The complementary region  $U = \widehat{\mathbb{C}} - J(f)$  is a John domain [CJK], but it is not uniformly connected. Indeed, under suitable blowups around the origin,  $U$  converges to a planar region with 4 components, divided by 4 limiting arms of the Julia set. Compare [Tan].

6. The snowflake in Figure 6 is also a locally connected dendrite, arising as the limit set  $\Lambda$  of a geometrically infinite Kleinian group. In this example  $\Gamma$  is isomorphic to  $\langle a, b : [a, b]^3 = 1 \rangle$ , the fundamental group of a 2-dimensional orbifold  $S$  of genus one with a singular point of order 3. This  $\Gamma$  lies on the boundary of Bers' embedding of  $\text{Teich}(S)$ , where it behaves as the attracting fixed-point for the pseudo-Anosov mapping class  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ . An extended discussion of such groups can be found in [Mc2, §3] and [Mc3, §7].

The domain of discontinuity  $U = S_\infty^2 - \Lambda$  is uniformly connected, but not a John domain, as is evident from the narrow fjords reaching towards the center of the picture. In fact  $\partial U$  has measure zero [Th] but Hausdorff dimension two [Sul1], [BJ].

The center of symmetry  $c$  of the picture is a cut point of the limit set;  $\Lambda - \{c\}$  has six components. However, under blowups about  $c$ , the limit set converges to the plane and the region  $U$  converges to the empty set [Mc2, p.68], in contrast to the Julia set of example 5. Indeed, the furriness of  $\Lambda$  near any cut point is necessary by uniform connectivity of  $U$ .



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