# MAT213A: Homework 8 – Solutions

1. Recall that
$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\Big(\frac{1}{z-n} + \frac{1}{z+n}\Big).$$
Differentiating three times, we find that
$$\frac{-2\pi^4(\cos(2\pi z)+2)}{\sin^4(\pi z)} = \frac{1}{z^4} + \sum_{n=1}^{\infty}\Big(\frac{1}{(z-n)^4} + \frac{1}{(z+n)^4}\Big).$$
Subtracting $1/z^4$ from both sides and dividing by 2, we find that
$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{2}\lim_{z\to 0}\left(\frac{-2\pi^4(\cos(2\pi z)+2)}{\sin^4(\pi z)} - \frac{1}{z^4}\right) = \frac{\pi^4}{90}.$$

2. (Solution by Sam Watson) Recall Stirling's formula says that
$$\Gamma(x) \sim \sqrt{\frac{2\pi}{x}}\Big(\frac{x}{e}\Big)^x$$
Taking logarithms, we find that
$$\log\Gamma(x) \sim \log(\sqrt{2\pi}) - \frac{1}{2}\log x + x\log x - x.$$
By L'Hôpital's rule, asymptotic formulas are preserved under differentiation. We find
$$\frac{\Gamma'(x)}{\Gamma(x)} \sim \log x - \frac{1}{2x}.$$
Hence,
$$\Gamma'(x) \sim \Gamma(x)\Big(\log x - \frac{1}{2x}\Big) \sim \sqrt{\frac{2\pi}{x}}\Big(\frac{x}{e}\Big)^x\Big(\log x - \frac{1}{2x}\Big).$$

3. The function $g(w) = Arg(w)/\pi$ on the upper half plane has the boundary values 0 on the positive real ray, 1 on the negative real ray. Transferring it to the disk by a Möbius transformation, we obtain a function $v(z)$ which is 1 on the upper boundary and 0 on the lower boundary. The function $u(z) = v(z^2)$ is the desired function. It is now clear that $u(z) = 1/2$ on the real and imaginary lines.

4. For $z \in \Delta$, let $\delta_z = 1 - |z|$ and $B_z$ be the ball centered at $z$ of radius $\delta_z$. Applying the mean value property and then Hölder's inequality

$$f(z) = \int_{B_z} f(z) \cdot 1 \; |dz|^2 = C(\delta_z) \cdot ||f||_p$$

where $C(\delta_z)$ is some function of $\delta_z$ (which tends to $\infty$ as $\delta_z$ tends to 0). The above shows that if $f_n \to f$ in $L^p$ then it converges uniformly on compact sets, so the limit function would be analytic. The set $\mathcal{F}$ is obviously closed, and the above also shows the functions are locally uniformly bounded, hence $\mathcal{F}$ is compact.

5. For $\rho/\delta = 2$, take $(\Delta, 0)$; for $\rho/\delta = 1/2$, take $(\mathbb{C} \setminus (-\infty, -1/4], 0)$. In the second case the conformal radius can be measured by the Koebe map.

6. By the area theorem, $b_2 = 1/\sqrt{2}$ would force all other coefficients to be 0, so the map would reduce to $z + (1/\sqrt{2})z^{-2}$. This map, sadly fails to be univalent.

7. Given $f$, let $f_r = (1/r)f(rz)$. It is easy to see that $f_r \in S$ and converge uniformly on compact sets to $f$. As $f$ is smooth in the unit disk, the images of $f_r$ are Jordan domains.

8. By Cauchy's bound and the Köbe distortion theorem,

$$|a_n| \leq \frac{M(r)}{r^n} = \frac{r/(1-r)^2}{r^n}$$

In particular, if take $r = 1/2$, we see that $|a_n| \leq 2^{n+1}$.