

# Math 213b HW1 Solutions

February 10, 2019

## Problem 8

Use the relation between the spherical metric and the inner product on  $\mathbb{C}^2$  to compute the distance by air, in kilometers, between Boston (42 N 71 W) and Singapore (1 N 104 E). Assume the distance from the north pole to the equator (by air) is 10,000 km. What is the distance between these two points in the hyperbolic metric (of constant curvature  $-1$ ) on the northern hemisphere?

### Solution

Denote by  $S^2$  the unit sphere in  $\mathbb{R}^3$ , with coordinates  $(x, y, z)$ . Identify  $S^2$  with  $\widehat{\mathbb{C}}$  via stereographic projection to the  $(x, y)$  plane, and then identify  $\widehat{\mathbb{C}}$  with the complex projective line  $\mathbb{P}^1(\mathbb{C})$ . Given two points  $p_1, p_2 \in S^2$ , choose unit vectors  $v_1, v_2 \in \mathbb{C}^2$  spanning the complex lines corresponding to  $p_1, p_2$ , respectively. The distance  $d(p_1, p_2)$  in the spherical metric is related to the standard positive definite Hermitian inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{C}^2$  by the formula

$$|\langle v_1, v_2 \rangle|^2 = \cos^2(d(p_1, p_2)/2).$$

Given points  $p_j$  in spherical coordinates  $(\varphi_j, \theta_j)$ , where  $\varphi_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  is the latitude (the angle north/south of the equator) and  $\theta_j \in (-\pi, \pi]$  is the longitude (the angle west/east of the prime meridian), the corresponding unit vector in  $\mathbb{C}^2$  is

$$v_j = \frac{(1, \tan(\alpha_j) e^{i\theta_j})}{\sqrt{1 + \tan(\alpha_j)^2}},$$

where  $\alpha_j = \frac{\pi}{4} + \frac{\varphi_j}{2}$ . So the spherical distance between  $p_1$  and  $p_2$  is given by

$$\cos(d(p_1, p_2)/2) = \frac{|1 + \tan(\alpha_1) \tan(\alpha_2) e^{i(\theta_1 - \theta_2)}|}{\sqrt{(1 + \tan(\alpha_1)^2)(1 + \tan(\alpha_2)^2)}}$$

Plugging in  $(\varphi_1, \theta_1) = (\frac{42\pi}{180}, \frac{71\pi}{180})$  and  $(\varphi_2, \theta_2) = (\frac{\pi}{180}, \frac{104\pi}{180})$ , we get

$$\cos(d(p_1, p_2)/2) = 0.3684$$

$$d(p_1, p_2) = 2.3870$$

Since the distance from the north pole to the equator is given as 10,000 km, we scale by  $\frac{20000}{\pi}$  km to get a distance of  $\approx 15196$  km.

In the hyperbolic metric of constant curvature  $-1$  on the northern hemisphere, we similarly have

$$|\langle v_1, v_2 \rangle|^2 = \cosh^2(d(p_1, p_2)/2),$$

where now the  $v_i$  are normalized using the standard indefinite Hermitian inner product on  $\mathbb{C}^2$ . In spherical coordinates as above, we have

$$\cosh(d(p_1, p_2)/2) = \frac{|-1 + \tan(\alpha_1) \tan(\alpha_2) e^{i(\theta_1 - \theta_2)}|}{\sqrt{(-1 + \tan(\alpha_1)^2)(-1 + \tan(\alpha_2)^2)}}$$

Plugging in  $(\varphi_1, \theta_1) = (\frac{42\pi}{180}, \frac{71\pi}{180})$  and  $(\varphi_2, \theta_2) = (\frac{\pi}{180}, \frac{104\pi}{180})$ , we get

$$\cosh(d(p_1, p_2)/2) = 8.6607$$

$$d(p_1, p_2) = 5.6972$$

## Problem 9

Let  $X$  be a Riemann surface with universal cover  $\tilde{X} = \mathbb{H}, \mathbb{C},$  or  $\widehat{\mathbb{C}}$ . Then  $\pi_1(X)$  acts on  $\tilde{X}$  by deck transformations.

(a) How can you describe  $\text{Aut}(X)$  in terms of  $\pi_1(X) \subset \text{Aut}(\tilde{X})$ ?

(b) Find all Riemann surfaces with continuous symmetries. Equivalently, find all  $X$  for which there exists a group homomorphism  $\mathbb{R} \rightarrow \text{Aut}(X)$  other than the identity.

## Solution

(a) The universal covering map  $\tilde{X} \rightarrow X$  is regular, and the Deck group is given by  $\pi_1(X) \subset \text{Aut}(\tilde{X})$  acting by a subgroup of holomorphic automorphisms. Define a map

$$N_{\text{Aut}(\tilde{X})}\pi_1(X) \rightarrow \text{Aut}(X)$$

from the normalizer of  $\pi_1(X)$  in  $\text{Aut}(\tilde{X})$  to the automorphism group of  $X$ , by sending  $f \in N_{\text{Aut}(\tilde{X})}\pi_1(X)$  to the induced map on the quotient space  $\tilde{X}/\pi_1(X) \cong X$ . One then makes several checks to see that this is a well-defined surjective group homomorphism with kernel equal to  $\pi_1(X)$ , so that  $\text{Aut}(X) \cong (N_{\text{Aut}(\tilde{X})}\pi_1(X))/\pi_1(X)$ . Briefly, elements of the normalizer preserve the fibers of the covering map, and  $\pi_1(X)$  acts freely and transitively on these fibers, so it follows that the kernel of this map is equal to  $\pi_1(X)$ , and since  $\tilde{X}$  is simply connected, any automorphism of  $X$  lifts to an automorphism of  $\tilde{X}$ .

(b) The Riemann surfaces  $\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*,$  and  $\mathbb{C}/\Lambda$  (where  $\Lambda \subset \mathbb{C}$  is a lattice) have continuous symmetries, given by the maps

$$\mathbb{R} \rightarrow \text{Aut}(\widehat{\mathbb{C}}), \quad t \mapsto (z \mapsto z + t)$$

$$\mathbb{R} \rightarrow \text{Aut}(\mathbb{C}), \quad t \mapsto (z \mapsto z + t)$$

$$\mathbb{R} \rightarrow \text{Aut}(\mathbb{C}^*), \quad t \mapsto (z \mapsto e^{it}z)$$

$$\mathbb{R} \rightarrow \text{Aut}(\mathbb{C}/\Lambda), \quad t \mapsto (z + \Lambda \mapsto z + t + \Lambda)$$

By the uniformization theorem, any other Riemann surface  $X$  has a universal covering map  $\mathbb{H} \rightarrow X$ , with Deck group  $\pi_1(X)$  a discrete subgroup of  $PSL_2(\mathbb{R}) = \text{Aut}(\mathbb{H})$ . We assume that  $\pi_1(X) \neq 1$ . By part (a),  $\text{Aut}(X) \cong N_{\text{Aut}(\tilde{X})}\pi_1(X)$ , so a homomorphism  $\mathbb{R} \rightarrow \text{Aut}(X)$  lifts to a homomorphism

$$\mathbb{R} \rightarrow N_{\text{Aut}(\tilde{X})}\pi_1(X), \quad t \mapsto g_t$$

Given  $h \in \pi_1(X)$ , we have  $g_t h g_t^{-1} = h'_t$  for all  $t \in \mathbb{R}$ , with  $h'_t$  depending continuously on  $t$ . Since  $g_0 = \text{id}$ , we have  $h_0 = h$ , and since  $\pi_1(X)$  is discrete, it follows that  $h_t = h$  for all  $t \in \mathbb{R}$ . This means that  $g_t \in C_{\text{Aut}(\tilde{X})}\pi_1(X)$  lies in the centralizer in  $\text{Aut}(\tilde{X})$  of  $\pi_1(X)$  for all  $t \in \mathbb{R}$ .

By conjugating in  $PSL_2(\mathbb{R})$ , we may assume that the  $g_t$  all have the form  $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ , or  $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ . In each case, the fact that the elements of  $\pi_1(X)$  commute with all the  $g_t$  forces the elements of  $\pi_1(X)$  to be contained in the image  $\{g_t : t \in \mathbb{R}\}$ . The first case is forbidden, since rotations all fix  $i \in \mathbb{H}$ . In the second case,  $\pi_1(X)$  is discrete and therefore a cyclic group of translations, so  $X \cong \Delta^*$ . In the third case,  $\pi_1(X)$  is a cyclic group of dilations, so  $X \cong A(R)$  is an annulus of finite modulus.

The full list of Riemann surfaces with continuous symmetries is thus

$$\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*, \mathbb{C}/\Lambda, \mathbb{H}, \Delta^*, A(R)$$

## Problem 10

Find the unique metric  $\rho = \rho(z)|dz|$  on the punctured unit disk  $\Delta^*$  such that  $\pi^*(\rho) = \frac{|dz|}{\operatorname{Im}(z)}$ , where  $\pi : \mathbb{H} \rightarrow \Delta^*$  is the universal covering map. (This is the *hyperbolic metric* on  $\Delta^*$ .) Show that the volume  $V(r) = \int_{0 < |z| < r} \rho(z)^2 |dz|^2$  is finite for each  $0 < r < 1$  and compute its value. Verify that  $V(r) = 2V(r^2)$ , and give a conceptual explanation of this identity.

### Solution

Using the universal covering map  $\pi : \mathbb{H} \rightarrow \Delta^*$ ,  $z \mapsto e^{iz}$ , we get

$$\frac{|dz|}{\operatorname{Im}(z)} = \pi^* \rho = \rho(e^{iz}) |e^{iz}| |dz|$$

Since  $\log |e^{iz}| = -\operatorname{Im}(z)$ , the metric on  $\Delta^*$  must be

$$\rho(z) |dz| = -\frac{|dz|}{|z| \log |z|}$$

The volume  $V(r)$  is given by

$$V(r) = \int_{0 < |z| < r} \frac{|dz|^2}{(|z| \log |z|)^2} = \int_{0 < r < 1} \int_{0 \leq \theta < 2\pi} \frac{r dr d\theta}{(r \log r)^2} = \frac{-2\pi}{\log r}$$

where in the course of evaluation we see that the integral converges, so  $V(r)$  is finite and clearly satisfies  $V(r) = 2V(r^2)$ . A conceptual explanation is that the 2-to-1 covering map  $\Delta^* \rightarrow \Delta^*$ ,  $z \mapsto z^2$ , is a local isometry from the hyperbolic metric, and restricts to a 2-to-1 covering from the punctured disk of radius  $r$  to the punctured disk of radius  $r^2$ .

## Problem 11

Find the hyperbolic metric on  $A(R) = \{z : 1 < |z| < R\}$ .

### Solution

Using the universal covering map  $f : \mathbb{H} \rightarrow A(R)$ ,  $z \mapsto e^{\frac{-i \log R}{\pi} \log z}$  from class, we get

$$\frac{|dz|}{\operatorname{Im}(z)} = f^* \rho = \rho(f(z)) \frac{|f(z)| \log R}{\pi |z|} |dz|$$

so it must be that  $\rho(f(z)) = \frac{\pi}{|f(z)| \log R} \cdot \frac{|z|}{\operatorname{Im}(z)}$ , and solving for  $\rho(z)$  as in Problem 3 yields

$$\rho(z) |dz| = \frac{\pi}{\log R} \frac{|dz|}{|z| \sin\left(\frac{\pi}{\log R} \log |z|\right)}$$

## Problem 12

Let  $H = \{z \in \mathbb{C}^* : \operatorname{Im}(z) \geq 0\}$ . Let  $X$  be the Riemann surface obtained from  $H$  by gluing each positive boundary point  $x > 0$  to the point  $-x^2 < 0$ . Show that  $X$  can be realized explicitly as a domain  $X \subset \mathbb{C}$ . What is the image of  $\partial H$  in  $X$ ?

### Solution

The domain  $H$  can be realized as an annulus of modulus  $e^{-2\pi^2/\log 2}$ , and the boundary  $\partial H$  maps to a spiral winding into each boundary circle, as pictured below. The composition of the maps in the picture below takes  $H$  into  $\mathbb{C}$  and is given by

$$z \mapsto \log(-z)^{2\pi i/\log 2}$$

