Math 213b HW4 Solutions

March 7, 2019

Problem 24

Fix $n \geq 2$, and let $\zeta_n = \exp(2\pi i/n)$. Consider the action of $G \cong \mathbb{Z}/n$ on $\mathbb{C}^2$ generated by $(u, v) \mapsto (\zeta_n u, \zeta_n^{-1} v)$. (i) Show that $\mathbb{C}^2/G$ is isomorphic to the affine variety defined by $z^n = xy$. (ii) Conclude that the lens space $L(n, 1)$ is homeomorphic to an n-fold cover of S^3 branched over the Hopf link.

Solution

(i) Using the G-invariant polynomials u^n, v^n, uv on $\mathbb{C}^2$, we can define a map $\mathbb{C}^2 \to \mathbb{C}^3$, $(u, v) \to (u^n, v^n, uv)$, that factors through the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G$, and has image contained in the locus $\{z^n = xy\}$. One then checks that the resulting map $\mathbb{C}^2/G \to \{z^n = xy\}$ is an isomorphism of varieties.

(ii) We take as our definition of $L(n, 1)$ the quotient S^3/G, where $S^3 \subset \mathbb{C}^2$ is the real unit sphere and G acts as in part (i). The projection map $\mathbb{C}^3 \to \mathbb{C}^2$, $(x, y, z) \mapsto (x, y)$, restricts to

$$\{z^n = xy\} \cap \left\{|x|^{2/n} + |y|^{2/n} = 1\right\} \to \left\{|x|^{2/n} + |y|^{2/n} = 1\right\}$$

which is a degree n covering space over the complement of the locus $\{xy = 0\}$ (over this locus, the projection becomes 1-to-1). Restricting our map from (i) to $S^3 \subset \mathbb{C}^2$, factoring through G, and composing with this projection, we get

$$L(n, 1) = S^3/G \cong \{z^n = xy\} \cap \left\{|x|^{2/n} + |y|^{2/n} = 1\right\} \to \left\{|x|^{2/n} + |y|^{2/n} = 1\right\} \cong S^3$$

So we have a branched cover $L(n, 1) \to S^3$, branched over the locus $S^3 \cap \{xy = 0\}$, which we know to be the Hopf link.

Problem 25

Show that the largest negative value achieved by the orbifold Euler characteristic $2 - 2g + \sum_{i=1}^k (1/n_i - 1)$, $g \geq 0$, $n_i \geq 1$, is $-1/42$.

Solution

First, note that $-1/42$ is achieved by the Riemann sphere with 3 orbifold points of orders 2, 3, and 7, since

$$2 - 2 \cdot 0 + (1/2 - 1) + (1/3 - 1) + (1/7 - 1) = -1/42$$

We now whittle away at the possibilities until it becomes clear that this is the maximum negative value. Terms with $n_i = 1$ do not contribute to the Euler characteristic, so we assume all $n_i \geq 2$, so that $1/n_i - 1 \leq -1/2$. If $g \geq 2$, then $2 - 2g + \sum_{i=1}^k (1/n_i - 1) \leq -2$, and if $g = 1$, we need $k \geq 1$ to have negative Euler characteristic, and then $2 - 2g + \sum_{i=1}^k (1/n_i - 1) \leq -1/2$. So we can now assume $g = 0$. If $k \leq 2$, then $2 + \sum_{i=1}^k (1/n_i - 1) > 0$, and if $k \geq 5$, then $2 + \sum_{i=1}^k (1/n_i - 1) \leq -1/2$. If $k = 4$, then either all $n_i = 2$ and $2 + \sum_{i=1}^k (1/n_i - 1) = 0$, or at least one $n_i \geq 3$ and $2 + \sum_{i=1}^k (1/n_i - 1) \leq 2 - 3/2 - 1/3 = -1/6$. So suppose

$k=3$, and that $n_1 \leq n_2 \leq n_3$. If $n_1 \geq 3$, then either all $n_i = 3$ and $2+\sum_{i=1}^{k}(1/n_i - 1) = 0$, or $n_3 \geq 4$ and $2+\sum_{i=1}^{k}(1/n_i - 1) \leq 2 - 4/3 - 3/4 = -1/12$. So suppose $n_1 = 2$. If $n_2 \geq 4$, then either $n_2 = n_3 = 4$ and $2+\sum_{i=1}^{k}(1/n_i - 1) = 0$, or $n_3 \geq 5$ and $2+\sum_{i=1}^{k}(1/n_i - 1) \leq 2-1/2-3/4-4/5 = -1/20$. So suppose $n_2 = 3$. If $n_3 \leq 6$, then $2+\sum_{i=1}^{k}(1/n_i - 1) \geq 0$, and if $n_3 \geq 8$, then $2+\sum_{i=1}^{k}(1/n_i - 1) \leq 2-1/2-2/3-7/8 = -1/24$. The only remaining possibility is $g=0$, $k=3$, $(n_1, n_2, n_3) = (2,3,7)$, so we can now conclude that $-1/42$ is the maximum negative value, and it is achieved uniquely in this way.

Problem 26

Find a Belyi polynomial $p(z)$ of degree 5 such that $p^{-1}([0,1])$ is homeomorphic to the letter Y, with the fork at $z=0$. (The Belyi condition means $p(0)=0$, $p(1)=1$ and the critical values of p are contained in $\{0,1\}$).

Solution

Consider one possibility, a Belyi tree with a single vertex of valence 3, and two adjacent vertices of valence 2. In this case, $p'(z)$ vanishes to order 2 at $z=0$, to order 1 at $z=1$, and to order 1 at some other $b \neq 0,1$, so we have

$$p'(z) = az^2(z-1)(z-b) = a\left(z^4 - (b+1)z^3 + bz^2\right)$$

Since $p(0)=0$, we must have

$$p(z) = a\left(\frac{1}{5}z^5 - \frac{b+1}{4}z^4 + \frac{b}{3}z^3\right)$$

and since $p(1)=1$ and $p(b)=0$, we solve to get $a = \frac{45}{4}$ and $b = \frac{5}{3}$, thus

$$p(z) = \frac{1}{4}\left(9z^5 - 30z^4 + 25z^3\right)$$

Another possibility would be a tree with a single vertex of valence 3, but two non-adjacent vertices of valence 2. In this case, the resulting Belyi polynomial is

$$p(z) = \frac{1}{64}\left(9z^5 + 15z^4 + 40z^3\right)$$

Problem 27

Prove that the coefficients of any Belyi polynomial are algebraic numbers.

Solution

The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on polynomials in $\mathbb{C}[x]$ coefficient-wise, and this action restricts to an action on Belyi polynomials of a fixed degree d. Indeed, a Belyi polynomial $p(z)$ is a polynomial satisfying $p(0)=0$, $p(1)=1$, and $p(z) \in \{0,1\}$ whenever $p'(z)=0$, and these conditions are all expressible as algebraic equations in the coefficients of $p(z)$ with rational coefficients. Every Belyi polynomial is uniquely determined by its associated Belyi tree, which is a 2-colored tree with d edges, and 1 vertex of each color marked. As there are finitely many such trees, there are finitely many Belyi polynomials of degree d. Then the orbit of any Belyi polynomial under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ must be finite, so each coefficient has a finite $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-orbit and therefore lies in a finite extension of $\mathbb{Q}$.

Problem 32

For which values of g does there exist a compact Riemann surface X of genus g admitting a degree 5 map $f : X \to \widehat{\mathbb{C}}$ with $B(f) = \{0,1,\infty\}$? For each g which exists, give an explicit pair of elements $\sigma_0, \sigma_1 \in S_5$ describing the monodromy of X over 0 and 1. (In other words, describe the associated map $\rho : \pi_1\left(\widehat{\mathbb{C}} - B(f)\right) \to S_5$ by giving its values on a loop around 0 and a loop around 1).

Solution

Suppose we have such a map $f : X \to \widehat{\mathbb{C}}$. By the Riemann-Hurwitz formula,

$$\begin{aligned}2g-2 &= \deg(f)\left(g\left(\widehat{\mathbb{C}}\right)-2\right)+\sum_{x\in X}(\operatorname{mult}(f,x)-1) = -10+\sum_{y\in B(f)}\left(5-\left|f^{-1}(y)\right|\right)\\ &= 5-\left|f^{-1}(0)\right|-\left|f^{-1}(1)\right|-\left|f^{-1}(\infty)\right| \le 2\end{aligned}$$

so since $g \geq 0$ is an integer, the possibilities are $g = 0, 1, 2$. Examples for each possibility are:

$$\begin{aligned}g=0,&\quad \sigma_0=(123)(45),\ \sigma_1=(34),\ \sigma_\infty=(13542)\\ g=1,&\quad \sigma_0=(14)(253),\ \sigma_1=(12345),\ \sigma_\infty=(1245)\\ g=2,&\quad \sigma_0=(12345),\ \sigma_1=(12345),\ \sigma_\infty=(14253)\end{aligned}$$