Math 213A F23 Homework 10 Solutions

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If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Show that the set of Riemann mappings whose images are polygons are dense in $\mathcal{S}$, in the topology of uniform convergence on compact sets.

Solution. We do this in two steps. Define

$$\mathcal{S}^{\infty} := \{g \in \mathcal{S} : \mathrm{Im}(g) \text{ is bounded domain with real analytic boundary}\}$$

and

$$\mathcal{S}^{\mathrm{poly}} := \{g \in \mathcal{S} : \mathrm{Im}(g) \text{ is a polygon}\}.$$

We show that $\mathcal{S}^{\infty}$ is dense in $\mathcal{S}$[1] and $\mathcal{S}^{\mathrm{poly}}$ is dense in $\mathcal{S}^{\infty}$; this suffices.

(i) To show that $\mathcal{S}^{\infty}$ is dense in $\mathcal{S}$, let $f \in \mathcal{S}$ be arbitrary, and consider the family of functions $(f_r)_{r\in(0,1]}$ in $\mathcal{S}$ defined by

$$f_r(z) = \frac{1}{r} f(rz).$$

Clearly, $f_r \in J$ for each $r \in (0,1)$, so prove this result, it suffices to show that as $r \to 1^-$, we have $f_r \to f$ in $\mathcal{S}$, i.e. f_r converges to f uniformly on compact subsets, which amounts to showing that if $K \subset \Delta$ is compact, then

$$\lim_{r\to 1^-} \sup_{z\in K} |f(z) - f_r(z)| = 0$$

To do this, use the triangle inequality

$$|f(z) - f_r(z)| \le |f(z) - f(rz)| + \frac{1-r}{r}|f(rz)|$$

for $z \in \Delta$ and $r \in (0,1)$, along with the uniform continuity of f on $\overline{\Delta(0,\rho)}$, where $\rho \in (0,1)$ is chosen so that $K \subset \Delta(0,\rho)$ (with details left to the diligent reader).

(ii) To show that $\mathcal{S}^{\mathrm{poly}}$ is dense in $\mathcal{S}^{\infty}$, let $f \in \mathcal{S}^{\infty}$ be arbitrary. Let $U := \mathrm{Im}(f)$. By Theorem 4.5 in the class notes dates 11/23, we know that f extends to a homeomorphism between $\overline{\Delta}$ and $\overline{U}$.[2] The idea is to approximate the real analytic curve ∂U by polygonal paths. Here's one way to do this rigorously: fix a point $p \in \partial U$. Suppose that the length of ∂U is ℓ; for each integer $n \ge 3$ and $k = 0, \dots, n-1$, define p_k^n to the point at a distance $k\ell/n$ along ∂U from p measured counterclockwise, and let B_n be the polygonal path joining $p = p_0^n, p_1^n \dots, p_{n-1}^n$ in that order (see Figure 1).

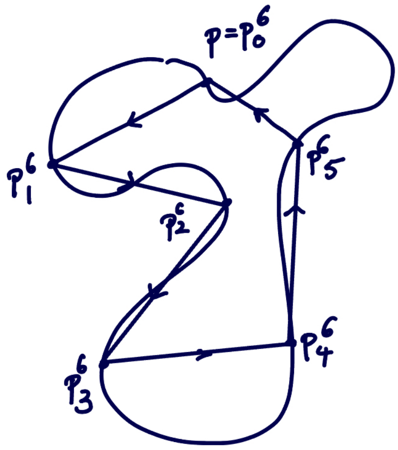

Figure 1: The polygonal approximation to a real analytic Jordan curve.

It is easy to see geometrically and prove rigorously from the real analyticity of ∂U that there is an $N \gg 1$ such that for all $n \ge N$ and k, the exterior vertex angles $\pi\mu_k^n$ at the p_k^n satisfy $\mu_k^n \in (-1,1)$

[1] The image of g for any $g \in \mathcal{S}$ is a real analytic Jordan domain. The density of J in $\mathcal{S}$ is proves the result of exercise 4.14.

[2] You can avoid the appeal to Theorem 4.5 by combining parts (i) and (ii) into one proof, where it is clear that $f_r(z)$ for $r \in (0,1)$ satisfies this property.

for $n \gg 1$ and the path B_n bounds a non self-intersecting polygon; see Figure (2) for an example of why this needs $N \gg 1$. Let's call this polygon P_n, so that the boundary $\partial P_n = B_n$. Clearly, P_n is simply connected. The key observation is the following lemma.

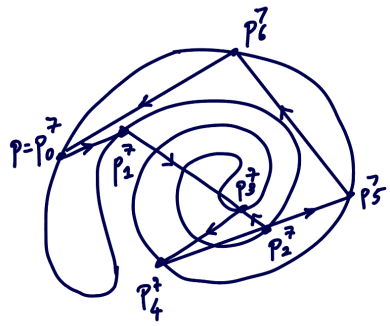

Figure 2: Why we need $N \gg 1$ for P_n to be an actual polygon for $n \geq 1$.

Lemma 0.0.1. Given any $q \in U$, there is an $N \gg 1$ (depending on q) such that for all $n \geq N$ we have $q \in P_n$. Similarly, given any $q \in \mathbb{C} \setminus \overline{U}$, there is an $N \gg 1$ such that for all $n \geq N$ we have $q \notin \overline{P}_n$.

Given this result, we finish the proof as follows. Let C be the convex hull of $\overline{U}$, and note that this is a compact subset of $\mathbb{C}$. Let $M := \max_{z \in C} |z| \in (0, \infty)$. Applying the lemma to $0 \in U$, we conclude that $0 \in P_n$ for all $n \gg 1$. Therefore, by the Riemann mapping theorem, for all $n \gg 1$, there is a normalized Riemann map $f_n : (\Delta, 0) \to (P_n, 0)$, i.e. a biholomorphism $f_n : \Delta \to P_n$ satisfying $f_n(0) = 0$ and $f_n'(0) > 0$. From $P_n \subset C$ it follows that $\sup_{z \in \Delta} |f_n(z)| \leq M$, so that each $f_n \in \mathcal{O}_\Delta(M)$. From a previous problem set, we know that $\mathcal{O}_\Delta(M)$ is compact in the compact-open topology, and so there is a subsequence f_{n_k} of f_n that converges uniformly on compact subsets to some function $g : \Delta \to \mathbb{C}$. Firstly, we must have $g(0) = 0$. Secondly, it follows from Lemma 0.0.1 and the argument principle that $\operatorname{Im} g = U$. Indeed, if $q \in U$ is arbitrary, then $f_{n_k} - q$ has a zero in Δ for $k \gg 1$ by the previous lemma, and hence $g - q$ has a zero in Δ by the argument principle; this shows $U \subseteq \operatorname{Im} g$. Conversely, if $q \notin \overline{U}$ then $g - q$ cannot have a zero in Δ; if it did, then by a previous homework problem (exercise 1.9 in the version of the notes dated 11/23), the function $f_{n_k} - q$ would have a zero for all $k \gg 1$, contradicting the result of the previous lemma; this shows $\operatorname{Im} g \subseteq \overline{U}$. Since $U \subseteq \operatorname{Im} g$, it follows that g is nonconstant and hence by the open mapping principle we have that $\operatorname{Im} g \subseteq \operatorname{Int} \overline{U} = U$, so that $\operatorname{Im} g = U$. Since g is nonconstant, it follows also that g is univalent, since each f_{n_k} is. Since $f_{n_k}'(0) > 0$ for all k, it follows also that $g'(0) \geq 0$ and hence by univalence that $g'(0) > 0$. Therefore, $g : (\Delta, 0) \to (U, 0)$ is a normalized Riemann map, and hence, by uniqueness, $g = f$. This tells us that $f_{n_k} \to f$ uniformly on compact sets. This doesn't quite suffice, because there is no guarantee that $f_{n_k} \in \mathcal{S}$; equivalently, we don't know that the conformal radius $r(P_{n_k}, 0) = 1$. However, we can consider the normalized sequence

$$z \mapsto \frac{1}{f_{n_k}'(0)} f_{n_k}(z)$$

of functions in $\mathcal{S}^{\text{poly}}$, which also converges uniformly on compact subsets to f, as we can check using an estimate similar to the one in (i) and that $\lim_{k \to \infty} f_{n_k}'(0) = f'(0) = 1$. Since $f \in \mathcal{S}^\infty$ is arbitary, this completes the proof, up to proving Lemma 0.0.1.

Proof of Lemma 0.0.1. For this, we use the notion of the Hausdorff distance between two subsets of the plane. Suppose X is a metric space and $A \subset X$ a subset. For any $\varepsilon \geq 0$, the ε-flattening of A in X is defined to be

$$A^\varepsilon := \bigcup_{a \in A} \overline{\Delta(a, \varepsilon)}$$

where $\overline{\Delta(a, \varepsilon)} := \{x \in X : d(x, a) \leq \varepsilon\}$. Given two subsets $A, B \subseteq X$, we define the Hausdorff distance between A and B to be

$$d_H(A, B) := \inf\{\varepsilon \geq 0 : A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}.$$

The key is that the "short dam, small lake" principle implies that if we take $X = \mathbb{C}$ with the Euclidean metric, then as $n \to \infty$, the set B_n approaches ∂U in the Hausdorff distance, i.e. that

$$\lim_{n\to\infty} d_H(\partial U, B_n) = 0.$$

(Do you see why rigorously?) Equivalently, this says that for any $\varepsilon > 0$, there is an $N \gg 1$ such that for all $n \geq N$ we have $\partial U \subseteq B_n^\varepsilon$ and $B_n \subseteq \partial U^\varepsilon$; this is the key property of the polygonal approximation that we will use. Given this, we proceed as follows: given $q \in U$ or $q \in \mathbb{C} \setminus \overline{U}$, and pick an $\varepsilon > 0$ such that $q \notin \partial U^\varepsilon$. By the above, we may pick an $N \gg 1$ such that for all $n \geq N$ we have $B_n \subseteq \partial U^\varepsilon$. Then for all $n \geq N$, we have $q \in \mathbb{C} \setminus \partial U^\varepsilon \subseteq \mathbb{C} \setminus B_n$, and $\mathbb{C} \setminus B_n$ has two connected components, namely P_n and $\mathbb{C} \setminus \overline{P}_n$. Since B_n is clearly homotopic to ∂U through a homotopy with image in ∂U^ε (can you write down a formula for it?), it follows that their winding numbers around q agree, i.e. $W(q, \partial U) = W(q, B_n)$ for all $n \geq N$. Since the regions U and $\mathbb{C} \setminus \overline{U}$ can be characterized as

$$U := \{x \in \mathbb{C} \setminus \partial U : W(x, \partial U) = 1\} \text{ and } \mathbb{C} \setminus \overline{U} = \{x \in \mathbb{C} \setminus \partial U : W(x, \partial U) = 0\},$$

and similarly for B_n, it follows that if $q \in U$, then $q \in P_n$ for all $n \geq N$, and if $q \in \mathbb{C} \setminus \overline{U}$, then $q \in \mathbb{C} \setminus \overline{P}_n$ for all $n \geq N$.

■

Remark 1. Here's an alternative argument. We know from class that f_n also extends to a homeomorphism $S^1 \to B_n$. One can then argue directly from the polygonal approximation that $f_n \to f$ uniformly on S^1 (this needs some work). Finally, using this and the Cauchy integral formula, we can conclude that $f_n \to f$ uniformly on compact sets. Indeed, if $K \subset \Delta$ is any compact set, then for any $z \in K$, we have that

$$|f_n(z) - f(z)| = \left| \frac{1}{2\pi \mathrm{i}} \oint_{\zeta \in S^1} \frac{f_n(\zeta) - f(\zeta)}{\zeta - z} \right| \leq \frac{1}{2\pi \cdot d(K, \partial U)} \oint_{\zeta \in S^1} |f_n(\zeta) - f(\zeta)|$$

and since $f_n \to f$ uniformly on S^1, this suffices to prove that $f_n \to f$ uniformly on K. Finally, we can normalize and consider $f_n(0)^{-1} f_n(z)$ instead, finishing the proof.

Remark 2. Quite a few solutions talked about "the Riemann map" $f_n : \Delta \to P_n$. There is is no "the Riemann map" unless you fix a choice of normalization. The usual choice of normalization is to take $f_n(0) = 0$ and $f_n'(0) > 0$, but for this to make sense you have to first show that $0 \in P_n$ for all $n \gg 1$. If you don't normalize appropriately, there is no guarantee that $f_n \to f$ uniformly on compact sets. Some solutions first fixed a compact $K \subset \Delta$, and then produced a sequence $f_n \in \mathcal{S}^{\text{poly}}$ (depending on K) such that $\sup_{z\in K} |f_n - f| < 2^{-n}$ (say). This doesn't work because you need to produce one sequence f_n that converges to f uniformly on any given compact set. Even if you replace K by an exhaustion K_n of Δ by compact sets, it is not clear to me that you can "diagonalize" appropriately to get this to work.

Q2. Let $P_n \subset \mathbb{C}$ be a regular polygon with n sides and let $f_n : P_n \to \Delta$ be a Riemann mapping (a bijective holomorphic map).

(a) Prove that if $n = 3$, then f_n extends to a meromorphic function on all of $\mathbb{C}$.
(b) Prove that this is false if $n = 6$.

Solution. The key to this problem is the Schwarz Reflection Principle.

(a) Here we use the following lemma:

Lemma 0.0.2. Let $T = \Delta ABC$ be an equilateral triangle in the complex plane, and let $f : \overline{T} \to \hat{\mathbb{C}}$ be a continuous function such that $f|_{\operatorname{Int} T} : \operatorname{Int} T \to \hat{\mathbb{C}}$ is holomorphic and $f(\partial T) \subset S^1$.[3] Let A' be the reflection of A in BC and define B' and C' similarly. Let $T' := \Delta A'B'C'$. Then f extends to a function of the same type on T', i.e. there is a continuous function $f' : \overline{T}' \to \hat{\mathbb{C}}$ such that $f'|_{\operatorname{Int} T'} : \operatorname{Int} T' \to \hat{\mathbb{C}}$ is holomorphic and $f'(\partial T') \subset S^1$ such that $f'|_{\overline{T}} = f$.

Proof. By a translation and rotation in the domain (and relabelling if necessary), we may assume without loss of generality that $A = 1, B = \omega, C = \omega^2$ (where $\omega = e^{2\pi i/3}$), and so $T' = -2T$. By the Schwarz Reflection Principle, the formula

$$f'(z) := \overline{\left(\frac{1}{f(-1-\overline{z})}\right)}$$

defines the extension of f to $\Delta A'BC$, and there are similar formulae for the reflections across AB and AC. See Figure (3). ■

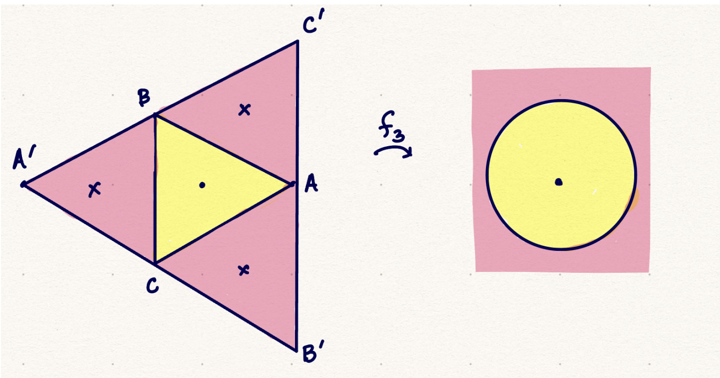

Figure 3: The extension of f_3.

Given this lemma, we proceed as follows: consider the map $f_3 : P_3 \to \Delta$. Since the boundary of P_3 is polygonal (and in particular locally connected), it follows from the discussion in class that f_3 extends to a homeomorphism $f_3 : \overline{P}_3 \to \overline{\Delta}$ with $f_3(\partial P_3) \subset S^1$. Now we may inductively apply the previous lemma to get an extension of f_3 across successively larger equilateral triangles centered at the center of P_3. Since these triangles cover $\mathbb{C}$, we therefore get an extension of f_3 to a meromoprhic function on the plane.

(b) Label the vertices of the hexagon as in Figure (4). Here the hexagon $P_6' := AB'C'D'E'F$ is obtained by reflecting the original hexagon $P_6 = ABCDEF$ about the side AF, and the hexagon $P_6'' := ABC''D''E''F''$ is obtained by reflecting P_6 acros the side AB. Firstly, as before, f_6 extends to a homeomorphism $f_6 : \overline{P}_6 \to \overline{\Delta}$ with $f_6(\partial P_6) \subset S^1$. As before, f_6 can be extended to a continuous map on $\operatorname{Int}(P_6 \cup P_6')$ and $\operatorname{Int}(P_6 \cup P_6'')$ obtained by Schwarz reflecting across AF and AB, and take $\operatorname{Int} P_6'$ and $\operatorname{Int} P_6''$ to $\hat{\mathbb{C}} \setminus \overline{\Delta}$. If f_6 extends to a meromorphic function on the plane, then these extensions must now also be related by Schwarz reflection across the common boundary $AB' = AF''$ by precomposition with the reflection across AB' and postcomposition with the reflection $z \mapsto 1/\overline{z}$ in S^1, and so in particular cannot both map to $\hat{\mathbb{C}} \setminus \overline{\Delta}$. This is the required contradiction. ■

[3] In other words, f is a meromorphic function on the interior of an equilateral triangle in the plane that extends continuously to the boundary to a map that takes the boundary to S^1.

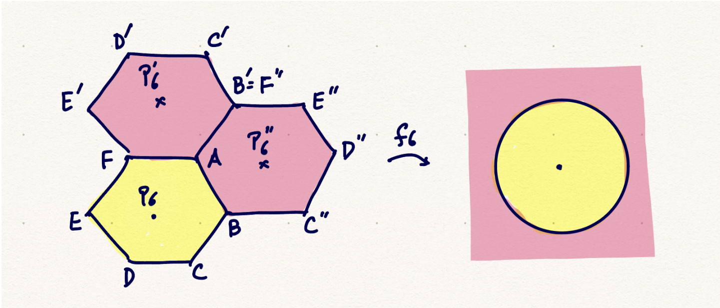

Figure 4: The non-existence of a meromorphic extension of f_6.

Remark 3. Note that $n \geq 3$ for the problem statement to even make sense. For such a meromorphic extension of f_n to be possible, we certainly need the reflections of the polygon across its edges to tile the plane (why?), and this only happens for $n \in \{3, 4, 6\}$ (why?). Of these, the functions for $n = 3$ and $n = 4$ extend to meromorphic functions on $\mathbb{C}$, whereas that for $n = 6$ does not. The functions so obtained for $n = 3$ and $n = 4$ are both examples of elliptic functions (for the hexagonal lattice and for the square lattice respectively). The function for $n = 3$ is also an example of (the inverse of) a Schwarzian triangle function. For more on these topics, see [1, Chapter 6, §2.4] and [2, §VI.5].

Remark 4. The idea is to tile the plane using successive reflections of the original polygon, and to check that for any two distinct sequences of Schwarz reflections leading to the same point, the corresponding extensions of the function agree. This needs each vertex to be surrounded by an even number of polygons, and in particular for us to be able to color the tiling of the plane using the polygons P_n using two colors such that adjacent polygons that share an edge get colored differently. This is possible for $n \in \{3, 4\}$ only.

Q3. Prove that if $f : \mathbb{H} \to \mathbb{C}$ is given by the Schwarz-Christoffel Formula, but $S = \sum \mu_i \neq 2$, then the image $P = f(\mathbb{H})$ is a polygon with a vertex at $f(\infty)$ with external angle $(2 - S)\pi$.

Solution. One could obtain this "from first principles" by arguing similarly to how we derived the Schwarz-Christoffel formula (see [2, §V.6]), or use that result follows to prove this; here we do the latter. We'll take the following result from class as known:

Theorem 0.0.3 (Schwarz-Christoffel)**.** For some integer $n \geq 1$, let $q_1, \ldots, q_n$ be distinct real numbers, and $\mu_1, \ldots, \mu_n$ be real numbers satisfying $-1 < \mu_i < 1$ for $i = 1, \ldots, n$ and $S := \sum_{i=1}^n \mu_i = 2$. Then for any $z_0 \in \overline{\mathbb{H}}$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, the formula

$$f(z) := \alpha \int_{z_0}^{z} \prod_{i=1}^{n} (\zeta - q_i)^{-\mu_i} \, \mathrm{d}\zeta + \beta \tag{1}$$

defines a holomorphic map $\mathbb{H} \to \mathbb{C}$, where we choose any branch of the integrand on $\mathbb{H}$ and the integral is taken along any path in $\mathbb{H}$. This map f extends continuously to a map $\overline{\mathbb{H}} \to \hat{\mathbb{C}}$, and is biholomorphic onto its image, which is a polygon with vertices at the points $p_1, \ldots, p_n$ defined by $p_i = f(q_i)$ for $i = 1, \ldots, n$, with the exterior vertex angle at p_i being $\pi\mu_i$. Conversely, any such Riemann map has this form.

Now suppose we are given an integer $n \geq 1$ and distinct real numbers $q_1, \ldots, q_n$, and real numbers $\mu_1, \ldots, \mu_n$ with $-1 < \mu_u < 1$ for $i = 1, \ldots, n$, but such that the sum $S := \sum_{i=1}^n \mu_i$ is not necessarily two, and we are asked to investigate the mapping properties of the function defined by the same expression as in (1) for some given α, β, z_0. The idea is simply to precompose with a biholomorphism $g : \mathbb{H} \to \mathbb{H}$ that takes the $\infty \in \overline{\mathbb{H}}$ to a point on $\mathbb{R}$, and express $g \circ f$ in the same form as in (1) but with the sum $S = 2$. For this, pick any $r \in \mathbb{R}$ distinct from all the q_i and z_0, and define $g : \mathbb{H} \to \mathbb{H}$ by $g(z) = r - (1/z)$.[4] Then

$$(f \circ g)(z) = \alpha \int_{z_0}^{r-(1/z)} \prod_{i=1}^{n} (\zeta - q_i)^{-\mu_i} \, \mathrm{d}\zeta + \beta.$$

In the integral, now make the substitution $\zeta = r - 1/\xi$ to get

$$(f \circ g)(z) = \alpha \int_{-1/(z_0 - r)}^{z} \prod_{i=1}^{n} \left(r - \frac{1}{\xi} - q_i \right)^{-\mu_i} \frac{-1}{\xi^2} \, \mathrm{d}\xi + \beta = \alpha' \int_{z_0'}^{z} (\xi - 0)^{-(2-S)} \prod_{i=1}^{n} \left(\xi - \frac{1}{r - q_i} \right)^{-\mu_i} \mathrm{d}\xi + \beta,$$

where $\alpha' := \alpha \prod_{i=1}^n (r - q_i)^{-\mu_i} \in \mathbb{C}^*$ (for some choice of branches) and $z_0' = -1/(z_0 - r) \in \overline{\mathbb{H}}$. This is now of the required form, and hence by the theorem we conclude that the image of $f \circ g$, or equivalently the image of f since g is a biholomorphism, is a polygon with vertices at the $(f \circ g)(1/(r - q_i)) = f(q_i)$ with vertex angles $\pi\mu_i$ and a vertex at $(f \circ g)(0) = f(\infty)$ with vertex angle $(2 - S)\mu$ as needed. ■

Remark 5. Technically the above only works if $2 - S \in (-1, 1)$, i.e. if $S \in (1, 3)$. When $S \notin (1, 3)$, then the polygon will "cross itself" at the vertex $f(\infty)$ (imagine a polygon with interior angle "greater than 2π).

[4] This sign choice ensures that g takes $\mathbb{H}$ to $\mathbb{H}$ (why?).

Q4. Find a formula (which may involve an integral) for a covering map $f : \mathbb{H} \to \mathbb{C} \setminus T$ whose image is the complement of a square T.

Solution. The Schwarz-Christoffel map

$$g(z) := \int_0^z \frac{\mathrm{d}w}{(1+w)^{1/4}(1-w)^{1/4}} = \int_0^z \frac{\mathrm{d}w}{(1-w^2)^{1/4}}$$

is biholomorphism from the upper half plane $\mathbb{H}$ to the region

$$R := \{z = x + \mathrm{i}y \in \mathbb{C} : x + y + \lambda > 0, -x + y + \lambda > 0, \text{ and } y > 0\},$$

where

$$\lambda := \int_0^1 \frac{\mathrm{d}w}{(1-w^2)^{1/4}} = \sqrt{\frac{2}{\pi}}\Gamma\left(\frac{3}{4}\right)^2 = 1.198\ldots.$$

See Figure (5). Here we choose the branch of $(1-w^2)^{1/4}$ that is positive on $w \in (-1,1)$. Since the map

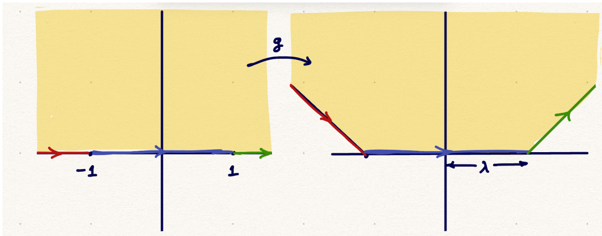

Figure 5: The mapping g.

$z \mapsto \sin z$ takes the vertical half-strip

$$S := \{z \in \mathbb{H} : -\pi/2 < \operatorname{Re} z < \pi/2\}$$

biholomorphically to the upper half plane, it follows that the map f defined by

$$f(z) := g(\sin z) = \int_0^{\sin z} \frac{\mathrm{d}w}{(1-w^2)^{1/4}}$$

is a biholomorphism from S to R. It follows then by the Schwarz reflection principle that this function f extends to give the universal covering map $f : \mathbb{H} \to \mathbb{C} \setminus T$ of the complement of the square $T := [-\lambda, \lambda] \times [-2\lambda, 0]\mathrm{i}$ with fundamental domain given by $\{z \in \mathbb{H} : -2\pi < \operatorname{Re} z \leq 2\pi\}$; see Figure (6).

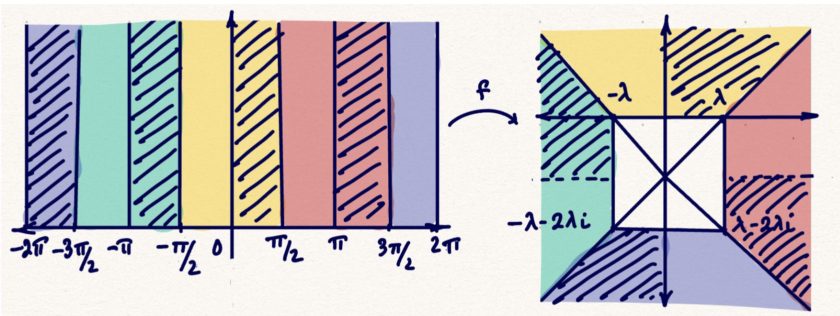

Figure 6: The extension of f by Schwarz reflection gives the covering map.

To write this in a more familiar form, we make the change of variables $w = \sin\theta$ to get

$$f(z) = \int_0^z \sqrt{\cos\theta}\,\mathrm{d}\theta,$$

where we have chosen the branch of $\sqrt{\cos\theta}$ that is positive on $\theta \in (-\pi/2, \pi/2)$. ■

Remark 6. The integral

$$E(z|m) := \int_0^z \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta$$

is called the incomplete elliptic integral of the second kind. With this notation, the the above formula can be written as

$$f(z) = \int_0^z \sqrt{1 - 2\sin^2(\theta/2)}\,\mathrm{d}\theta = 2E\left(\frac{z}{2}\,\middle|\,2\right).$$

This problem gives us a geometric interpretation of the the elliptic integral of the second kind as a covering map of the complement of a square. To relate this to the result of [2, Exercise 3, p. 196], note that the 4π-periodicity of $f(z)$ and the observation about its fundamental domain above implies that we can write

$$f(z) = h(\mathrm{e}^{iz/2})$$

for some biholomorphism

$$h : \Delta^* \to \mathbb{C} \smallsetminus T.$$

To find h, we can write[5]

$$h(z) = \int_0^{-2\mathrm{i}\log z} \sqrt{\cos\theta}\,\mathrm{d}\theta = \int_1^z \sqrt{\cos(-2\mathrm{i}\log u)}\left(\frac{-2\mathrm{i}}{u}\right)\mathrm{d}u = C\int_1^z \frac{\sqrt{1+u^4}}{u^2}\mathrm{d}u = D\int_{z_0}^{\zeta_8 z} \frac{\sqrt{1-w^4}}{w^2}\mathrm{d}w$$

for some choice of constants $C, D, z_0 \neq 0$, which is the statement in Exercise 3 (up to precomposition by a rotation $z \mapsto \zeta_8 z$.) This map h provides an explicit uniformation of the doubly connected region $\mathbb{C} \smallsetminus T$.

[5] Here we are a little less sloppy about choices of branches and constants, by an application of the identity principle.

Q5. Let f and g be entire functions solving Fermat's equation $f^n + g^n = 1$ for $n > 2$. Prove that f and g are constant. (Hint: consider f/g.)

Solution. Not both f and g can be identically zero, so, by relabelling if need, asumme that g is not. Then $h := f/g$ is a meromorphic function on the plane, and we have the identity of meromorphic functions

$$g^n(h^n + 1) = 1. \tag{2}$$

Since g is entire (and hence has no poles), it follows that $h^n + 1$ never attains the value 0, and hence h^n never attains the value -1. In particular, h misses all the n distinct n^{th} roots of -1. Therefore, if $n \geq 3$, then Picard's Little Theorem tells us that h is constant. Then it follows from Eq. (2) that $|g|$ is constant, and hence from the Open Mapping Principle that g is constant, whence $f = gh$ is constant as well. ■

Remark 7. The statement, as in the usual Fermat problem, is not true for $n = 1$ (clearly) or $n = 2$, as shown by $f(z) = \sin z$ and $g(z) = \cos z$. Indeed, $\tan^2 z = \pm \mathrm{i}$ has no solutions (why?).

Remark 8. If we allow f and g to be meromorphic, then there are also solutions for $n = 3$ but none for $n \geq 4$. You are asked to produce solutions for $n = 3$ on a problem on the next homework using elliptic functions. The nonexistence of solutions for $n \geq 4$ follows from noting that the Fermat curve $C_n := \mathbb{V}(x^n + y^n - z^n) \subset \mathbb{CP}^2$ for $n \geq 1$ has genus $g = (n-1)(n-2)/2$, which for $n \geq 4$ is at least 2, so, by uniformization, C_n has universal cover Δ for $n \geq 4$. If f, g are meromorphic solutions to $f^n + g^n = 1$, then $[f : g : 1] : \mathbb{C} \to C_n$ is a holomorphic map, and hence lifts to a map $\tilde{f} : \mathbb{C} \to \Delta$ of universal covers, which must be constant by Liouville's Theorem.

Remark 9. A lot of submissions to this problem were essentially correct, but a little sloppy with the details. For instance, the quotient f/g of two entire functions is meromorphic only if g is not identically the zero function; this might seem like a triviality, but it is actually fairly important because only by defining meromorphic functions as holomorphic maps to $\hat{\mathbb{C}}$ that are not identically ∞ can we make the set of meromorphic functions into a field. Therefore, unless you exclude this case (as in the solution above), you can't just divide by g as the hint suggests and claim that f/g is meromorphic.

Q6. Let $\Lambda \subset \mathbb{R}^n$ be a lattice, i.e. a discrete subgroup isomorphic to $\mathbb{Z}^n$.[6] Choose a sequence of vectors $a_1, \ldots, a_n \in \Lambda$ such that a_1 is the[7] shortest nonzero vector, and (for $i > 1$) a_i is the shortest vector linearly independent from $(a_1, \ldots, a_{i-1})$. Is it always the case that $\Lambda = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_n$?

Solution. This result is not always true, and the standard counterexample is given as follows. Take $n = 5$, and take

$$\Lambda := \left\{ \sum_{i=1}^{n} a_i \mathrm{e}_i : a_i \in \mathbb{Z}, a_1 \equiv a_2 \equiv \cdots \equiv a_5 \pmod 2 \right\}.$$

If $v = \sum_{i=1}^5 a_i \mathrm{e}_i$ with each $a_i \equiv 1 \pmod 2$, then $|v| \geq \sqrt{5} > 2$. On the other hand, Λ contains at least five linearly independent vectors of norm 2, namely $2\mathrm{e}_1, 2\mathrm{e}_2, \ldots, 2\mathrm{e}_5$. Therefore, we can select $a_i = 2\mathrm{e}_i$ for $i = 1, \ldots, 5$; this will satisfy the conditions in the problem statement, but Λ is not generated by the a_i because $\bigoplus_{i=1}^5 \mathbb{Z}a_i = 2\mathbb{Z}^5$ and $\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5 \in \Lambda \setminus 2\mathbb{Z}^5$.[8] ■

Remark 10. In fact, this procedure always works if $n \leq 3$, can be chosen to work if $n = 4$, and doesn't always work if $n \geq 5$. To state this precisely, we say that a sequence $a_1, \ldots, a_n \in \Lambda$ is a minimal sequence if for each $i \geq 1$, a_i is some vector of minimal length that is linearly independent from $a_1, \ldots, a_{i-1}$. Note that any minimal sequence for a full rank lattice has to be a basis of the ambient $\mathbb{R}^n$. Then the definitive result is:

Theorem 0.0.4.

(a) If $n \leq 3$, then for any full rank lattice $\Lambda \subset \mathbb{R}^n$, any minimal sequence in Λ is a $\mathbb{Z}$-basis.
(b) If $n = 4$, then for all lattices $\Lambda \subset \mathbb{R}^4$ except one (up to isomorphism), called D_4, any minimal sequence in Λ is a $\mathbb{Z}$-basis. For $\Lambda \cong D_4$, some minimal sequences form a $\mathbb{Z}$-basis, whereas others don't.
(c) If $n \geq 5$, then there are lattices $\Lambda \subset \mathbb{R}^n$ for which no minimal sequence is a $\mathbb{Z}$-basis.

Proof. For $n \geq 5$, an analogous construction to the one above works as a counterexample. The case $n \leq 4$ is classical, and some versions of proofs can be found in [3, Chapter 12] or [4, Chapter 15, §10]. Here, I include a proof of the result when $n \leq 2$, since that is the case relevant to us in this class; for a uniform modern exposition of the result for $n \leq 4$, see [5], a Stack Exchange answer written by very own Prof. Noam Elkies. The case $n = 1$ is clear, since if $\Lambda = \mathbb{Z}v \subset \mathbb{R}$ for some $v \neq 0$, then $|nv| = |n| \cdot |v|$ for all $n \in \mathbb{Z}$, so any minimal sequence must be either $\{v\}$ or $\{-v\}$. For the case $n = 2$, let $\{\alpha, \beta\}$ be a minimal sequence, and suppose we have a $\lambda \in \Lambda \setminus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. By translation by some element of $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, we can assume without loss of generality that $\lambda = s\alpha + t\beta$ with $s, t \in (-1/2, 1/2]$ and $t \neq 0$ (else $\lambda \in \mathbb{R}\alpha$, and then by $n = 1$ we have $\lambda \in \mathbb{Z}\alpha$, a contradiction). Then λ lies in the translate of the (closed) fundamental parallelogram centered at the origin, and so the distance of λ from the origin is at most the largest distance of any point on this parallelogram, which we can bound above by the triangle inequality. In formulae,

$$|\lambda| \leq \sup_{(s,t) \in \left(-\frac{1}{2}, \frac{1}{2}\right]^2} |s\alpha + t\beta| < \frac{1}{2}\left(|\alpha| + |\beta|\right) \leq |\beta|,$$

where in the second step we have used the triangle inequality, and the inequality is sharp because α and β are linearly independent. However, this contradicts that $|\beta| \leq |\lambda|$, which is required by the condition that $\{\alpha, \beta\}$ is a minimal system (since λ is linearly independent from α). This contradiction proves that there can't be any $\lambda \in \Lambda \setminus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, i.e. that α, β is a basis for Λ. The cases $n = 3, 4$ proceed similarly, by replacing the use of the triangle inequality by a clever inequality on Euclidean lengths. ■

[6] This is what is usually called a full-rank lattice.

[7] If there are multiple such vectors of the same smallest length, we assume that we pick one. This arbitrary breaking of ties only has consequences for when $n = 4$.

[8] If you are uncomfortable with our assumptions about breaking ties, you can replace Λ with the deformation

$$\Lambda_t := \left\{ \sum_{i=1}^{n} a_i t^i \mathrm{e}_i : a_i \in \mathbb{Z}, a_1 \equiv a_2 \equiv \cdots \equiv a_5 \pmod 2 \right\}.$$

for $t > 1$ with $t - 1$ small enough so that $\sqrt{t^2 + t^4 + t^6 + t^8 + t^{10}} > 2t^5$. (This means roughly $t \lesssim 1.0611$.) Then the smallest possible lengths of vectors in Λ_t are $2t < 2t^2 < 2t^3 < 2t^4 < 2t^5$, so we are forced to take $a_i := \pm 2t^i \mathrm{e}_i$, and again these do not form a $\mathbb{Z}$-basis of Λ_t.

Q7. Let $\omega = \exp(2\pi \mathrm{i}/3)$, and let $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\mathrm{i}$ and $L = \mathbb{Z} \oplus \mathbb{Z}\omega$. Prove that

$$\sum_{\lambda \in \Lambda}{}' \lambda^{-6} = \sum_{\lambda \in L}{}' \lambda^{-4} = 0,$$

where the prime means that $\lambda = 0$ is omitted.

Proof. We showed in class that these infinite sums, the Eisenstein series $G_3(\mathrm{i}) = G_3(\Lambda)$ and $G_4(\omega) = G_4(L)$ respectively, are absolutely convergent, and hence all rearrangement in what follows is justified. The lattice Λ is invariant under the map $z \mapsto \mathrm{i}z$, and hence it follows that

$$G_3(\mathrm{i}) = \sum_{\lambda \in \Lambda}{}' \lambda^{-6} = \sum_{\lambda \in \Lambda}{}' (\mathrm{i}\lambda)^{-6} = \mathrm{i}^{-6} G_3(\mathrm{i}) = -G_3(\mathrm{i}),$$

so that $G_3(\mathrm{i}) = 0$. Similarly, the lattice L is invariant under $z \mapsto \omega z$, and hence it follows that

$$G_4(\omega) = \sum_{\lambda \in L}{}' \lambda^{-4} = \sum_{\lambda \in L}{}' (\omega\lambda)^{-4} = \omega^2 G_4(\omega),$$

from which again it follows that $G_4(\omega) = 0$. ■

Remark 11. In the above, we have used the very special property of the lattices Λ and L, namely that they admit automorphisms other than ± 1. In fact, for all values of $\tau \in \mathbb{H}/\operatorname{PSL}_2 \mathbb{Z}$ except $\tau = \mathrm{i}$ and $\tau = \omega$, the lattice $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$ satisfies $\operatorname{Aut}(\Lambda_\tau) = \{\pm 1\}$. (For the square and hexagonal lattices we have

$$\operatorname{Aut}(\Lambda_{\mathrm{i}}) = \mu_4 = \{\pm 1, \pm \mathrm{i}\} \text{ and } \operatorname{Aut}(\Lambda_\omega) = \mu_6 = \{\pm 1, \pm\omega, \pm\omega^2\},$$

and these correspond to j-invariants $j(\mathrm{i}) = 1728$ and $j(\omega) = 0$ respectively.)

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