

## Math 275 Week 8 Homework Solutions

**Exercise. (82)** Let  $\Lambda$  be the limit set of a nonelementary discrete group  $\Gamma \subset G = SL_2(\mathbb{R})/(\pm I)$ . (i) Show that the fixed points of hyperbolic elements of  $\Gamma$  are dense in  $\Lambda$ . (ii) Suppose  $\Lambda = S_\infty^1$ . Show there exists a dense geodesic on  $X = \Gamma \backslash \mathbb{H}$ .

*Proof.* (i) Since  $\Gamma$  is nonelementary, it contains a hyperbolic element. If  $g \in \Gamma$  is hyperbolic, then any forward orbit  $g^n p \in \mathbb{H}$  converges as  $n \rightarrow +\infty$  to a fixed point  $p_\infty \in \Lambda \subset S_\infty^1$  of  $g$ . For any  $h \in \Gamma$ , the point  $hp_\infty$  is a fixed point of the hyperbolic element  $hgh^{-1} \in \Gamma$ . So, the set  $F \subset S_\infty^1$  of fixed points of hyperbolic elements of  $\Gamma$  is a nonempty  $\Gamma$ -invariant subset of its limit set  $\Lambda$ . The same holds for the closure  $\overline{F}$ , and thus by minimality  $\overline{F} = \Lambda$ .

(ii) For any open ball  $B \subset \mathbb{H}$ , the subset  $S(B)$  of  $x \in S_\infty^1$  such that the geodesic ray from  $i$  to  $x$  meets  $\Gamma B$  is nonempty and open. Since  $\Lambda = S_\infty^1$ , for any  $p \in B$  there is a sequence  $\gamma_n \in \Gamma$  such that  $\gamma_n p \rightarrow x$ , so any neighborhood of  $x \in S_\infty^1$  meets  $S(B)$ . Therefore,  $S(B) \subset S_\infty^1$  is dense and open. Take a countable collection of balls  $B_n$  generating the topology on  $\mathbb{H}$  (for instance, balls with rational radius and center in  $\mathbb{Q}(i) \cap \mathbb{H}$ ). By the Baire category theorem, the countable intersection  $\bigcap_n S(B_n)$  is dense in  $S_\infty^1$ , and in particular nonempty. For any  $x \in \bigcap_n S(B_n)$ , the geodesic in  $\mathbb{H}$  containing the ray from  $i$  to  $x$  projects to a dense geodesic in  $X = \Gamma \backslash \mathbb{H}$ .  $\square$

**Exercise. (86)** Discuss the moduli space  $M = A \backslash G/N$  of pairs  $(\gamma, \eta)$ , where  $\gamma$  is an oriented geodesic and  $\eta$  is a horocycle. In particular, construct a continuous map  $D : M \rightarrow \mathbb{R}$  that is generically one-to-one; and show  $M$  is not Hausdorff. For what values of  $D(\gamma, \eta)$  is  $\gamma$  tangent to  $\eta$ ? For what values do  $\gamma$  and  $\eta$  cross? When does one endpoint of  $\gamma$  coincide with the center (at infinity) of  $\eta$ ? Let  $\iota : M \rightarrow M$  be the map that reverses the orientation of  $\gamma$ . Show that  $M/\iota$  is Hausdorff and describe it as a topological space.

*Proof.* We view  $M = A \backslash G/N$  as the space of  $A$ -orbits on  $G/N \cong (\mathbb{R}^2 - \{(0,0)\})/(\pm I)$ . Under this identification, the left action of  $G$  on  $G/N$  is sent to the linear action of  $PSL_2(\mathbb{R})$  on  $(\mathbb{R}^2 - \{(0,0)\})/(\pm I)$ . The subgroup  $A$  consists of diagonal matrices  $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ , which preserve the values of the function  $(x, y) \mapsto xy$ , so this function descends to a well-defined continuous map  $D : M \rightarrow \mathbb{R}$ , which is 1-to-1 whenever  $xy \neq 0$ . As  $xy \rightarrow 0$ , the  $A$ -orbits have 2 distinct limits, namely the vertical rays from the origin and the horizontal rays from the origin, so  $M$  is not Hausdorff. We have  $D(\gamma, \eta) = \pm \frac{1}{2}$  if and only if  $\gamma$  is tangent to  $\eta$ ;  $D(\gamma, \eta) \in (-\frac{1}{2}, \frac{1}{2})$  if and only if  $\gamma$  and  $\eta$  cross; and  $D(\gamma, \eta) = 0$  if and only if one endpoint of  $\gamma$  coincides with the center at infinity of  $\eta$ . The map  $\iota : M \rightarrow M$  lifts to the map  $(x, y) \mapsto (-y, x)$  on  $\mathbb{R}^2 - \{(0,0)\}$ , so in the quotient  $M/\iota \cong \mathbb{R}_{\geq 0}$ , the vertical and horizontal rays from the origin represent a single point, and the hyperbolas in  $xy = \pm x_0 y_0 \neq 0$  represent a single point.  $\square$

**Exercise. (89)** Find a hyperbolic element  $g \in SL_2(\mathbb{Z})$  that fixes  $\sqrt{5} \in \mathbb{R}$ .

*Proof.* For any hyperbolic element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ , we have  $c \neq 0$  and  $|a+d| > 2$ , and its two fixed points in  $\mathbb{R}$  are the Galois conjugate quadratic irrationals

$$x, x' = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$$

When  $a = d$ , the fixed points are just  $\pm\sqrt{a^2 - 1}/c$ , so to realize  $\sqrt{5}$  as a fixed point, it is enough to find integer solutions to

$$a^2 - 5c^2 = 1$$

This is an instance of Pell's equation, and a solution can be obtained from the convergents of the continued fraction expansion

$$\sqrt{5} = [2; \overline{4}] = 2 + \frac{1}{4 + \frac{1}{4 + \dots}}$$

The first few convergents are  $2/1, 9/4, 42/17, \dots$ , and in fact  $(9, 4)$  is a solution. Translating back, the hyperbolic element  $\begin{pmatrix} 9 & 20 \\ 4 & 9 \end{pmatrix} \in SL_2(\mathbb{Z})$  fixes  $\sqrt{5} \in \mathbb{R}$ .  $\square$

**Exercise. (90)** Let  $\Gamma = SL_2(\mathbb{Z})$ . Show that for any  $g \in GL_2(\mathbb{Q})$ , the group  $\Delta = \Gamma \cap g\Gamma g^{-1}$  has finite index in  $SL_2(\mathbb{Z})$ .

*Proof.* Pick  $N \in \mathbb{Z}$  such that the matrices  $Ng, Ng^{-1}$  have entries in  $\mathbb{Z}$ . The subgroup

$$\Gamma(N^2) = \{M \in SL_2(\mathbb{Z}) : M \equiv I \pmod{N^2}\}$$

is the kernel of a homomorphism  $SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/N^2\mathbb{Z})$  to a finite group, so it is a normal subgroup of finite index in  $SL_2(\mathbb{Z})$ . For any  $I + N^2A \in \Gamma(N^2)$ ,

$$g^{-1}(I + N^2A)g = I + (Ng^{-1})A(Ng)$$

has integer entries and determinant 1, so it is an element of  $\Gamma$ . Then  $\Gamma(N^2) \subset g\Gamma g^{-1}$ , so  $\Gamma \cap g\Gamma g^{-1}$  has finite index in  $\Gamma$ .  $\square$

**Exercise. (91)** Show that the pairs of fixed points  $(x, y)$  of hyperbolic elements of  $SL_2(\mathbb{Z})$  form a dense subset of  $\mathbb{R}^2$ . Conclude from this that closed geodesics are dense in  $X = SL_2(\mathbb{Z}) \backslash \mathbb{H}$ .

*Proof.* For any hyperbolic element  $g$  and  $0 \neq n \in \mathbb{Z}$ , the fixed points of  $g$  and  $g^n$  are the same, so the subset  $S \subset (\mathbb{R} \cup \infty) \times (\mathbb{R} \cup \infty)$  of pairs of fixed points of hyperbolic elements is preserved under passing to finite-index subgroups. In the case of  $SL(2, \mathbb{Z})$ , we have  $S \subset (\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q})$ . By Exercise (90),  $S$  is invariant under the action of  $GL_2(\mathbb{Q})$  by fractional linear transformations. Given, e.g., that  $(\sqrt{5}, -\sqrt{5}) \in S$  (by Exercise (89)), we then have that  $S$  contains any pair of Galois conjugate irrationals in  $\mathbb{Q}(\sqrt{5})$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , these pairs form a dense subset of  $\mathbb{R}^2$ . Any geodesic in  $\mathbb{H}$  with endpoints  $(x, y) \in S$  projects to a closed geodesic in  $X$ . Since  $S$  is dense in  $\mathbb{R}^2$ , the corresponding geodesics are dense in  $\mathbb{H}$ , and they project to a dense subset of  $X$  consisting of closed geodesics.  $\square$