Math 275 Week 10 Homework Solutions

Exercise. (98) Explain the fallacy in the following "proof" that the horocycle flow for a compact hyperbolic surface X is uniquely ergodic. (i) by mixing of the geodesic flow, any large circle $S^1(x,r)$ is nearly equidistributed in T_1X; (ii) horocycles are limits of spheres as $r \to \infty$, so they too are uniformly distributed.

Proof. The fallacy is in step (ii). Horocycles are "limits" of spheres $S(x_n, r_n)$ with both $r_n \to \infty$ and $x_n \to \infty$, and this is not enough to imply that horocycles must be equidistributed. In the upper half-plane, a large circle tangent to a horocycle will spend most of its time far away from that horocycle. In the case of where X has finite-volume and a cusp, large circles are still nearly equidistributed on X, but there are large circles tangent at a point to closed horocycles on X. □

Exercise. (99) Prove that the solvable Lie group $AN \subset G = SL_2(\mathbb{R})$ contains no lattice.

Proof. Let $\Gamma \subset AN$ be a discrete subgroup. Every element of Γ fixes $\infty \in \partial\mathbb{H}$, so Γ contains no nontrivial elliptic elements, and the parabolic elements in Γ form a cyclic subgroup $\Gamma \cap N$. A lattice in AN cannot be cyclic, so Γ contains a hyperbolic element γ, and an element $\rho \notin \gamma^{\mathbb{Z}}$. By conjugating, we may assume $\gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ for some $a > 0$, and $\rho = \begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix}$ for some $b, c \neq 0$. But then the conjugates $\gamma^{-n}\rho\gamma^n = \begin{pmatrix} b & ca^{-2n} \\ 0 & b^{-1} \end{pmatrix}$ converge to $\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$, contradicting discreteness of Γ. □

Exercise. (100) Let $\alpha \in \mathbb{R}$ be a quadratic irrational. Show there exists $C > 0$ such that $|\alpha - p/q| \geq C/q^2$ for all $p/q \in \mathbb{Q}$.

Proof. Let α' be the Galois conjugate of α. The norm $N(\alpha) = \alpha\alpha'$ and trace $T(\alpha) = \alpha + \alpha'$ are rational numbers, and since $\alpha \neq 0$, $N(\alpha) \neq 0$. There is an integer $m \geq 2$ such that $mN(\alpha)$ and $mT(\alpha)$ are integers. Since $\alpha \notin \mathbb{Q}$, for any $p/q \in \mathbb{Q}$,

$$mq^2 N\left(\alpha - \frac{p}{q}\right) = mq^2 N(\alpha) - mpqT(\alpha) + mp^2 \in \mathbb{Z} \setminus 0,$$

so in particular

$$\frac{1}{mq^2} \leq \left| N\left(\alpha - \frac{p}{q}\right) \right| = \left|\alpha - \frac{p}{q}\right| \cdot \left| \left(\alpha - \frac{p}{q}\right) + (\alpha' - \alpha) \right|.$$

One of the factors on the right-hand side must be at least $\frac{1}{2}|\alpha' - \alpha|$, which depends only on α, so suppose $|\alpha - p/q| < \frac{1}{2}|\alpha' - \alpha|$. Then

$$\frac{1}{mq^2} \leq \frac{1}{2}\left|\alpha - \frac{p}{q}\right| \left(\left|\alpha - \frac{p}{q}\right| + |\alpha' - \alpha| \right) \leq \left|\alpha - \frac{p}{q}\right| \cdot |\alpha' - \alpha|$$

Thus,

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{mq^2|\alpha' - \alpha|}$$

and we can take $C = \frac{1}{m|\alpha'-\alpha|}$. □

Exercise. (101) Show there exists an irrational quadratic form $Q(x, y) = x^2 - \beta y^2$, such that $|Q(x, y)| > \varepsilon > 0$ for all nonzero $(x, y) \in \mathbb{Z}^2$.

Proof. Pick $\beta = \alpha^2 \notin \mathbb{Q}$, with $\alpha > 0$ a quadratic irrational. Clearly $|Q(x, 0)| \geq 1$ whenever $x \neq 0$, so we may consider $(x, y) \in \mathbb{Z}^2$ with $y \neq 0$. Either $\left|\frac{x}{y} - \alpha\right| \geq \alpha$ or $\left|\frac{x}{y} + \alpha\right| \geq \alpha$, so by Exercise (100), there is $C > 0$ depending only on α such that

$$\frac{|Q(x, y)|}{y^2} = \left|\frac{x}{y} - \alpha\right| \left|\frac{x}{y} + \alpha\right| \geq \frac{C}{y^2} \cdot \alpha$$

Then $|Q(x, y)| \geq C\alpha > 0$. □

Exercise. (102) Prove that the quadratic form $Q(x) = x_1^2 + x_2^2 - D(x_3^2 + x_4^2)$ does not represent zero when $D = 3 \mod 4$.

Proof. Suppose $Q(x) = 0$ with $x_i \in \mathbb{Z}$. Since $Q(x/2) = 0$ as well, we may assume some x_i is odd. Consider $Q(x) \mod 8$, and note that $x_i = 1 \mod 8$, $D \in \{3, 7\} \mod 8$, and $x_j \in \{0, 1, 4\} \mod 8$ for $j = 1, 2, 3, 4$. So $x_1^2 + x_2^2 \in \{0, 1, 2, 4, 5\} \mod 8$, and $x_3^2 + x_4^2 \in \{0, 1, 2, 4, 5\} \mod 8$. Since $D^2 = 1 \mod 8$, by multiplying by D we may assume $i \in \{1, 2\}$. Then $x_1^2 + x_2^2 \in \{1, 2, 5\} \mod 8$, and $D(x_3^2 + x_4^2) \in \{0, 3, 4, 6, 7\} \mod 8$, a contradiction. □