

# Rational maps and Kleinian groups

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## 1 Introduction

There are many parallels between the theory of iterated rational maps  $f$  and that of Kleinian groups  $\Gamma$ , considered as dynamical systems on the Riemann sphere  $\widehat{\mathbb{C}}$ .<sup>1</sup> In this paper we will survey three chapters of this developing theory, and the Riemann surface techniques they employ:

1. The combinatorics of critically finite rational maps and the geometrization of Haken 3-manifolds via iteration on Teichmüller space.
2. Renormalization of quadratic polynomials and 3-manifolds which fiber over the circle.
3. Boundaries and laminations — Teichmüller space in Bers' embedding and the Mandelbrot set.

## 2 The theme of short geodesics

What are the possible topological forms for a conformal dynamical system?

Part of the answer is provided by two theorems, due to Thurston, which employ *iteration on Teichmüller space* to construct rational maps and Kleinian groups of a given topological form. More precisely, the iteration either *finds a geometric model* or *reveals a topological obstruction* to its existence. This dichotomy stems from:

**Theorem 1 [Mum].** *Let  $X_n$  be a sequence of points in the moduli space  $\mathcal{M}_{g,n}$  of hyperbolic Riemann surfaces of genus  $g$  with  $n$  punctures. After passing to a subsequence, either*

- $X_n$  converges to  $X$  in  $\mathcal{M}_{g,n}$ , or
- there is a collection of disjoint simple closed geodesics  $S_n$  on  $X_n$  such that the hyperbolic length of  $S_n$  tends to zero.

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<sup>1</sup> See [Sul2] for part of the dictionary.

## 2.1 Critically finite rational maps

Let  $f : \mathbf{S}^2 \rightarrow \mathbf{S}^2$  be a branched covering of the sphere of degree greater than one, and let  $P$  denote the *post-critical set* of  $f$ , i.e.

$$P = \bigcup_{n=1}^{\infty} f^n(B),$$

where  $B$  denotes the branch points (at which  $f$  is locally many-to-one). If  $|P| < \infty$  we say  $f$  is *critically finite*. Two such maps  $f$  and  $g$  are *combinatorially equivalent* if there is a homeomorphism  $h : (\mathbf{S}^2, P_f) \rightarrow (\mathbf{S}^2, P_g)$  such that  $hfh^{-1}$  and  $g$  are isotopic rel  $P_g$ .

A critically finite map is a generalization to the complex domain of the *kneading sequence* for maps of the interval.

The following theorem provides a topological characterization of critically finite rational maps.

**Theorem 2 [Th3], [DH3].** *Let  $f : \mathbf{S}^2 \rightarrow \mathbf{S}^2$  be critically finite with hyperbolic orbifold. Then either*

- *$f$  is combinatorially equivalent to a rational map  $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ , unique up to automorphisms of  $\widehat{\mathbb{C}}$ , or*
- *there is an  $f$ -invariant system of disjoint simple closed curves  $\Gamma$  in  $\mathbf{S}^2 - P$  providing a topological obstruction to such an equivalence.*

The technical condition “with hyperbolic orbifold” rules out certain elementary cases (which are also understood). It is satisfied, for example, if  $|P| > 4$ .

**Sketch of the proof.** The space of Riemann surface structures on  $(\mathbf{S}^2, P)$ , up to isotopy rel  $P$  is exactly the Teichmüller space of the sphere with  $|P|$  distinguished points, denoted  $\text{Teich}(\mathbf{S}^2, P)$ . Given such a structure, pull it back by  $f$  to obtain a new structure on the same space: this defines a map

$$T_f : \text{Teich}(\mathbf{S}^2, P) \rightarrow \text{Teich}(\mathbf{S}^2, P).$$

A fixed point for  $T_f$  gives an invariant complex structure and therefore a rational map  $g$  combinatorially equivalent to  $f$ .

This iteration has two fundamental features:

- $T_f^k$  *contracts* the Teichmüller metric (for some fixed iterate  $k$ ); and
- the contraction at a point  $X$  in  $\text{Teich}(\mathbf{S}^2, P)$  is less than  $c[X] < 1$  where  $c[X]$  is a continuous function depending only on the location of  $X$  in *moduli space*.

Now try to locate a fixed point of  $T_f$  by studying the sequence of iterates  $X_n = T_f^n(X_0)$  of an arbitrary starting guess  $X_0$ . If  $[X_n]$  returns infinitely often to a compact subset  $K$  of moduli space, then due to uniform contraction over  $K$ , the sequence converges to a fixed point and  $f$  is equivalent to a rational map.

Otherwise, by Mumford’s theorem, the length of the shortest geodesic on  $X_n$  tends to zero. Set  $\Gamma = \{\text{isotopy classes of very short geodesics on } X_n\}$  for  $n$

sufficiently large. Since the Teichmüller distance from  $X_n$  to  $X_{n+1}$  is bounded, lengths change by only a bounded factor. Therefore  $\Gamma$  is  $f$ -invariant, in the sense that any geodesic representing a component of  $f^{-1}(\gamma)$  is again in  $\Gamma$ .

Let  $A : \mathbb{R}^F \rightarrow \mathbb{R}^F$  be defined by  $A_{\delta\gamma} = \sum_{\alpha} \deg(f : \alpha \rightarrow \gamma)^{-1}$ , where the sum is over components  $\alpha$  of  $f^{-1}(\gamma)$  homotopic to  $\delta$ . By analyzing the geometry of short geodesics, one shows the leading eigenvalue of  $A$  is  $\geq 1$ . This provides the desired topological obstruction.

Indeed, if  $f$  is equivalent to a rational map  $g$ , then one can thicken the curves in  $\Gamma$  to disjoint annuli, with conformal moduli  $m_\gamma > 0$ . By considering inverse images of these annuli under  $g$ , one finds the same curves can be represented by annuli with moduli  $m'_\delta > \sum_\gamma A_{\delta\gamma} m_\gamma$ . It follows that some curve can be thickened to an annulus of arbitrarily large conformal modulus, a contradiction.  $\square$

## 2.2 Haken 3-manifolds

There is a parallel theory of iteration in Thurston's construction of hyperbolic structures on Haken manifolds. For simplicity we stick to closed manifolds.

To a Kleinian group  $\Gamma$  one associates the 3-dimensional *Kleinian manifold*  $N = (\mathbb{H}^3 \cup \Omega)/\Gamma$ , where  $\Omega \subset \hat{\mathbb{C}}$  is the domain of discontinuity. Then  $N$  has a hyperbolic structure on its interior and a conformal structure on its boundary.

**Theorem 3** [Th1], [Mor] . *Let  $M^3$  be a closed Haken 3-manifold. Then either*

- *$M^3$  is diffeomorphic to a unique hyperbolic manifold  $\mathbb{H}^3/\Gamma$ , or*
- *there is a map of a torus into  $M^3$ , injective on  $\pi_1$ , providing a topological obstruction to a hyperbolic structure on  $M^3$ .*

**Sketch of the proof.** We combine Thurston's original approach with the Riemann surface techniques of [Mc2] and emphasize the parallel with the geometrization of rational maps.

A 3-manifold is *Haken* if it can be constructed by starting with 3-balls, and repeatedly gluing along incompressible submanifolds of the boundary. The idea of the proof is to carry out the construction geometrically, at each stage providing the pieces with hyperbolic structures. An orbifold technique [Mor, Fig. 14.6] reduces the problem to the case of gluing along the entire boundary.

Iteration enters at the inductive step: given a compact 3-manifold  $M^3$  and gluing instructions encoded by an orientation-reversing involution  $\tau : \partial M^3 \rightarrow \partial M^3$ , we must construct a hyperbolic structure on  $M^3/\tau$ . By induction  $M^3$  is diffeomorphic to a Kleinian manifold  $N^3$ . Unlike the case of a closed manifold, which admits at most one hyperbolic structure by Mostow rigidity, the manifold  $N^3$  is flexible. The set of possible shapes for  $N^3$  is parameterized by the Teichmüller space of the boundary of  $M$ .

Which structure descends to  $M/\tau$ ? The answer can be formulated as a fixed point problem on Teichmüller space. Using the topology of  $M^3$ , Thurston defines the *skinning map*

$$\sigma : \text{Teich}(\partial M) \rightarrow \text{Teich}(\overline{\partial M})$$

by forming quasifuchsian covering spaces for each component of the boundary, and recording the conformal structure on the new ends which appear. The gluing instructions determine an isometry

$$\tau : \text{Teich}(\overline{\partial M}) \rightarrow \text{Teich}(\partial M),$$

and a fixed point for

$$T = \sigma \circ \tau$$

solves the gluing problem.

Here the parallel with the construction of critically finite maps emerges. The completion of the proof will follow [Mc2].

Assume  $M^3$  is not an interval bundle over a surface (this special case is discussed in the next section). Then some fixed iterate  $T^k$  contracts the Teichmüller metric; in fact:

- The contraction of  $T^k$  at a point  $X$  in  $\text{Teich}(\partial M)$  is bounded by  $c[X] < 1$  where  $c[X]$  depends only on the location of  $X$  in *moduli* space.

As before, this reduces the proof to an analysis of short geodesics. Let  $X_n = T^n(X_0)$  be the forward orbit of a starting guess  $X_0$  in  $\text{Teich}(\partial M)$ . If  $[X_n]$  returns infinitely often to a compact subset of moduli space, the sequence converges and the gluing problem is solved.

Otherwise  $X_n$  develops short geodesics. With further analysis one finds these short geodesics bound cylinders in  $M^3$ , joined by  $\tau$  to form an incompressible torus in  $M^3/\tau$ . A closed hyperbolic manifold contains no such torus (it must correspond to a cusp), so we have located a topological obstruction to a hyperbolic structure.  $\square$

The bound on contraction  $c[X]$  comes from a general result in the theory Riemann surfaces.

Let  $Y \rightarrow X$  be a covering space of a hyperbolic Riemann surface  $X$  of finite area. Then there is a natural map  $\theta : \text{Teich}(X) \rightarrow \text{Teich}(Y)$ , defined by lifting complex structures from  $X$  to  $Y$ . Consider the case of the universal covering  $\Delta \rightarrow X$  where  $\Delta$  is the unit disk;  $G$  denotes the Fuchsian group of deck transformations.

**Theorem 4 [Mc1].** *The map  $\theta : \text{Teich}(X) \rightarrow \text{Teich}(\Delta)$  is a contraction for the Teichmüller metric. Moreover  $\|d\theta\| < c[X] < 1$  where  $c$  depends continuously on the location of  $X$  in moduli space.*

This theorem is related to classical Poincaré series, as follows. For any Riemann surface  $R$ , let  $Q(R)$  denote the Banach space of integrable holomorphic quadratic differentials  $\phi(z)dz^2$  with  $\|\phi\| := \int_R |\phi| < \infty$ . Starting with  $\phi$  in  $Q(\Delta)$ , we can construct an automorphic form for  $G$  by the Poincaré series [Poin]:

$$\Theta(\phi) = \sum_G g^*(\phi).$$

Since  $\Theta(\phi)$  is  $G$ -invariant, it determines an element of  $Q(X)$ .

In Teichmüller theory,  $Q(X)$  is naturally identified with the cotangent space to  $\text{Teich}(X)$  at  $X$ , its norm is dual to the Teichmüller metric, and the operator  $\Theta : Q(\Delta) \rightarrow Q(X)$  is the coderivative  $d\theta^*$ . This gives:

**Corollary 5 (Kra's Theta conjecture)**  $\|\Theta\| < 1$  for classical Poincaré series.

On a global level the theorem says that lifts of Teichmüller mappings can be relaxed (isotoped to mappings of less dilatation):

**Corollary 6** Let  $f : X_0 \rightarrow X_1$  be a Teichmüller mapping between distinct points in  $\text{Teich}(X)$ . Then the map  $\tilde{f} : \Delta \rightarrow \Delta$  obtain by lifting  $f$  to the universal covers of domain and range is not extremal among quasiconformal with the same boundary values on  $S^1$ .

More generally, these contraction principles apply to a covering  $Y \rightarrow X$  iff the covering is *nonamenable*; see [Mc1].

Now for a typical (acylindrical)  $M^3$  the skinning map  $\sigma$  can be described as follows. Given a Riemann surface  $X$  in  $\text{Teich}(\partial M)$ ,

- (1) form countably many copies of its universal cover  $\tilde{X}$ , then
- (2) glue them together in a pattern determined by the combinatorics of  $M^3$  to obtain a new Riemann surface  $\sigma(X)$ .

The surface  $\sigma(X)$  contains a dense full measure set of open disks each of which is canonically identified with the universal cover of  $X$ . By the results above, step (1) is a contraction for the Teichmüller metric. Step (2) is at worst an isometry, so  $\|d\sigma\| \leq \|d\theta\| < c[X] < 1$ . (A more detailed expository account appears in [Mc5].)

### 3 Renormalization and 3-manifolds which fiber over the circle

For the special case of Haken manifolds presented as surface bundles over the circle, the construction of a hyperbolic structure is different in spirit and finds parallels with the construction of fixed points for renormalization.

#### 3.1 Surface bundles

Let  $S$  be a closed oriented surface of genus  $g > 1$ , and let  $\phi : S \rightarrow S$  be a *mapping class*, that is a diffeomorphism determined up to isotopy. From this data one can construct a 3-manifold by starting with  $M^3 = S \times [0, 1]$  and gluing the ends together by  $\phi$ . Every 3-manifold which fibers over the circle admits such a description;  $\phi$  is the *monodromy* of the fibration.

**Theorem 7 [Th2].** A 3-manifold  $M^3$  which fibers over the circle admits a hyperbolic structure iff the monodromy  $\phi$  is pseudo-Anosov.

As in the preceding section, the construction of a hyperbolic structure can be formulated as a fixed point problem. There are two essential differences: (1) the desired fixed point lies on the boundary of Teichmüller space, rather than in its interior, and (2) it is dynamically hyperbolic rather than attracting.

**Construction of the hyperbolic structure.** We will work in the representation variety  $\mathcal{V} = \text{Hom}(\pi_1(S), \text{Isom}(\mathbb{H}^3))/\text{conjugation}$ . Let  $AH(S) \subset \mathcal{V}$  denote the closed subset of discrete faithful representations. The idea is to find in  $AH(S)$  the  $\mathbb{Z}$ -covering space of  $M^3$  carrying the fundamental group of a fiber. The deck transformation acts by isometry on this covering space, so it is characterized as a fixed point in  $AH(S)$  for the map

$$\Phi : \mathcal{V} \rightarrow \mathcal{V}$$

given by  $\Phi(\rho) = \rho \circ \pi_1(\phi)$ .

The construction of the fixed point can be organized into two steps. Let  $QF(S) \subset AH(S)$  be the open subset of quasifuchsian groups; it is holomorphically parameterized by  $\text{Teich}(S) \times \text{Teich}(\overline{S})$  (here  $\overline{S}$  indicates reversal of orientation) and we denote by  $\rho(X, Y)$  the marked group corresponding to a pair of Riemann surfaces  $X$  and  $Y$ .

(Step 1) Form the limit  $\rho_\infty = \lim_{n \rightarrow \infty} \rho(X, \phi^{-n}(Y))$ . Here  $X$  and  $Y$  are arbitrary Riemann surfaces and  $\phi(Y)$  denotes the action of the mapping class on Teichmüller space. The representations  $\rho(X, \phi^{-n}(Y))$  range in a Bers' slice, which has compact closure in  $AH(S)$ , so the existence of some accumulation point is clear. Logically one can work with any accumulation point  $\rho_\infty$ ; in fact, the sequence converges [CT, §7].

(Step 2) Form the limit  $\rho = \lim \Phi^n(\rho_\infty)$ ; this is a fixed point for  $\Phi$ . Existence of this limit depends on

**Compactness:** the double limit theorem of [Th2], which assures there is some accumulation point  $\rho$ ; and

**Rigidity:** Sullivan's quasiconformal rigidity theorem [Sul1], which gives  $\Phi(\rho) = \rho$  for any accumulation point.

The first step produces a point on the stable manifold of a fixed point of  $\Phi$ , and the second iterates it to find the fixed point.

The limit in (Step 2) can be lifted to the level of marked groups  $G_n \subset \text{Aut}(\mathbb{H}^3)$  (rather than groups up to conjugacy) such that  $G_n$  tends algebraically to  $G = \text{Image}(\rho)$ . The groups  $G_n$  (conjugate to  $\text{Image}(\Phi^n(\rho_\infty))$ ) are all isomorphic; we are viewing a single dynamical system from a changing perspective. As  $n \rightarrow \infty$  the limit set of  $G_n$  becomes denser and denser, and the limit set of  $G$  is the full sphere.

The group  $G_{n+1}$  is obtained from  $G_n$  by a  $K$ -quasiconformal deformation with uniform  $K$ . By compactness of  $K$ -quasiconformal maps, one obtains a quasiconformal map  $\psi$  equivariant with respect to  $G$  and inducing the automorphism  $\Phi$ . Since the limit set is the whole sphere,  $\psi$  is conformal by Sullivan's result. The group generated by  $G$  and  $\psi$  together is then a Kleinian group isomorphic to  $\pi_1(M^3)$ .

### 3.2 Quadratic-like maps

This discussion parallels the emerging complex viewpoint on *renormalization* of quadratic-like maps. For concreteness we will discuss the case of period doubling; see [Cvi], [Milnor], [Sul3], and [Sul4] for background and more details.

Consider the family of quadratic maps  $z \mapsto z^2 + c$  as the parameter  $c$  decreases along the real axis, starting at  $c = 0$ . One finds a sequence of parameter values  $c(n)$  for which the attractor of  $f_n(z) = z^2 + c(n)$  bifurcates from a cycle of order  $2^n$  to  $2^{n+1}$ ; at  $c(\infty) = \lim c(n)$  the attractor becomes a Cantor set. This *cascade of period doublings* was observed by Feigenbaum to have many universal features around  $f_\infty(z) = z^2 + c(\infty)$ . For example

$$\frac{c(n) - c(\infty)}{c(n+1) - c(\infty)} \rightarrow \lambda = 4.669201609\dots$$

and this value of  $\lambda$  (as well as the fine structure of  $f_\infty$ , such as the Hausdorff dimension of its attracting Cantor set) is the same for other families of smooth mappings with the same topological form as  $z^2 + c$ .

This universality is part of a larger renormalization picture proposed by Feigenbaum and established rigorously by Lanford and others. We present a version with the complex quadratic-like maps of Douady and Hubbard [DH2]; cf. [Sul3].

A *quadratic-like mapping*  $f : U \rightarrow V$  is a proper degree two holomorphic map between open disks with  $\bar{U} \subset V \subset \mathbb{C}$ . Its *filled-in Julia set*  $K(f)$  is  $\bigcap_1^\infty f^{-n}(V)$ . When  $K(f)$  is connected, there is a unique quadratic polynomial  $I(f)$  (the *inner class*) conjugate to  $f$  near  $K(f)$  by a quasiconformal map which is conformal on  $K(f)$ . Thus  $I$  takes values in the Mandelbrot set  $M$  of polynomials  $z^2 + c$  with connected Julia sets.

Let  $\mathcal{Q}$  be the space of all analytic maps  $f : \Omega_f \rightarrow \mathbb{C}$  defined on a region  $\Omega_f$  containing the origin, such that  $f'(0) = 0$  and  $f$  is quadratic-like on some neighborhood of zero. We identify  $f(z)$  and  $g(z)$  if some rescaling  $\alpha f(z/\alpha)$  agrees with  $g(z)$  on their common domain of definition. Finally  $f_i \rightarrow f$  if there are representatives of  $f_i$  which converge to  $f$  uniformly on compact subsets of  $\Omega_f$ .

The *renormalization operator*  $\mathcal{R} : \mathcal{Q}' \rightarrow \mathcal{Q}$  is given by  $\mathcal{R}(f) = f \circ f$ . It is defined on an open set  $\mathcal{Q}'$  such that  $f \circ f$  is still quadratic-like near the origin.

Central to the picture is the existence of a unique fixed point  $g$  for  $\mathcal{R}$ . Since  $g$  and  $\mathcal{R}(g)$  are equivalent,  $g$  satisfies the Cvitanović-Feigenbaum functional equation  $\alpha g \circ g(z) = g(\alpha z)$ . The universal constant  $\lambda$  above is the unique expanding eigenvalue for  $\mathcal{R}$  at  $g$ .

We will sketch a construction of this fixed point  $g$  which parallels the geometrization of surface bundles.

Douady and Hubbard define a *tuning map*  $\tau : M \rightarrow M$  such that  $I(\mathcal{R}(f)) = \tau^{-1}(I(f))$  when defined. Thus  $\tau$  describes the inverse of renormalization as it acts on the inner class. One finds that  $\tau(f_n) = f_{n+1}$  and  $\tau$  fixes the Feigenbaum polynomial  $f_\infty$ . This accomplishes:

(Step 1) Form the limit  $f_\infty = \lim \tau^n(f_0)$ .

Now let  $\mathcal{Q}_\infty$  denote those  $f$  with inner class  $I(f) = f_\infty$ . Then  $\mathcal{R}(\mathcal{Q}_\infty) \subset \mathcal{Q}_\infty$  and  $f$  and  $\mathcal{R}(f)$  are quasiconformally conjugate near  $K(f)$  for any  $f$  in  $\mathcal{Q}_\infty$ .

(Step 2) Form the limit  $g = \lim \mathcal{R}^n(f_\infty)$ . This is a fixed point for  $\mathcal{R}$ , and in fact all  $f$  in  $\mathcal{Q}_\infty$  are attracted to  $g$ .

The proof of (Step 2) again appeals to two principles.

**Compactness:** For any  $f$  in  $\mathcal{Q}_\infty$ ,  $\langle \mathcal{R}^n(f) \rangle$  ranges in a compact subset of  $\mathcal{Q}$ . This is a fundamental result of Sullivan [Sul4]. Thus there is a subsequence of  $n$  such that  $\mathcal{R}^n(f) \rightarrow g_0$ ,  $\mathcal{R}^{n-1}(f) \rightarrow g_1$ ,  $\dots$ ,  $\mathcal{R}^{n-k}(f) \rightarrow g_k$  and the tower  $\langle g_0, g_1, \dots \rangle$  satisfies  $\mathcal{R}(g_k) = g_{k-1}$ . We can then apply:

**Rigidity:** Such a tower admits no quasiconformal deformations [Mc4].

Since  $\langle g_0, g_1, \dots \rangle$  is conjugate to  $\langle g_1, g_2, \dots \rangle$  by a suitable limit of a quasiconformal conjugacy between  $f$  and  $\mathcal{R}(f)$ , these towers are conformally identical, and in particular  $\mathcal{R}(g_0) = g_0$ . By rigidity of all limiting towers, the full sequence  $\mathcal{R}^n(f) \rightarrow g_0$  and this fixed point is unique.

**Geometric limits.** The fixed point of renormalization  $g$  is *not* itself rigid. Its universal structure is a result of being embedded deep in the dynamics of  $f$ . The tower  $\langle g_0, g_1, \dots \rangle$  can be thought of as a geometric limit of the dynamical system generated by  $f$  as one rescales about its critical point. The limiting dynamic is *divisible* ( $g_n = g_{n+1} \circ g_{n+1}$ ), and its Julia set fills the whole plane. If we set  $g_{-n}$  = the  $2^n$ th iterate of  $g_0$ , then renormalization acts as a shift on the bi-infinite tower  $\langle \dots g_{-1}, g_0, g_1 \dots \rangle$  in a manner reminiscent of the deck transformation acting on the  $\mathbb{Z}$ -covering space of  $M^3$  constructed before.

**Self-similarity in the Mandelbrot set and in Bers' slice.** In the polynomial-like setting, one can actually define countably many renormalization and tuning operators  $\mathcal{R}_c, \tau_c$ , one for each  $c$  such that the critical point of  $z^2 + c$  is periodic. Milnor has made a detailed computer study of these operators, supporting many conjectures [Milnor]; among them, that  $\tau_c$  has a unique fixed point and is differentiable there, with derivative given by the inverse of the expanding eigenvalue of  $\mathcal{R}_c$  at *its* fixed point.

Similarly, we conjecture (in the case of one dimensional Teichmüller spaces) that Bers' boundary is self-similar about the point  $\rho_\infty$  constructed in (Step 1), with similarity factor given by the expanding eigenvalue of the mapping class  $\Phi$  at the fixed point of (Step 2). Dave Wright's computer study of the closely related Maskit boundary for the Teichmüller space of a punctured torus supports this conjecture [Wr]. In this case the mapping class group is  $SL_2\mathbb{Z}$ , and the expanding eigenvalue is algebraic (but different from the eigenvalues of the matrix). For example, when  $\Phi = \begin{pmatrix} 21 & \\ & 11 \end{pmatrix}$ , the boundary scales by  $\lambda = 4.79129\dots = \frac{5+\sqrt{21}}{2}$ ; see Figure 1 for two blowups around  $\rho_\infty$  computed by Wright.

**Remark.** Sullivan has established a compactness theorem for arbitrary compositions of a finite number of renormalization operators  $\mathcal{R}_c$ , with the condition that  $c$  is real. Our rigidity argument applies whenever such a compactness result is available. Much progress on a conceptual understanding of the full renormalization picture, including a different approach to rigidity, appears in [Sul4].

**Fig. 1.** Self-similarity at the edge of Teichmüller space.

## 4 Boundaries and laminations.

We conclude with a very brief account of progress on the boundary of Teichmüller space and the boundary of the Mandelbrot set.

**Conjecture.** (Douady–Hubbard). The Mandelbrot set  $M$  is locally connected. Its boundary is homeomorphic to a quotient of the circle by an explicit combinatorial equivalence relation. (Cf. [DH1], [Dou], [Lav] and [Th3].)

**Conjecture.** (Thurston) Bers’ boundary for Teichmüller space, modulo quasiconformal equivalence, is homeomorphic to the space  $\mathbb{P}\mathcal{ML}$  of projective measured laminations, modulo forgetting the measure.

Both conjectures express the hope that certain geometrically infinite dynamical systems can be uniquely described by a lamination on the circle — invariant

under  $z^2$  in the first case, and under the action of a surface group in the second.

Measures supported on maximal systems of disjoint simple closed curves are dense in  $\mathbb{PML}$ ; these correspond to *maximal cusps* in Bers' boundary, that is geometrically finite limits of quasifuchsian groups where these curves have been pinched to form rank one cusps. Thus Thurston's conjecture is supported by:

**Theorem 8 [Mc3].** *Maximal cusps are dense in Bers' boundary.*

This result was conjectured by [Bers]. The proof is by via an explicit estimate for the algebraic effect of a quasiconformal deformation supported in the thin part.

Also relevant is Bonahon's result: a general geometrically infinite surface group admits an ending lamination [Bon], supporting the conjecture that geometrically finite groups are dense in  $AH(S)$ .

Progress on the Mandelbrot set includes the following breakthrough:

**Theorem 9 (Yoccoz).**  *$M$  is locally connected at every quadratic polynomial which is not in the image of a tuning map.*

Yoccoz's result brings us a step closer to resolving the well-known:

**Conjecture.** Hyperbolic dynamics is open and dense in the space of complex quadratic polynomials.

It seems likely that Yoccoz's theorem generalizes to the case of polynomials lying in the image of only finitely many tuning maps. If so, by [MSS], the density of hyperbolic dynamics is equivalent to the quasiconformal rigidity of infinitely renormalizable polynomials.

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