MAT213A: Homework 9 – Solutions

1. Suppose $f \in S$ takes $(\Delta, 0) \to (\Omega, p)$. We can write Ω as the increasing union of polygons P_i. Let the Riemann maps $f_i : (\Delta, 0) \to (P_i, p)$ be chosen such that $f_i'(0) > 0$. (Strictly speaking, the f_i are not in S but they are almost in S). Recall that the limit of conformal maps is either conformal or constant. By the nesting principle, the $f_i'(0)$ are increasing, and are bounded by 1 from above. Thus any subsequential limit F cannot be constant, and is thus conformal. By the argument principle, one can see that the image of any such F would be Ω; therefore, the f_i converge to f. As $f_i'(0) \to 1$, we can rescale the f_i slightly so they would be in S.

2. Let Ω_n be constructed as follows: take two circles $D_1(0), D_1(10)$ and connect them by a thin bridge along the real axis of width $1/n$. Consider the sequence of Riemann maps f_n taking the unit disk to Ω_n mapping the origin to the origin (and normalized so that $f_n'(0) > 0$). The image of the limiting map has to be contained in the intersection of the Ω_i, but as the image is open and connected, we are forced to lose the second disk. Finally, the limiting map is not constant since all Ω_n contain $D_1(0)$ so the derivatives $f_n'(0)$ are bounded from below.

3. By looking at the power series expansion, it is easy to see that $f(iz) = if(z)$ implies that
$$f(z) = z + a_5 z^5 + a_9 z^9 + \dots$$
The function $g = f(\sqrt[4]{z})^4$ is well-defined and univalent, and furthermore belongs in $\mathcal{S}$. By Koebe's 1/4 theorem, we see that the image of $g(\Delta)$ contains $B(0, 1/4)$ and hence the image of $f(\Delta)$ contains $B(0, 1/\sqrt{2})$.

4. This follows from the compactness of $\mathcal{S}$ and the fact that you can connect any two points in U by a chain of discs.

5. We compute the Riemann map from $\mathbb{H} \to U$. The polygon in question has two vertices, one with an angle of 2π at 0, and a -2π at infinity. From this $\tilde{f}(w) = \alpha \int \frac{d\zeta}{\zeta^{-1}} + \beta = \alpha w^2 + \beta$. Take $\tilde{f}(w) = w^2$. Transfering the map to disk, we get $f(z) = 4z/(1-z)^2$ which satisfied the desired conditions.

6. If $\omega = df$, by Stokes theorem, the integral of ω over any loop is 0. Conversely, if the integral of ω over any loop is zero, we can define $f(z) = \int_*^z \omega$.

7. If f is affine, then $f''(z) = 0$, and so $\mathcal{N}f = 0$. Conversely, if $\mathcal{N}f = 0$ identically, then $f''(z) = 0$ identically, and so f is affine. We compute

$$\begin{aligned}
\mathcal{N}(f(g(z))) &= \frac{(f' \circ g \cdot g'(z))'}{f' \circ g \cdot g'(z)} dz \\
&= \frac{f'' \circ g \cdot (g')^2 + f' \circ g \cdot g''}{f' \circ g \cdot g'(z)} dz \\
&= \frac{f'' \circ g}{f' \circ g}(g' dz) + \frac{g''}{g'} dz \\
&= g^* \mathcal{N} f + \mathcal{N} g.
\end{aligned}$$

8. (Solution of Eric Larson) If f'' vanishes identically, then f is affine and in particular, a Möbius transformation. Otherwise, f is Möbius if and only if there is a constant a such that

$$d/dz^2\Big((z+a)f(z)\Big) = 0$$

Expanding, one finds that this is equivalent to

$$\frac{f'(z)}{f''(z)} = -\frac{1}{2}(z+a)$$

which is the same as

$$\frac{d}{dz}\Big(\frac{f'(z)}{f''(z)}\Big) = -\frac{1}{2}.$$

We can compute

$$\frac{d}{dz}\Big(\frac{f'(z)}{f''(z)}\Big) + \frac{1}{2} = \frac{3f''(z)^2 - 2f'(z)f'''(z)}{2f''(z)^2} = -\mathcal{S}f \cdot \frac{f'(z)^2}{2f''(z)^2},$$

i.e $f(z)$ is Möbius if its Schwarzian derivative vanishes.

9. To see that the determinant is constant, we compute

$$W'(z) = (u_1 u_2' - u_2 u_1')' = u_1 u_2'' - u_2 u_1'' = u_1(-\varphi u_2) - u_2(-\varphi u_1) = 0.$$

We now calculate $\mathcal{S}f$. As

$$\frac{f''}{f'} = -2\frac{u_2'}{u_2},$$

we see that $\mathcal{S}f = -2u_2''/u_2 = 2\varphi$. Taking different solutions for u_1, u_2, amounts to taking linear combinations, and f would be changed by a Möbius transformation. By problem 8 and the cocycle property of the Schwarzian derivative, $\mathcal{S}f$ would be unchanged.